On maximum packings of $\lambda$-fold complete 3-uniform hypergraphs with triple-hyperstars of size 4

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Abstract

A symmetric triple-hyperstar is a connected, 3-uniform hypergraph where, for some edge $\{a, b, c\}$, vertices $a$, $b$, and $c$ all have degree $k > 1$ and all other edges contain exactly 2 vertices of degree 1. Let $H$ denote the symmetric triple-hyperstar with 4 edges and, for positive integers $\lambda$ and $v$, let $\lambda K_v^{(3)}$ denote the $\lambda$-fold complete 3-uniform hypergraph on $v$ vertices. We find maximum packings of $\lambda K_v^{(3)}$ with copies of $H$.

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1. Introduction

A hypergraph $H$ consists of a finite, nonempty set $V$ of vertices and a finite collection $E = \{e_1, e_2, \ldots, e_m\}$ of nonempty subsets of $V$ called hyperedges or simply edges. For a given hypergraph $H$, we use $V(H)$ and $E(H)$ to denote the vertex set and the edge set (or multiset) of $H$, respectively. We call $|V(H)|$ and $|E(H)|$ the order and size of $H$, respectively. A hypergraph $H$ is simple if no edge appears more than once in $E(H)$. If for each $e \in E(H)$ we have $|e| = t$, then $H$ is said to be $t$-uniform. Thus $t$-uniform hypergraphs are generalizations of the concept...
of a graph (where \( t = 2 \)). Graphs with repeated edges are often called multigraphs. If \( H \) is a simple hypergraph and \( \lambda \) is a positive integer, then \( \lambda \)-fold \( H \), denoted \( \lambda H \), is the multi-hypergraph obtained from \( H \) by repeating each edge exactly \( \lambda \) times. The hypergraph with vertex set \( V \) and edge set the set of all \( t \)-element subsets of \( V \) is called the complete \( t \)-uniform hypergraph on \( V \) and is denoted by \( K^{(t)}_V \). If \( v = |V| \), then \( \lambda K^{(t)}_v \) is called the \( \lambda \)-fold complete \( t \)-uniform hypergraph of order \( v \) and is used to denote any hypergraph isomorphic to \( \lambda K^{(t)}_V \). When \( t = 2 \), we will use \( \lambda K_v \) in place of \( \lambda K^{(2)}_V \). Similarly, if \( \lambda = 1 \), then we will use \( K^{(t)}_v \) in place of \( 1 K^{(t)}_V \). If \( H' \) is a subhypergraph of \( H \), then \( H \setminus H' \) denotes the hypergraph obtained from \( H \) by deleting the edges of \( H' \). We may refer to \( H \setminus H' \) as the hypergraph \( H \) with a hole \( H' \). The vertices in \( H' \) may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs or multigraphs into edge-disjoint subgraphs. A decomposition of a multigraph \( K \) is a set \( \Delta = \{G_1, G_2, \ldots, G_s\} \) of subgraphs of \( K \) such that \( \{E(G_1), E(G_2), \ldots, E(G_s)\} \) is a partition of \( E(K) \). If each element of \( \Delta \) is isomorphic to a fixed graph \( G \), then \( \Delta \) is called a \( G \)-decomposition of \( K \). If exactly one element \( L \in \Delta \) is not isomorphic to \( G \), then \( \Delta \) is called a \( G \)-packing of \( K \) with leave \( L \). Such a \( G \)-packing is maximum if no other possible \( G \)-packing of \( K \) has a leave of a smaller size than that of \( L \). Clearly, if \( |E(L)| < |E(G)| \), then the \( G \)-packing is maximum. Moreover, a \( G \)-decomposition of \( K \) can be viewed as a maximum \( G \)-packing with an empty leave.

A \( G \)-decomposition of \( \lambda K_v \) is also known as a \( G \)-design of order \( v \) and index \( \lambda \). A \( K_v \)-design of order \( v \) and index \( \lambda \) is usually known as a \( 2-(v, k, \lambda) \) design or as a balanced incomplete block design of index \( \lambda \) or a \( (v, k, \lambda) \)-BIBD. The problem of determining all \( v \) for which there exists a \( G \)-design of order \( v \) is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A decomposition of a hypergraph \( K \) is a set \( \Delta = \{H_1, H_2, \ldots, H_s\} \) of subhypergraphs of \( K \) such that \( \{E(H_1), E(H_2), \ldots, E(H_s)\} \) is a partition of \( E(K) \). Any element of \( \Delta \) isomorphic to a fixed hypergraph \( H \) is called an \( H \)-block. If all elements of \( \Delta \) are \( H \)-blocks, then \( \Delta \) is called an \( H \)-decomposition of \( K \). If exactly one element \( L \in \Delta \) is not an \( H \)-block, then \( \Delta \) is called an \( H \)-packing of \( K \) with leave \( L \), where we again define such a packing to be maximum if \( L \) has the fewest edges possible. An \( H \)-decomposition of \( \lambda K^{(t)}_v \) is called an \( H \)-design of order \( v \) and index \( \lambda \). The problem of determining all \( v \) for which there exists an \( H \)-design of order \( v \) and index \( \lambda \) is called the \( \lambda \)-fold spectrum problem for \( H \)-designs.

A \( K^{(t)}_k \)-design of order \( v \) and index \( \lambda \) is a generalization of \( 2-(v, k, \lambda) \) designs and is known as a \( t-(v, k, \lambda) \) design or simply as a \( t \)-design. A summary of results on \( t \)-designs appears in [16]. A \( t-(v, k, 1) \) design is also known as a Steiner system and is denoted by \( S(t, v, k) \) (see [9] for a summary of results on Steiner systems). Keevash [15] has recently shown that for all \( t \) and \( k \) the obvious necessary conditions for the existence of an \( S(t, k, v) \)-design are sufficient for sufficiently large values of \( v \). Similar results were obtained by Glock, Kühn, Lo, and Osthus [10, 11] and extended to include the corresponding asymptotic results for \( H \)-designs of order \( v \) for all uniform hypergraphs \( H \). These results for \( t \)-uniform hypergraphs mirror the celebrated results of Wilson [24] for graphs. Although these asymptotic results assure the existence of \( H \)-designs for sufficiently large values of \( v \) for any uniform hypergraph \( H \), the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.
In the study of graph decompositions, a fair amount of the focus has been on $G$-decompositions of $K_v$, where $G$ is a graph with a relatively small number of edges (see [1] and [6] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [4], the 1-fold spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the 1-fold spectrum problem was settled in [5] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [5], they also settle the 1-fold spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered $H$-designs where $H$ is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let $T$, $O$, and $I$ denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph $T$ is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [12]. In another paper [13], Hanani settled the spectrum problem for $O$-designs and gave necessary conditions for the existence of $I$-designs. The 1-fold spectrum problem is also settled for a type of 3-uniform hyperstars which is part of a larger class of hypergraphs known as delta-systems. For a positive integer $m$, let $S_m^{(3)}$ denote the 3-uniform hypergraph of size $m$ that consists of one vertex of degree $m$ and $2m$ vertices of degree one. Necessary and sufficient conditions for the existence of $S_m^{(3)}$-decompositions of $K_v^{(3)}$ are given in [22] for $m \in \{4, 5, 6\}$ and settled in [19] for any $m$. Some results on maximum $S_m^{(3)}$-packings of $K_v^{(3)}$ are given in [20]. Perhaps the best known general result on decompositions of complete $t$-uniform hypergraphs is Baranyai’s result [3] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers $m$. There are, however, several articles on decompositions of complete $t$-uniform hypergraphs (see [2] and [21]) and of $t$-uniform $t$-partite hypergraphs (see [17] and [23]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [14] and [18]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this paper we are interested in maximum $H$-packings of $\lambda K_v^{(3)}$, where $H$ is a 3-uniform symmetric triple-hyperstar with 4 edges. A triple-hyperstar is a connected 3-uniform hypergraph where, for some edge $\{a, b, c\}$, vertices $a$, $b$, and $c$ all have degree greater than 1 and all other edges contain exactly two vertices of degree 1. That is, if the degrees of vertices $a$, $b$, and $c$ in the triple-hyperstar are $m_1 + 1$, $m_2 + 1$, and $m_3 + 1$, respectively, then the removal of edge $\{a, b, c\}$ would result in the hypergraph consisting of three components, namely $S_{m_1}^{(3)}$, $S_{m_2}^{(3)}$, and $S_{m_3}^{(3)}$. We call such a triple-hyperstar symmetric if $m_1 = m_2 = m_3 = m$. Thus a symmetric triple-hyperstar has $6m + 3$ vertices and $3m + 1$ edges. We are interested in the case $m = 1$.

