Lower bounds for the algebraic connectivity of graphs with specified subgraphs

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Abstract

The second smallest eigenvalue of the Laplacian matrix of a graph $G$ is called the algebraic connectivity and denoted by $a(G)$. We prove that

$$a(G) > \frac{\pi^2}{3} \left( p \frac{12 \overline{g}(n_1, n_2, \ldots, n_p)^2 - \pi^2}{4 \overline{g}(n_1, n_2, \ldots, n_p)^4} + 4(q - p) \frac{3 \overline{g}(n_{p+1}, n_{p+2}, \ldots, n_q)^2 - \pi^2}{\overline{g}(n_{p+1}, n_{p+2}, \ldots, n_q)^4} \right),$$

holds for every non-trivial graph $G$ which contains edge-disjoint spanning subgraphs $G_1, G_2, \ldots, G_q$ such that, for $1 \leq i \leq p$, $a(G_i) \geq a(P_{n_i})$, with $n_i \geq 2$, and, for $p + 1 \leq i \leq q$, $a(G_i) \geq a(C_{n_i})$, where $P_{n_i}$ and $C_{n_i}$ denote the path and the cycle of the corresponding order, respectively, and $\overline{g}$ denotes the geometric mean of given arguments. Among certain consequences, we emphasize the following lower bound

$$a(G) > \pi^2 \frac{12(4q - 3p)n^2 - (16q - 15p)\pi^2}{12n^4},$$

referring to $G$ which has $n$ ($n \geq 2$) vertices and contains $p$ Hamiltonian paths and $q - p$ Hamiltonian cycles, such that all of them are edge-disjoint. We also discuss the quality of the obtained lower bounds.

Keywords: edge-disjoint subgraphs, Laplacian matrix, algebraic connectivity, geometric mean, Hamiltonian cycle

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1. Introduction

The Laplacian of a graph $G$ is the positive semidefinite matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the standard adjacency matrix. Among all eigenvalues of the Laplacian of a graph, one of the most popular is the second smallest called, by Fiedler [5], the algebraic connectivity of a graph. The algebraic connectivity is usually denoted by $a(G)$. Its significance is due to the fact that it measures (to a certain extent) how well a graph is connected. For example, a graph $G$ is connected if and only if $a(G) > 0$.

The number of vertices (also known as the order) and the number of edges of a graph $G$ are denoted by $n$ and $m$ (or $m(G)$), respectively. We also use $d$ for the diameter of a graph. A path and a cycle of order $n$ are denoted by $P_n$ and $C_n$, respectively. A graph is Hamiltonian if it contains a spanning subgraph which is a cycle, while every such cycle is referred to as a Hamiltonian cycle. Similarly, every spanning path is referred to as a Hamiltonian path.

There is a significantly large number of bounds for the algebraic connectivity expressed in terms of other graph invariants. One of them is a classical result of Mohar [8] stating that

$$a(G) \geq \frac{4}{dn},$$

where, as said above, $d$ is the diameter of $G$. Some others can be found in [1, 4, 10]. In this study we obtain a lower bound for $a(G)$ which relies on the assumption that $G$ contains edge-disjoint spanning subgraphs such that the algebraic connectivity of each of them is not less than the algebraic connectivity of either a fixed path or a fixed cycle. This result yields the lower bound for $a(G)$ expressed in terms of orders of the longest paths or cycles contained in the corresponding spanning subgraphs. In particular, we establish a lower bound when $G$ contains the set of edge-disjoint Hamiltonian paths and cycles.

Our contribution is reported in the forthcoming sections. Precisely, theoretical results are given in Section 2, a concluding discussion is given in Section 3, while in the Appendix we observe the existence of an upper bound for the algebraic connectivity (which is implicitly proved in [2]).

2. Results

We use the following lemma referred to Fiedler.

**Lemma 2.1.** [5] Let $G_1, G_2, \ldots, G_k$ be edge-disjoint spanning subgraphs of a non-trivial signed graph $G$ such that $m(G) = \sum_{i=1}^{k} m(G_i)$. Then

$$a(G) \geq \sum_{i=1}^{k} a(G_i).$$

We also use the following limit point without reference:

$$\lim_{x \to 0} \left( \frac{\sum_{i=1}^{k} t_i^x}{k} \right)^{\frac{1}{x}} = \left( \prod_{i=1}^{k} t_i \right)^{\frac{1}{k}},$$

for positive $t_1, t_2, \ldots, t_k$. 258
Theorem 2.1. Assume that a graph $G$ with $n$ ($n \geq 2$) vertices contains edge-disjoint spanning subgraphs $G_1, G_2, \ldots, G_q$ such that for $1 \leq i \leq p$ it holds $a(G_i) \geq a(P_{n_i})$ with $n_i \geq 2$ and for $p + 1 \leq i \leq q$ it holds $a(G_i) \geq a(C_{n_i})$. Then