Let $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$ denote the symmetric triple-hyperstar $H$ with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and edge set $\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}$ as seen Figure 1. Here we show that for all $v \geq 9$ and $\lambda \geq 1$, there exists a maximum $H$-packing of $\lambda K_v^{(3)}$ where the leave has fewer than 4 edges.

1.1. Additional Notation and Terminology

Let $\mathbb{Z}_n$ denote the group of integers modulo $n$. We next define some notation for certain types of 3-uniform hypergraphs.
Let $U_1, U_2, U_3$ be pairwise disjoint sets. The hypergraph with vertex set $U_1 \cup U_2 \cup U_3$ and edge set consisting of all 3-element sets having exactly one vertex in each of $U_1, U_2, U_3$ is denoted by $K_{U_1,U_2,U_3}^{(3)}$. The hypergraph with vertex set $U_1 \cup U_2$ and edge set consisting of all 3-element sets having at most 2 vertices in each of $U_1, U_2$ is denoted by $L_{U_1,U_2}^{(3)}$. If $|U_i| = u_i$ for $i \in \{1, 2, 3\}$, we may use $K_{u_1,u_2,u_3}^{(3)}$ or $L_{u_1,u_2}^{(3)}$ to denote any hypergraph that is isomorphic to $K_{U_1,U_2,U_3}^{(3)}$ or $L_{U_1,U_2}^{(3)}$, respectively.

2. Main Results

2.1. Decompositions and Packings of Simple Hypergraphs

We begin by giving necessary conditions for the existence of an $H$-decomposition of $K_v^{(3)}$. An obvious necessary condition is that 4 must divide the number of edges in $K_v^{(3)}$, and thus we must have $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$. Since $K_1^{(3)}$ and $K_2^{(3)}$ contain no edges, it is vacuously true that $H$ decomposes $K_1^{(3)}$ and $K_2^{(3)}$. Also, since $H$ has order 9, there is no $H$-decomposition of $K_4^{(3)}$, $K_6^{(3)}$, or $K_8^{(3)}$. Hence, we have the following.

**Lemma 1.** There exists an $H$-decomposition of $K_v^{(3)}$ only if $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$ and $v \not\in \{4, 6, 8\}$.

We intend to prove that the above conditions are sufficient by showing how to construct $H$-decompositions of $K_v^{(3)}$ for all $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$ with $v \geq 9$. Our constructions are dependent on the many small examples given in the Appendix. We begin by proving a lemma that is fundamental to our constructions.

**Lemma 2.** Let $n$, $x$, and $r$ be nonnegative integers such that $nx + r \geq 3$. There exists a decomposition of $K_{nx+r}^{(3)}$ that is comprised of isomorphic copies of each of the following under the given conditions:

- $K_r^{(3)}$ if $x = 0$,
- $K_{n+r}^{(3)}$ if $x \geq 1$, 

Figure 1. The symmetric triple-hyperstar $H$ of size 4, denoted by $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$. 
Lemma 2 it suffices to find $K_r$ in Lemma 1. Thus we need only to establish their sufficiency. Let $v$ only consider when $x \geq 2$, $K_{r,n,n} \cup L_{n,n}^{(3)}$ if $x \geq 2$, and $K_{n,n,n}$ if $x \geq 3$.

Furthermore, if $x \geq 1$ and $r \geq 3$, then the decomposition contains exactly one isomorphic copy of $K_{n+r}^{(3)}$.

Proof. If $x \in \{0, 1\}$, the decomposition is trivial. Similarly, if $n = 0$, then $r \geq 3$, and the result is trivial because $K_r^{(3)} = K_{n+r}^{(3)} = K_{n+n+r}$ while $K_{n+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,n,n} \cup L_{n,n}^{(3)}$, and $K_{n,n,n}$ are all empty (i.e., contain no edges). For the remainder of the proof, we assume that $x \geq 2$ and $n \geq 1$.

Let $V_0, V_1, \ldots, V_x$ be pairwise disjoint sets of vertices with $|V_0| = r$ and $|V_1| = |V_2| = \cdots = |V_x| = n$. Then, the decomposition of $K_{n+r}^{(3)}$ results from the fact that the complete 3-uniform hypergraph on the vertex set $V_0 \cup V_1 \cup \cdots \cup V_x$, which is $nx + r$ vertices, can be viewed as the (edge-disjoint) union

$$K_{V_0 \cup V_0}^{(3)} \cup \bigcup_{2 \leq i \leq x} \left( \bigcup_{K_{V_0 \cup V_0}^{(3)} \setminus K_{V_0}^{(3)}} \bigcup_{1 \leq i < j \leq x} \left( K_{V_0 \cup V_0}^{(3)} \cup L_{V_0 \cup V_0}^{(3)} \right) \bigcup_{1 \leq i < j < k \leq x} \left( K_{V_0 \cup V_0}^{(3)} \right) \right).$$

In addition, if $r \geq 3$, the single isomorphic copy of $K_{n+r}^{(3)}$ in the decomposition is $K_{V_0 \cup V_0}^{(3)}$.

We now give our main results.

Theorem 3. There exists an $H$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$ and $v \not\in \{4, 6, 8\}$.