$$a(G) > \frac{\pi^2}{3} \left( p \frac{12\bar{g}(n_1, n_2, \ldots, n_p)^2 - \pi^2}{4\bar{g}(n_1, n_2, \ldots, n_p)^4} + 4(q - p) \frac{3\bar{g}(n_{p+1}, n_{p+2}, \ldots, n_q)^2 - \pi^2}{\bar{g}(n_{p+1}, n_{p+2}, \ldots, n_q)^4} \right),$$

where $\bar{g}$ denotes the geometric mean of given arguments.

Proof. By Lemma 2.1, $a(G) \geq \sum_{i=1}^q a(G_i)$, i.e., $a(G) \geq \sum_{i=1}^p a(P_{n_i}) + \sum_{i=p+1}^q a(C_{n_i})$. It holds $a(P_{n_i}) = 2(1 - \cos(\frac{\pi}{n_i}))$ and $a(C_{n_i}) = 2(1 - \cos(\frac{2\pi}{n_i}))$; see, for example, [1].

Using the Taylor series, we get

$$a(P_{n_i}) > 2 \left( 1 - 1 - \frac{\pi^2}{2n_i^2} - \frac{\pi^4}{24n_i^4} \right) = \frac{\pi^2}{12n_i^4} (12n_i^2 - \pi^2)$$

and

$$a(C_{n_i}) > 2 \left( 1 - 1 - \frac{4\pi^2}{2n_i^2} - \frac{16\pi^4}{24n_i^4} \right) = \frac{4\pi^2}{3n_i^4} (3n_i^2 - \pi^2)$$

that gives

$$a(G) > \frac{\pi^2}{3} \left( \frac{1}{4} \sum_{i=1}^p \frac{12n_i^2 - \pi^2}{n_i^4} + 4 \sum_{i=p+1}^q \frac{3n_i^2 - \pi^2}{n_i^4} \right).$$

We consider the first sum of (4). For $\alpha \geq 2$, we define the function

$$f_\alpha(x) = \frac{12x^\alpha - \pi^2}{x^{2\alpha}}.$$ 

It holds $f_\alpha''(x) = \frac{2\alpha(\alpha + 1)x^\alpha - \pi^2(2\alpha + 1)}{x^{2\alpha + 1}}$, and so, for $x \geq 2$, $f_\alpha$ is convex. Using the Jensen’s inequality, we get

$$\sum_{i=1}^p \frac{12n_i^2 - \pi^2}{n_i^4} \geq pf_\alpha \left( \frac{\sum_{i=1}^p n_i^{2/\alpha}}{p} \right) = \frac{12 \left( \frac{\sum_{i=1}^p n_i^{2/\alpha}}{p} \right)^{\alpha}}{\left( \frac{\sum_{i=1}^p n_i^{2/\alpha}}{p} \right)^{2\alpha}} - \pi^2.$$ 

If $\alpha \to \infty$, by (2), we have

$$\sum_{i=1}^p \frac{12n_i^2 - \pi^2}{n_i^4} \geq p \frac{12\bar{g}(n_1, n_2, \ldots, n_p)^2 - \pi^2}{\bar{g}(n_1, n_2, \ldots, n_p)^4}. \quad (5)$$

The second sum of (4) is considered in a similar way. For $\alpha \geq 3$, we define the function

$$h_\alpha(x) = \frac{3x^\alpha - \pi^2}{x^{2\alpha}},$$
which is convex for \( x \geq 3 \) (as \( h''_\alpha(x) = \frac{\alpha}{x^{2(\alpha+1)}} \left( 3(\alpha + 1)x^\alpha - 2\pi^2(2\alpha + 1) \right) \)). This leads to

\[
\sum_{i=p+1}^{q} \frac{3n_i^2 - \pi^2}{n_i^4} \geq (q - p) h_\alpha \left( \frac{\sum_{i=p+1}^{q} n_i^{2/\alpha}}{q - p} \right) = (q - p) \frac{3 \left( \frac{\sum_{i=p+1}^{q} n_i^{2/\alpha}}{q - p} \right)^{\alpha}}{2^{2\alpha}}.
\]

Letting \( \alpha \to \infty \), we get

\[
\sum_{i=p+1}^{q} \frac{3n_i^2 - \pi^2}{n_i^4} \geq (q - p) \frac{3g(n_{p+1}, n_{p+2}, \ldots, n_q)^2 - \pi^2}{\bar{g}(n_{p+1}, n_{p+2}, \ldots, n_q)^4}. \tag{6}
\]

The inequality (4), in conjunction with (5) and (6), gives (3).