Proof. The necessary conditions for the existence of an $H$-decomposition of $K_v^{(3)}$ are established in Lemma 1. Thus we need only to establish their sufficiency. Let $v = 8x + r$ where $x \geq 1$ and $r \in \{1, 2, 4, 6, 8\}$. By Lemma 2 it suffices to find $H$-decompositions of $K_{8+r}^{(3)}$, $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8,8}^{(3)}$, and $K_{r,8,8}^{(3)}$. We note that if $r \in \{1, 2\}$ then $K_{8+r}^{(3)} \setminus K_r^{(3)}$ is isomorphic to $K_{8+r}^{(3)}$. Also, $K_{r,8,8}^{(3)}$ decomposes $K_{6,8,8}^{(3)}$, and $K_{r,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, it suffices to find $H$-decompositions of $K_3^{(3)}$, $K_9^{(3)}$, $K_{10}^{(3)}$, $K_{12}^{(3)}$, $K_{14}^{(3)}$, $K_{16}^{(3)}$, $K_{18}^{(3)} \setminus K_{4}^{(3)}$, $K_{14}^{(3)} \setminus K_{6}^{(3)}$, $K_{16}^{(3)} \setminus K_{8}^{(3)}$, $K_{18,8}^{(3)} \cup L_{8,8}^{(3)}$, $K_{2,8,8}^{(3)} \cup L_{8,8}^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which are each shown to exist within Examples 1–16.

Theorem 4. If $v \geq 9$ is an integer, then there exists a maximum $H$-packing of $K_v^{(3)}$ where the leave has fewer than four edges.

Proof. If $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$, then the result follows from the $H$-decomposition result in Theorem 3, which translates to a maximum $H$-packing with an empty leave. Hence, we need only consider when $v \equiv 3, 5, \text{ or } 7 \pmod{8}$. Let $v = 8x + r$ where $x \geq 1$ and $r \in \{3, 5, 7\}$. By Lemma 2 it suffices to find

- a maximum $H$-packing of $K_{8+r}^{(3)}$, with a leave consisting of fewer than four edges and
- $H$-decompositions of $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{r,8,8}^{(3)}$.
We note that an $H$-decomposition of $K_{11}^{(3)} \setminus K_3^{(3)}$ is a subset of an $H$-packing of $K_{11}^{(3)}$ with a leave consisting of the single edge in the hole, which is necessarily then a maximum $H$-packing of $K_{11}^{(3)}$. Also, $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, it suffices to find maximum $H$-packings (with leaves of fewer than four edges) of $K_{11}^{(3)}$, $K_{13}^{(3)}$, and $K_{15}^{(3)}$, which are each shown to exist in Examples 17–19, and $H$-decompositions of $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, $K_{5,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which are each shown to exist within Examples 6–15.

2.2. Results for any Positive Index

We show here the necessary conditions for an $H$-decomposition of $\lambda$-fold $K_v^{(3)}$ for any positive integer $\lambda$. This will inform our choice on which combinations of $\lambda$ and $v$ we search for decompositions of $\lambda K_v^{(3)}$ versus finding maximum packings.

**Lemma 5.** Let $v \geq 9$ be an integer. There exists an $H$-decomposition of $\lambda$-fold $K_v^{(3)}$ only if the following hold:

- if $\gcd(\lambda, 4) = 1$, then $v \equiv 0$, 1, 2, 4, or 6 (mod 8);
- if $\gcd(\lambda, 4) = 2$, then $v \equiv 0$, 1, or 2 (mod 4);
- if $\gcd(\lambda, 4) = 4$, then $v \geq 9$.

**Proof.** Suppose there exists an $H$-decomposition of $\lambda K_v^{(3)}$. Since $|E(H)| = 4$, we must have $4 \mid \lambda(v^2 - 1)(v - 2)/6$, and thus $8 \mid \lambda v(v - 1)(v - 2)$. First, if $\gcd(\lambda, 4) = 1$, then $8 \mid v(v - 1)(v - 2)$, and thus $v \equiv 0$, 1, 2, 4, or 6 (mod 8). Second, if $\gcd(\lambda, 4) = 2$, then $4 \mid v(v - 1)(v - 2)$, and thus $v \equiv 0$, 1, or 2 (mod 4). Finally, if $\gcd(\lambda, 4) = 4$, then $2 \mid v(v - 1)(v - 2)$, which is true for any $v \geq 9$.

Next, we settle the decomposition and maximum packing results for some small values of $\lambda$.

**Theorem 6.** Let $v \geq 9$ be an integer. There exists an $H$-decomposition of 2-fold $K_v^{(3)}$ if $v \equiv 0$, 1, or 2 (mod 4).

**Proof.** If $v \equiv 0$, 1, 2, 4, or 6 (mod 8), then the result follows from 2 copies of an $H$-decomposition of $K_v^{(3)}$, which exists by Theorem 3. Hence, we need only consider when $v \equiv 5$ (mod 8).

First, we consider when $v = 13$. Let $v_1, v_2, \ldots, v_9 \in V(K_v^{(3)})$. By Example 18, there exist both a maximum $H$-packing, say $\Delta_1$, of $K_{13}^{(3)}$ with a leave consisting of two edges that share a single vertex and a maximum $H$-packing, say $\Delta_2$, of $K_{13}^{(3)}$ with a leave consisting of two vertex-disjoint edges. Let $L_1$ and $L_2$ be the leaves of $\Delta_1$ and $\Delta_2$, respectively. Without loss of generality, we may assume that

$$E(L_1) = \{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}\},$$
$$E(L_2) = \{\{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}.$$
Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2)$. Hence, $L'$ is isomorphic to $H$, and the (multi-)set
\[(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}\]
is a collection of $H$-blocks such that each edge of $K^{(3)}_v$ is represented exactly twice. Therefore, we have an $H$-decomposition of $2K^{(3)}_v$.

Now, let $v = 8x + 5$ where $x \geq 2$. By Lemma 2 it suffices to find $H$-decompositions of (2-fold) $K^{(3)}_1, K^{(3)}_2 \setminus K^{(3)}_5, K^{(3)}_{5,8,8} \cup L^{(3)}_8$, and $K^{(3)}_{8,8,8}$. We note that $K^{(3)}_{4,8,8}$ decomposes $K^{(3)}_{8,8,8}$, and we already have that $H$ decomposes $2K^{(3)}_3$. Thus, we need only additionally find $H$-decompositions of $K^{(3)}_1 \setminus K^{(3)}_5, K^{(3)}_{5,8,8}, L^{(3)}_8$, and $K^{(3)}_{8,8,8}$, which exist by Examples 13, 11, 6, and 10, respectively.

**Theorem 7.** If $v \geq 9$ is an integer, then there exists a maximum $H$-packing of 2-fold $K^{(3)}_v$ where the leave has no edges or two vertex-disjoint edges.

**Proof.** If $v \equiv 0$, 1, or 2 (mod 4), then the result follows from the $H$-decomposition result in Theorem 6, which translates to a maximum $H$-packing with an empty leave. Hence, we need only consider when $v \equiv 3$ (mod 4).