Here are some consequences.

**Corollary 2.1.** Under the assumptions of Theorem 2.1, we have

\[
a(G) > \frac{\pi^2}{3} \left( p \frac{12\bar{\pi}(n_1, n_2, \ldots, n_p)^2 - \pi^2}{4\bar{\pi}(n_1, n_2, \ldots, n_p)^4} + 4(q - p) \frac{3\bar{\pi}(n_{p+1}, n_{p+2}, \ldots, n_q)^2 - \pi^2}{\bar{\pi}(n_{p+1}, n_{p+2}, \ldots, n_q)^4} \right), \tag{7}
\]

where \( \bar{\pi} \) denotes the arithmetic mean of given arguments.

**Proof.** The function \( \frac{12x^2 - \pi^2}{4x^4} \) decreases for \( x \geq 2 \), and so

\[
\frac{12\bar{\pi}(n_1, n_2, \ldots, n_p)^2 - \pi^2}{4\bar{\pi}(n_1, n_2, \ldots, n_p)^4} \geq \frac{12\bar{\pi}(n_1, n_2, \ldots, n_p)^2 - \pi^2}{4\bar{\pi}(n_1, n_2, \ldots, n_p)^4}.
\]

Similarly, as \( \frac{3x^2 - \pi^2}{x^4} \) decreases for \( x \geq 3 \), we have

\[
\frac{3\bar{\pi}(n_{p+1}, n_{p+2}, \ldots, n_q)^2 - \pi^2}{\bar{\pi}(n_{p+1}, n_{p+2}, \ldots, n_q)^4} \geq \frac{3\bar{\pi}(n_{p+1}, n_{p+2}, \ldots, n_q)^2 - \pi^2}{\bar{\pi}(n_{p+1}, n_{p+2}, \ldots, n_q)^4},
\]

and the proof follows.

**Corollary 2.2.** Under the assumptions of Theorem 2.1, we have

\[
a(G) > q \pi^2 \frac{12\bar{g}(n_1, n_2, \ldots, n_q)^2 - \pi^2}{12\bar{g}(n_1, n_2, \ldots, n_q)^4} \geq q \pi^2 \frac{12\bar{\pi}(n_1, n_2, \ldots, n_q)^2 - \pi^2}{12\bar{\pi}(n_1, n_2, \ldots, n_q)^4}. \tag{8}
\]

where \( \bar{g} \) and \( \bar{\pi} \) denote the geometric mean and the arithmetic mean of given arguments, respectively.

**Proof.** In the notation of Theorem 2.1, since \( a(C_n) > a(P_n) \), we have \( a(G_i) \geq a(P_n) \), for \( 1 \leq i \leq q \). The first inequality follows by setting \( p = q \) in (3), and then the second follows by the previous corollary.

We proceed with the following particular result.

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Theorem 2.2. If a non-trivial graph $G$ contains $p$ Hamiltonian paths and $q - p$ Hamiltonian cycles, such that all of them are edge-disjoint, then

$$a(G) > \pi^2 \frac{12(4q - 3p)n^2 - (16q - 15p)\pi^2}{12n^4}. \quad (9)$$

Proof. Obviously, $G$ contains edge-disjoint spanning subgraphs $G_1, G_2, \ldots, G_q$ such that the first $p$ of them contain a Hamiltonian path and the remaining ones contain a Hamiltonian cycle. By Lemma 2.1, the algebraic connectivity of $G_i$ is at least the algebraic connectivity of its spanning subgraph, i.e., all the assumptions of Theorem 2.1 are satisfied (with $n_i = n$, for $1 \leq i \leq q$). By (3), we compute

$$a(G) > \frac{\pi^2}{3} \left( p\frac{12n^2 - \pi^2}{4n^4} + 4(q - p)\frac{3n^2 - \pi^2}{n^4} \right),$$

giving the desired inequality. \hfill \square

Since, for a connected graph $G$, we have $a(G) \geq 2\epsilon(1 - \cos \frac{\pi}{n})$ (see [5]), where $\epsilon$ denotes the edge connectivity of $G$, it follows that Theorem 2.1 can be applied to any connected non-trivial graph with itself in the role of the unique spanning subgraph. Here is another criterion concerning graphs with small diameter.

Theorem 2.3. If a connected graph $G$ with $n$ ($n \geq 2$) vertices and diameter $d$ contains a path $P_k$ (resp. a cycle $C_k$) such that $4k^2 \geq dn\pi^2$ (resp. $k^2 \geq dn\pi^2$), then $a(G) > a(P_k)$ (resp. $a(G) > a(C_k)$).