First, we consider when $v = 11$. Let $\Delta_1$ and $\Delta_2$ be maximum $H$-packings of $K^{(3)}_1$ with leaves $L_1$ and $L_2$, respectively, which exist by Example 17. Now, let $L'$ be the hypergraph with edge (multi-)set $E(L_1) \cup E(L_2)$. Hence, $L'$ consists of two edges. In fact, we further note that $L'$ can be any hypergraph with two edges, including $2K^{(3)}_3$. Hence, the (multi-)set
\[(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}\]
is a maximum $H$-packing of $2K^{(3)}_3$ with a leave, $L'$, consisting of two (possibly vertex-disjoint) edges.

Second, we consider when $v = 15$. Let $v_1, v_2, \ldots, v_9 \in V(K^{(3)}_v)$. By Example 19, there exist maximum $H$-packings of $K^{(3)}_v$ where the leaves consist of three disjoint edges. Let $\Delta_1$ and $\Delta_2$ be such $H$-packings of $K^{(3)}_v$ with leaves $L_1$ and $L_2$, respectively. Without loss of generality, we may assume that
\[
E(L_1) = \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \{v_7, v_8, v_9\}\},
E(L_2) = \{\{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}.
\]
Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2)$. We note that $L'$ is decomposable into copies of $K^{(3)}_3$ and $H$. That is, if we let $L''$ be the hypergraph with edge set $\{\{v_4, v_5, v_6\}, \{v_7, v_8, v_9\}\}$, then $L' \setminus L''$ is isomorphic to $H$, and the (multi-)set
\[(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L' \setminus L'', L''\}\]
is a maximum $H$-packing of $2K^{(3)}_3$ with a leave, $L''$, consisting of two (disjoint) edges.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 7\}$. By Lemma 2 it suffices to find
• a maximum $H$-packing of (2-fold) $K_{8+r}^{(3)}$ with a leave consisting of fewer than four edges and

• $H$-decompositions of $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_r^{(3)} \cup L_{8,8,8}$, and $K_{8,8,8}^{(3)}$.

We already have the maximum $H$-packing results. Also, we note that $K_{7,8,8}^{(3)}$ of copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, we need only additionally find $H$-decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, and $L_{8,8,8}^{(3)}$, which exist by Examples 17, 15, 9, 10, and 6, respectively.

**Theorem 8.** If $v \geq 9$ is an integer, then there exists a maximum $H$-packing of 3-fold $K_v^{(3)}$ where the leave has fewer than four edges.

**Proof.** If $v \equiv 0, 1, 2, 4, 5, 6 \pmod{8}$, then the result follows from the $H$-decomposition result in Theorem 3, which translates to a maximum $H$-packing with an empty leave. Hence, we need only consider when $v \equiv 3, 5, 6 \pmod{8}$.

First, we consider when $v = 11$. Let $\Delta_1$ be a maximum $H$-packing of $K_{11}^{(3)}$ with leave $L_1$ consisting of a single edge, which exists by Example 17, and let $\Delta_2$ be a maximum $H$-packing of $2K_{11}^{(3)}$ with leave $L_2$ consisting of two edges, which exists by Theorem 7. Now, let $L'$ be the hypergraph with edge (multi-)set $E(L_1) \cup E(L_2)$. Hence, $L'$ consists of three edges. In fact, we further note that $L'$ can be any hypergraph with three edges, including $3K_3^{(3)}$. Hence, the (multi-)set

$$
(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}
$$

is a maximum $H$-packing of $3K_{11}^{(3)}$ with a leave, $L'$, consisting of three edges.

Second, we consider when $v = 13$. Let $\Delta_1$ be a maximum $H$-packing of $K_{13}^{(3)}$ with leave $L_1$ consisting of two edges, which exists by Example 18, and let $\Delta_2$ be an $H$-decomposition of $2K_{13}^{(3)}$, which exists by Theorem 6. Hence, the (multi-)set $\Delta_1 \cup \Delta_2$ is a maximum $H$-packing of $3K_{13}^{(3)}$ with a leave, $L_1$, consisting of two edges.

Third, we consider when $v = 15$. Let $v_1, v_2, \ldots, v_9 \in V(K_v^{(3)})$, let $\Delta_1$ be a maximum $H$-packing of $K_{15}^{(3)}$ with leave $L_1$ consisting of a three vertex-disjoint edges, which exists by Example 19, and let $\Delta_2$ be a maximum $H$-packing of $2K_{15}^{(3)}$ with leave $L_2$ consisting of two vertex-disjoint edges, which exists by Theorem 7. Without loss of generality, we may assume that

$$
E(L_1) = \{\{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\},
$$

$$
E(L_2) = \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}.
$$

Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2)$. We note that $L'$ is decomposable into copies of $K_3^{(3)}$ and $H$. That is, if we let $L''$ be the hypergraph with the single edge $\{v_4, v_5, v_6\}$, then $L' \setminus L''$ is isomorphic to $H$, and the (multi-)set

$$
(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L' \setminus L'', L''\}
$$

is a maximum $H$-packing of $3K_{15}^{(3)}$ with a leave, $L''$, consisting of one edges.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 5, 7\}$. By Lemma 2 it suffices to find

150

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• a maximum $H$-packing of (3-fold) $K^{(3)}_{8+r}$ with a leave consisting of fewer than four edges and
• $H$-decompositions of $K^{(3)}_{8+r} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$.

We already have the maximum $H$-packing results. Also, we note that $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{r,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, we need only additionally find $H$-decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}$, $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{5,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which exist by Examples 17, 13, 15, 9, 11, 10, and 6, respectively.

**Theorem 9.** Let $v \geq 9$ be an integer. There exists an $H$-decomposition of 4-fold $K_v^{(3)}$.

**Proof.** If $v \equiv 0$, 1, or 2 (mod 4), then the result follows from 2 copies of an $H$-decomposition of $2K_v^{(3)}$, which exists by Theorem 6. Hence, we need only consider when $v \equiv 3$ (mod 4). Let $v_1, v_2, \ldots, v_9 \in V(K_v^{(3)})$.

First, we consider when $v = 11$. For $i \in \{1, 2, 3, 4\}$, let $\Delta_i$ be a maximum $H$-packing of $K^{(3)}_{11}$ with leave $L_i$ consisting of a single edge, which exists by Example 17, Without loss of generality, we may assume that

\[
E(L_1) = \{\{v_1, v_2, v_3\}\}, \quad E(L_2) = \{\{v_1, v_4, v_5\}\},
\]
\[
E(L_3) = \{\{v_2, v_6, v_7\}\}, \quad E(L_4) = \{\{v_3, v_8, v_9\}\}.
\]

Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2) \cup E(L_3) \cup E(L_4)$. Hence, $L'$ is isomorphic to $H$, and the (multi-)set

\[
L' \cup \bigcup_{i=1}^{4} (\Delta_i \setminus \{L_i\})
\]

is a collection of $H$-blocks such that each edge of $K_{11}^{(3)}$ is represented exactly four times. Therefore, we have an $H$-decomposition of $4K_{11}^{(3)}$.

Second, we consider when $v = 15$. Let $\Delta_1$ be a maximum $H$-packing of $K_{15}^{(3)}$ with leave $L_1$ consisting of a three vertex-disjoint edges, which exists by Example 19, and let $\Delta_2$ be a maximum $H$-packing of $3K_{15}^{(3)}$ with leave $L_2$ consisting of a single edge, which exists by Theorem 8, Without loss of generality, we may assume that

\[
E(L_1) = \{\{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\},
\]
\[
E(L_2) = \{\{v_1, v_2, v_3\}\}.
\]

Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2)$. Hence, $L'$ is isomorphic to $H$, and the (multi-)set

\[
(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}
\]

is a collection of $H$-blocks such that each edge of $K_{15}^{(3)}$ is represented exactly four times. Therefore, we have an $H$-decomposition of $4K_{15}^{(3)}$. 

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Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 7\}$. By Lemma 2 it suffices to find $H$-decompositions of $(4\text{-fold})$ $K_{8+r}^{(3)}$, $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup I_{8,8}^{(3)}$, and $K_{r,8,8,8,8}^{(3)}$. We note that $K_{7,8,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$. Also, $K_{3,8,8}^{(3)}$ decomposes $K_{4,8,8}^{(3)}$, and we already have that $H$ decomposes $4K_{11}^{(3)}$ and $4K_{15}^{(3)}$. Thus, we need only additionally find $H$-decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{4,8,8}^{(3)}$, $K_{4,8,8,8}^{(3)}$, and $I_{8,8}^{(3)}$, which exist by Examples 17, 15, 9, 10, and 6, respectively.

Finally, we show that the necessary conditions for the existence of an $H$-decomposition of $\lambda$-fold $K_v^{(3)}$ are sufficient.

**Theorem 10.** Let $\lambda$ and $v$ be positive integers with $v \geq 9$. There exists an $H$-decomposition of $\lambda$-fold $K_v^{(3)}$ if and only if the following hold:

- if $\gcd(\lambda, 4) = 1$, then $v \equiv 0, 1, 2, 4,$ or $6 \; (\text{mod } 8)$;
- if $\gcd(\lambda, 4) = 2$, then $v \equiv 0, 1, 2 \; (\text{mod } 4)$;
- if $\gcd(\lambda, 4) = 4$, then $v \geq 9$.

**Proof.** The necessary conditions are established in Lemma 5. For sufficiency, we consider the following cases.

**Case 1.** $\lambda \equiv 0 \; (\text{mod } 4)$

Let $\lambda = 4t$ for some positive integer $t$. Then the result follows from $t$ copies of an $H$-decomposition of $4K_v^{(3)}$, which exists by Theorem 9.

**Case 2.** $\lambda \equiv 1$ or $3 \; (\text{mod } 4)$

Since $\gcd(\lambda, 4) = 1$, we have that $v \equiv 0, 1, 2, 4,$ or $6 \; (\text{mod } 8)$. Let $\lambda = 4t + r$ for some integers $t \geq 0$ and $r \in \{1, 3\}$. Then the result follows from $t$ copies of an $H$-decomposition of $4K_v^{(3)}$, which exists by Theorem 9, and $r$ copies of an $H$-decomposition of $K_v^{(3)}$, which exists by Theorem 3.

**Case 3.** $\lambda \equiv 2 \; (\text{mod } 4)$

Since $\gcd(\lambda, 4) = 2$, we have that $v \equiv 0, 1, 2 \; (\text{mod } 4)$. Let $\lambda = 4t + 2$ for some nonnegative integer $t$. Then the result follows from $t$ copies of an $H$-decomposition of $4K_v^{(3)}$, which exists by Theorem 9, and 1 copy of an $H$-decomposition of $2K_v^{(3)}$, which exists by Theorem 6.

**Theorem 11.** If $v \geq 9$ is an integer, then there exists a maximum $H$-packing of $\lambda$-fold $K_v^{(3)}$ where the leave has fewer than four edges.

**Proof.** If $1 \leq \lambda \leq 3$, then the result follows from Theorems 4, 7, and 8. If $\lambda = 4$, then the result follows from the $H$-decomposition result in Theorem 9, which translates to a maximum $H$-packing with an empty leave. For the remainder of the proof, we assume that $\lambda \geq 5$. Let $\lambda = 4t + r$ for some integers $t \geq 1$ and $r \in \{1, 4\}$. Then the result follows from $t$ copies of an $H$-decomposition of $4K_v^{(3)}$, which exists by Theorem 9, and 1 copy of a maximum $H$-packing of $r$-fold $K_v^{(3)}$. 

460
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References


**Appendix: Some Small Examples**

We give several examples of $H$-decompositions and $H$-packings that are used in proving our main result.

**Decomposition Examples**

**Example 1.** Let $V\left(K_9^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$ and let

$B = \{H[0, 1, 4, 5, 6, \infty_1, 3, \infty_2, 2], H[\infty_1, \infty_2, 0, 3, 6, 1, 2, 4, 5], H[0, 2, 5, \infty_2, 4, \infty_1, 1, 6, 3]\}.$

Then an $H$-decomposition of $K_9^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{7}$.  

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Example 2. Let $V(K_{10}^{(3)}) = \mathbb{Z}_{10}$ and let
\[
B = \{ H[0, 2, 4, 8, 9, 3, 6, 5, 1], H[0, 2, 7, 1, 6, 5, 8, 9, 3], H[0, 1, 5, 7, 9, 2, 4, 8, 3] \}.
\]
Then an $H$-decomposition of $K_{10}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{10}$.

Example 3. Let $V(K_{12}^{(3)}) = \mathbb{Z}_{11} \cup \{ \infty \}$ and let
\[
B = \{ H[0, 1, 3, 8, 10, 2, 5, 6, 7], H[0, 1, 5, \infty, 6, 2, 8, 10, 3], H[0, 6, 9, 2, 5, 10, 3, \infty, 8],
H[\infty, 0, 3, 8, 10, 2, 4, 6, 9], H[0, 1, 2, \infty, 7, 5, 10, 3, 8] \}.
\]
Then an $H$-decomposition of $K_{12}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{11}$.

Example 4. Let $V(K_{14}^{(3)}) = \mathbb{Z}_{13} \cup \{ \infty \}$ and let
\[
B = \{ H[0, 1, 3, 10, 12, 2, 5, 6, 7], H[0, 1, 5, 7, 12, 2, 10, 6, 11], H[\infty, 4, 6, 0, 1, 2, 3, 5, 12],
H[\infty, 4, 8, 0, 3, 7, 12, 11, 1], H[\infty, 6, 11, 12, 5, 8, 10, 2, 7], H[0, 2, 7, 6, 10, 4, 11, 12, 1],
H[0, 2, 5, 8, 11, 6, 12, 3, 9] \}.
\]
Then an $H$-decomposition of $K_{14}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{13}$.