Proof. We use the inequality (1). Considering the existence of a path $P_k$, we get

$$a(G) \geq \frac{4}{dn} \geq \frac{4}{\frac{4k^2}{\pi^2}} = \frac{\pi^2}{k^2} = 2 \left( 1 - 1 + \frac{\pi^2}{2k^2} \right) > 2 \left( 1 - \cos \frac{\pi}{k} \right).$$

The existence of a cycle satisfying the assumption of the theorem is considered in the same way. \hfill \square

3. Remarks

The bound (3) and its consequences (7)–(9) are always non-trivial, in the sense that they are never negative. An easy consequence of (9) is the following lower bound

$$a(G) > 4q\pi^2 \frac{3n^2 - \pi^2}{3n^4}, \quad (10)$$

where $q$ stands for the number of edge-disjoint Hamiltonian cycles. In general, the bound (10) is incomparable with (1), but it gives a better estimate whenever

$$q \geq \frac{3n^3}{d\pi^2(n^2 - \pi^2)}. \quad (11)$$

In particular, this occurs for every Hamiltonian graph with $d \geq \frac{3n^3}{\pi^2(3n^2 - \pi^2)}$, as then the right hand side of (11) is at most 1; this lower bound for $d$ is asymptotically $n/\pi^2$.  

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Example 1. Consider the graph $G$ obtained by inserting an edge between every pair of vertices at distance 2 of a cycle $C_{2k+1}$, for $k \geq 2$. Obviously, $G$ has exactly 2 edge-disjoint Hamiltonian cycles, and thus due to (10) we have $a(G) > 8\pi^2 \frac{2(2k+1)^2 - \pi^2}{3(2k+1)^4}$. Say, for $k = 4$, we get $2.12 \approx a(G) > 0.94$.

As the right hand side of (10) increases with the number of edge-disjoint Hamiltonian cycles, it would be natural to consider it in conjunction with a lower bound for the number of such cycles. In this context, we recall that Nash-Williams proved that the assumptions of the well-known Dirac’s theorem guarantee the existence of many edge-disjoint Hamiltonian cycles. Precisely, every graph with $n$ vertices and minimum vertex degree at least $n/2$ contains at least $\lfloor 5n/22 \rfloor$ edge-disjoint Hamiltonian cycles [9]. It is conjectured in the same reference that every $r$-regular graph with at most $2r$ vertices contains $r/2$ Hamiltonian cycles. This conjecture is still open; an approximate version stating that every $r$-regular graph with $n$ ($14 \leq n \leq 2r + 1$) vertices contains $\lfloor (3r - n + 1)/6 \rfloor$ edge-disjoint Hamiltonian cycles is proved by Jackson [7]. For some asymptotic results, we refer to Christofides, Kühn and Osthus [3]. Particular constructions of arbitrarily large graphs with a specified number of Hamiltonian cycles can be found in Haythorpe’s [6].

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References


Appendix

We recall an interesting upper bound, obtained by Bollobás and Nikiforov [2], for the sum of the \( k - 1 \) least eigenvalues of a Hermitian matrix. Namely, if \( N_1 \sqcup N_2 \sqcup \cdots \sqcup N_k \) is a partition of a Hermitian matrix \( M = (m_{ij}) \) with eigenvalues \( \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \), then

\[
\sum_{p=n-k+2}^{n} \nu_p \leq \sum_{p=1}^{k} \frac{1}{|N_p|} \sum_{(i,j) : i,j \in N_p} m_{ij} - \frac{1}{n} \text{sum}(M),
\]  

(12)

where \( \text{sum}(M) \) denotes the sum of the entries of \( M \).

By considering the Laplacian matrix of a graph in the role of \( M \) and inserting \( k = 3 \) in (12), we get

\[
a(G) \leq \sum_{p=1}^{3} \frac{c(N_p)}{|N_p|},
\]

(13)

where, clearly \( N_1 \sqcup N_2 \sqcup N_3 \) is a vertex set partition, while \( c(N_p) \) denotes the cut of \( N_p \), i.e., the number of edges with exactly one end in \( N_p \). Indeed, if \( L = (l_{ij}) \) is the Laplacian matrix, then \( \sum_{(i,j) : i,j \in N_p} l_{ij} = c(N_p) \) and \( \text{sum}(L) = 0 \), so we get (13). This upper bound can be used to estimate the algebraic connectivity of graphs with given tripartion of a vertex set. For example, if \( G \) contains at least two cut-edges, then we have

\[
a(G) \leq \frac{1}{|N_1|} + \frac{2}{|N_2|} + \frac{1}{|N_3|},
\]

where cut-edges are located between \( N_1 \) and \( N_2 \), and \( N_2 \) and \( N_3 \).