Example 5. Let $V(K_{16}^{(3)}) = \mathbb{Z}_{15} \cup \{ \infty \}$ and let
\[
B_1 = \{ H[0, 1, 3, 5, 6, 2, 14, 4, 9], H[0, 2, 5, 3, 11, 4, 14, 8, 12], H[0, 1, 4, \infty, 7, 2, 13, 8, 12],
H[0, 2, 6, 3, 9, 4, 13, \infty, 11], H[0, 2, 8, 7, 14, 4, 11, \infty, 10], H[0, 1, 7, 4, 9, 2, 10, \infty, 13],
H[0, 1, 5, 3, 6, 2, 12, \infty, 8], H[0, 2, 7, \infty, 1, 4, 12, 9, 11], H[0, 3, 8, 4, 10, 6, 13, \infty, 12] \},
B_2 = \{ H[0, 5, 10, 1, 2, 6, 7, 11, 12], H[1, 6, 11, 2, 3, 7, 8, 12, 13], H[2, 7, 12, 3, 4, 8, 9, 13, 14],
H[3, 8, 13, 4, 5, 9, 10, 14, 0], H[4, 9, 14, 5, 6, 10, 11, 0, 1] \}.
\]
Then an $H$-decomposition of $K_{16}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{15}$ along with the $H$-blocks in $B_2$.

Example 6. Let $V(I_{8,8}^{(3)}) = \mathbb{Z}_{16}$ with vertex partition \( \{ \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\} \}$ and let
\[
B = \{ H[0, 1, 2, 7, 9, 4, 14, 8, 13], H[0, 1, 3, 5, 14, 2, 15, 8, 10], H[0, 4, 5, 7, 15, 8, 3, 12, 13],
H[0, 6, 13, 3, 7, 12, 15, 1, 4], H[0, 1, 7, 9, 15, 3, 14, 2, 4], H[0, 1, 6, 10, 15, 8, 12, 13, 2],
H[0, 1, 4, 3, 11, 2, 14, 12, 15] \}.
\]
Then an $H$-decomposition of $I_{8,8}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{16}$.
Example 7. Let \( V\left(L_{8,8}^{(3)} \cup K_{1,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty\} \) with vertex partition \( \{\infty\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\} \) and let
\[
B = \{H[0, 1, 3, 5, 14, 2, 15, 8, 10], H[0, 4, 5, 7, 15, 8, 3, 12, 13], H[0, 6, 13, 3, 7, 12, 15, 1, 4], H[0, 1, 7, 9, 15, 3, 14, 2, 4], H[0, 1, 6, 10, 15, 8, 12, 13, 2], H[0, 1, 4, 3, 11, 2, 14, 12, 15], H[\infty, 0, 9, 10, 13, 7, 14, 15, 4], H[\infty, 0, 11, 3, 4, 1, 2, 5, 8]\}.
\]
Then an \( H \)-decomposition of \( L_{8,8}^{(3)} \cup K_{1,8,8}^{(3)} \) consists of the orbits of the \( H \)-blocks in \( B \) under the action of the map \( \infty \mapsto \infty \) and \( j \mapsto j + 1 \pmod{16} \).

Example 8. Let \( V\left(L_{8,8}^{(3)} \cup K_{2,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2\} \) with vertex partition \( \{\infty_1, \infty_2\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\} \) and let
\[
B = \{H[0, 1, 3, 5, 14, 2, 15, 8, 10], H[0, 4, 5, 7, 15, 8, 3, 12, 13], H[\infty_1, 0, 15, 3, 10, 1, 4, 11, 14], H[0, 1, 7, 9, 15, 3, 14, 2, 4], H[0, 1, 6, 10, 15, 8, 12, 13, 2], H[\infty_1, 0, 3, 9, 14, 5, 13, 8, 11], H[0, 6, 13, 3, 7, 12, 15, 1, 4], H[\infty_2, 0, 9, 5, 6, 14, 15, 11, 2], H[\infty_2, 0, 13, 5, 10, 3, 6, 2, 7]\}.
\]
Then an \( H \)-decomposition of \( L_{8,8}^{(3)} \cup K_{2,8,8}^{(3)} \) consists of the orbits of the \( H \)-blocks in \( B \) under the action of the map \( \infty_i \mapsto \infty_i \), for \( i \in \{1, 2\} \), and \( j \mapsto j + 1 \pmod{16} \).

Example 9. Let \( V\left(K_{3,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2, \infty_3\} \) with vertex partition \( \{\infty_1, \infty_2, 7, 3, 6\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\} \) and let
\[
B = \{H[\infty_1, 0, 1, 2, 5, 7, 9, 3, 6], H[\infty_2, 0, 1, 2, 5, 7, 9, 11, 6], H[\infty_3, 0, 1, 2, 5, 7, 9, 12, 6]\}.
\]
Then an \( H \)-decomposition of \( K_{3,8,8}^{(3)} \) consists of the orbits of the \( H \)-blocks in \( B \) under the action of the map \( \infty_i \mapsto \infty_i \), for \( i \in \{1, 2, 3\} \), and \( j \mapsto j + 1 \pmod{16} \).

Example 10. Let \( V\left(K_{4,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\} \) with vertex partition \( \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\} \) and let
\[
B = \{H[\infty_1, 0, 1, 2, 5, 7, 3, 4, 6], H[\infty_2, 0, 1, 2, 5, 7, 9, 4, 6], H[\infty_3, 0, 1, 2, 5, 7, 9, 4, 6], H[\infty_4, 0, 1, 2, 5, 7, 9, 4, 6]\}.
\]
Then an \( H \)-decomposition of \( K_{4,8,8}^{(3)} \) consists of the orbits of the \( H \)-blocks in \( B \) under the action of the map \( \infty_i \mapsto \infty_i \), for \( i \in \{1, \ldots, 4\} \), and \( j \mapsto j + 1 \pmod{16} \).

Example 11. Let \( V\left(K_{5,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \) with vertex partition \( \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\} \) and let
\[
B = \{H[\infty_1, 0, 1, 2, 5, 7, 3, 4, 6], H[\infty_2, 0, 1, 2, 5, 7, 9, 4, 6], H[\infty_3, 0, 1, 2, 5, 7, 9, 4, 6], H[\infty_4, 0, 1, 2, 5, 7, 9, 4, 6], H[\infty_5, 0, 1, 2, 5, 7, 9, 4, 6]\}.
\]
Then an $H$-decomposition of $K_{5,8,8}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \ldots, 5\}$, and $j \mapsto j + 1 \pmod{16}$.

**Example 12.** Let $V\left(K_{12}^{(3)} \setminus K_4^{(3)}\right) = \mathbb{Z}_{12}$ with 0, 3, 6, 9 being the vertices in the hole and let

$$B_1 = \left\{ H[0, 3, 7, 2, 5, 6, 11, 9, 4], H[0, 2, 6, 1, 11, 4, 10, 8, 9], H[0, 1, 6, 7, 11, 2, 8, 10, 5], H[0, 1, 4, 8, 11, 3, 5, 9, 2]\right\} ,$$

$$B_2 = \left\{ H[7, 8, 10, 1, 4, 2, 5, 0, 9], H[1, 2, 4, 7, 10, 8, 11, 3, 6], H[8, 9, 11, 0, 4, 6, 7, 2, 5], H[1, 10, 11, 5, 9, 4, 7, 0, 2], H[2, 3, 5, 6, 10, 0, 1, 8, 11], H[4, 5, 7, 1, 10, 6, 8, 11, 3]\right\} .$$

Then an $H$-decomposition of $K_{12}^{(3)} \setminus K_4^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $j \mapsto j + 1 \pmod{12}$ along with the $H$-blocks in $B_2$.

**Example 13.** Let $V\left(K_{13}^{(3)} \setminus K_5^{(3)}\right) = \mathbb{Z}_8 \cup \{ \infty_1, \infty_2, \infty_3, \infty_4, \infty_5 \}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5$ being the vertices in the hole and let

$$B_1 = \left\{ H[\infty_3, \infty_5, 0, 1, 3, \infty_2, 7, \infty_1, \infty_4], H[\infty_4, \infty_5, 0, \infty_3, 6, \infty_1, 7, 2, 3], H[\infty_2, \infty_4, 0, 5, 7, 1, 4, \infty_5, 2], H[\infty_4, 0, 2, 4, 5, \infty_1, 7, \infty_5, 3]\right\} ,$$

$$B_2 = \left\{ H[\infty_1, \infty_3, 0, 3, 5, 1, 2, 4, 6], H[\infty_1, \infty_3, 1, 4, 6, 2, 3, 5, 7], H[\infty_1, \infty_3, 2, 5, 7, 3, 4, 6, 0], H[\infty_1, \infty_3, 3, 6, 0, 4, 5, 7, 1], H[\infty_1, \infty_3, 4, 7, 1, 5, 6, 0, 2], H[\infty_1, \infty_3, 5, 0, 2, 6, 7, 1, 3], H[\infty_1, \infty_3, 6, 1, 3, 7, 0, 2, 4], H[2, 4, 7, \infty_2, \infty_3, \infty_1, 1, 5, 6], H[3, 5, 0, \infty_2, \infty_3, \infty_1, 2, 6, 7], H[4, 6, 1, \infty_2, \infty_3, \infty_1, 3, 7, 0], H[5, 7, 2, \infty_2, \infty_3, \infty_1, 4, 0, 1], H[6, 0, 3, \infty_2, \infty_3, \infty_1, 5, 1, 2], H[7, 1, 4, \infty_2, \infty_3, \infty_1, 6, 2, 3], H[0, 2, 5, \infty_2, \infty_3, \infty_1, 7, 3, 4], H[2, 3, 7, \infty_3, 5, \infty_1, \infty_2, 1, 6], H[3, 4, 0, \infty_3, 6, \infty_1, \infty_2, 2, 7], H[4, 5, 1, \infty_3, 7, \infty_1, \infty_2, 3, 0], H[5, 6, 2, \infty_3, 0, \infty_1, \infty_2, 4, 1], H[6, 7, 3, \infty_3, 1, \infty_1, \infty_2, 5, 2], H[7, 0, 4, \infty_3, 2, \infty_1, \infty_2, 6, 3], H[0, 1, 5, \infty_3, 3, \infty_1, \infty_2, 7, 4], H[0, 2, \infty_3, 1, 0, 4, \infty_1, 7, 2, 6], H[\infty_2, 1, 5, \infty_1, 2, \infty_3, 0, 6, 4], H[1, 3, 6, \infty_3, 4, 5, 7, \infty_2, 2], H[\infty_1, 0, 3, 2, 4, 5, 6, \infty_2, 7], H[\infty_2, 0, 1, 3, 6, \infty_1, 4, \infty_3, 5], H[\infty_2, 1, 2, 4, 7, \infty_1, 5, \infty_3, 6], H[\infty_2, 2, 3, 5, 0, \infty_1, 6, \infty_3, 7], H[\infty_2, 3, 4, 6, 1, \infty_1, 7, \infty_3, 0], H[\infty_2, 4, 5, 7, 2, \infty_4, 0, \infty_5, 1], H[\infty_2, 5, 6, 0, 3, \infty_4, 1, \infty_5, 2], H[\infty_2, 6, 7, 1, 4, \infty_4, 2, \infty_5, 3], H[\infty_2, 7, 0, 2, 5, \infty_4, 3, \infty_5, 4], H[\infty_5, 0, 2, 1, 3, 4, 5, 6, 7], H[\infty_5, 3, 5, 2, 4, 7, 0, 6, 1], H[\infty_5, 4, 6, 5, 7, 0, 1, 2, 3], H[\infty_5, 1, 7, 0, 6, 2, 5, 3, 4]\right\} .$$

Then an $H$-decomposition of $K_{13}^{(3)} \setminus K_5^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \ldots, 5\}$, and $j \mapsto j + 1 \pmod{8}$ along with the $H$-blocks in $B_2$. 

465
Example 14. Let $V\left(K_{14}^{(3)} \setminus K_{6}^{(3)}\right) = \mathbb{Z}_{12} \cup \{\infty_1, \infty_2\}$ with $0, 3, 6, 9, \infty_1, \infty_2$ being the vertices in the hole and let

$$B_1 = \{ H[0, 1, 5, 7, 11, 2, 10, 6, 9], H[\infty_1, 0, 1, 6, 8, 10, 11, \infty_2, 2], H[\infty_2, 0, 4, 1, 3, 9, 11, \infty_1, 8], H[0, 1, 6, \infty_2, 7, 2, 8, \infty_1, 11], H[0, 2, 5, 7, 10, 4, 8, 9, 11], H[0, 2, 4, 3, 8, 5, 9, 6, 11]\},$$

$$B_2 = \{ H[\infty_1, 2, 8, 5, 11, 1, 4, 7, 10], H[\infty_2, 5, 11, 2, 8, 4, 7, 1, 10], H[0, 1, 3, 4, 8, \infty_1, 7, 2, 5], H[3, 4, 6, 7, 11, \infty_1, 10, 5, 8], H[6, 7, 9, 2, 10, \infty_2, 1, 8, 11], H[0, 9, 10, 2, 11, 1, 5, \infty_2, 4], H[\infty_1, \infty_2, 1, 2, 5, 8, 11, 4, 7], H[\infty_1, \infty_2, 2, 1, 4, 7, 10, 5, 8], H[\infty_1, \infty_2, 4, 5, 8, 2, 11, 7, 10], H[\infty_1, \infty_2, 5, 4, 7, 1, 10, 8, 11], H[\infty_1, \infty_2, 7, 8, 11, 2, 5, 1, 10], H[\infty_1, \infty_2, 8, 7, 10, 1, 4, 2, 11], H[\infty_1, \infty_2, 10, 2, 11, 5, 8, 1, 4], H[\infty_1, \infty_2, 11, 1, 10, 4, 7, 2, 5]\}.$$

Then an $H$-decomposition of $K_{14}^{(3)} \setminus K_{6}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty_i \mapsto i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{12}$ along with the $H$-blocks in $B_2$.

Example 15. Let $V\left(K_{15}^{(3)} \setminus K_{7}^{(3)}\right) = \mathbb{Z}_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7$ being the vertices in the hole and let

$$B_1 = \{ H[\infty_1, \infty_2, 0, \infty_5, 5, 3, 4, \infty_6, \infty_7], H[\infty_2, \infty_4, 0, \infty_5, 5, 2, 4, \infty_7, 3], H[\infty_3, \infty_4, 0, \infty_5, 5, \infty_7, 4, \infty_6, 3], H[\infty_4, \infty_5, 0, 4, 7, \infty_6, 1, \infty_3, 2], H[\infty_5, 0, \infty_7, 6, 3, 4, \infty_1, 1], H[\infty_5, \infty_7, 0, 5, 6, \infty_1, 4, \infty_6, 2], H[\infty_6, 0, \infty_7, 6, \infty_4, 2, \infty_5, 3], H[\infty_7, 3, 5, 0, 1, \infty_1, \infty_6, \infty_2, 7]\},$$

$$B_2 = \{ H[\infty_1, \infty_3, 0, 3, 5, 1, 2, 4, 6], H[\infty_1, \infty_3, 1, 4, 6, 2, 3, 5, 7], H[\infty_1, \infty_3, 2, 5, 7, 3, 4, 6, 0], H[\infty_1, \infty_3, 3, 6, 0, 4, 5, 7, 1], H[\infty_1, \infty_3, 4, 7, 1, 5, 6, 0, 2], H[\infty_1, \infty_3, 5, 0, 2, 6, 7, 1, 3], H[\infty_1, \infty_3, 6, 1, 3, 7, 0, 2, 4], H[2, 4, 7, \infty_2, \infty_3, \infty_1, 1, 5, 6], H[3, 5, 0, \infty_2, \infty_3, \infty_1, 2, 6, 7], H[4, 6, 1, \infty_2, \infty_3, \infty_1, 3, 7, 0], H[5, 7, 2, \infty_2, \infty_3, \infty_1, 4, 0, 1], H[6, 0, 3, \infty_2, \infty_3, \infty_1, 5, 1, 2], H[7, 1, 4, \infty_2, \infty_3, \infty_1, 6, 2, 3], H[0, 2, 5, \infty_2, \infty_3, \infty_1, 7, 3, 4], H[2, 3, 7, \infty_3, 5, \infty_1, \infty_2, 1, 6], H[3, 4, 0, \infty_3, 6, \infty_1, \infty_2, 2, 7], H[4, 5, 1, \infty_3, 7, \infty_1, \infty_2, 3, 0], H[5, 6, 2, \infty_3, 0, \infty_1, \infty_2, 4, 1], H[6, 7, 3, \infty_3, 1, \infty_1, \infty_2, 5, 2], H[7, 0, 4, \infty_3, 2, \infty_1, \infty_2, 6, 3], H[0, 1, 5, \infty_3, 3, \infty_1, \infty_2, 7, 4], H[\infty_2, \infty_3, 1, 0, 4, \infty_1, 7, 2, 6], H[\infty_2, 1, 5, \infty_1, 2, \infty_3, 0, 6, 4], H[1, 3, 6, \infty_3, 4, 5, 7, \infty_2, 2], H[\infty_1, 0, 3, 2, 4, 5, 6, \infty_2, 7], H[\infty_2, 0, 1, 3, 6, \infty_1, 4, \infty_3, 5], H[\infty_2, 1, 2, 4, 7, \infty_1, 5, \infty_3, 6], H[\infty_2, 2, 3, 5, 0, \infty_1, 6, \infty_3, 7]\}.$
Then an $H$-decomposition of $K_{15}^{(3)} \setminus K_7^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \ldots, 7\}$, and $j \mapsto j + 1 \pmod{8}$ along with the $H$-blocks in $B_2$.

**Example 16.** Let $V\left( K_{16}^{(3)} \setminus K_8^{(3)} \right) = \mathbb{Z}_8 \cup \{ \infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8 \}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8$ being the vertices in the hole and let

$$B_1 = \{ H[\infty_1, \infty_2, 0, \infty_3, 1, \infty_5, 2, \infty_4, \infty_8], H[\infty_2, \infty_3, 0, \infty_4, 1, \infty_6, 2, \infty_1, \infty_5], \ldots \},$$

$$B_2 = \{ H[0, 1, 2, \infty_1, 4, \infty_5, 3, \infty_2, 6], H[1, 2, 3, \infty_1, 5, \infty_5, 4, \infty_2, 7], \ldots \}.$$

Then an $H$-decomposition of $K_{16}^{(3)} \setminus K_8^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \ldots, 8\}$, and $j \mapsto j + 1 \pmod{8}$ along with the $H$-blocks in $B_2$.

**Maximum Packing Examples**

**Example 17.** Let $V\left( K_{11}^{(3)} \right) = \mathbb{Z}_{10} \cup \{ \infty \}$ and let

$$B_1 = \{ H[0, 2, 7, 1, 4, \infty, 9, 3, 6], H[0, 3, 6, 1, 5, \infty, 9, 7, 2], H[0, 2, 5, 1, 3, \infty, 4, 7, 8] \},$$

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Then a maximum $H$-packing of $K_{11}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{10}$ along with the $H$-blocks in $B_2$ and a leave consisting of the edge $\{1, 3, 9\}$.

**Example 18.** Let $V(K_{13}^{(3)}) = \mathbb{Z}_{13}$ and let

$$B_1 = \{ H[0, 3, 7, 6, 10, 5, 11, 9, 1], H[0, 2, 11, 1, 7, 5, 12, 3, 8], H[0, 3, 5, 8, 10, 7, 1, 9, 11],$$
$$H[0, 1, 5, 8, 12, 2, 7, 10, 11], H[0, 1, 3, 10, 12, 2, 5, 6, 7] \}.$$

$$B_2 = \{ H[0, 4, 8, 1, 12, 5, 6, 9, 10], H[1, 5, 9, 2, 3, 6, 7, 10, 11], H[2, 6, 10, 3, 4, 7, 8, 11, 12],$$
$$H[3, 4, 5, 7, 11, 8, 12, 10, 1], H[7, 8, 9, 11, 2, 12, 3, 0, 4], H[11, 12, 0, 2, 6, 3, 7, 5, 9] \}.$$

Then a maximum $H$-packing of $K_{13}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $j \mapsto j + 1 \pmod{13}$ along with the $H$-blocks in $B_2$ and a leave consisting of the edges $\{0, 1, 2\}$ and $\{1, 6, 10\}$, which share a single vertex. Additionally, let

$$B_2' = (B_2 \setminus \{ H[2, 6, 10, 3, 4, 7, 8, 11, 12] \}) \cup \{ H[2, 6, 10, 0, 1, 7, 8, 11, 12] \}.$$

Then a maximum $H$-packing of $K_{13}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $j \mapsto j + 1 \pmod{13}$ along with the $H$-blocks in $B_2'$ and a leave consisting of the edges $\{1, 6, 10\}$ and $\{2, 3, 4\}$, which are vertex-disjoint.

**Example 19.** Let $V(K_{15}^{(3)}) = \mathbb{Z}_{15}$ and let

$$B_1 = \{ H[0, 4, 9, 6, 11, 7, 14, 12, 2], H[0, 4, 8, 3, 6, 7, 13, 10, 12], H[0, 1, 3, 12, 14, 2, 5, 6, 7],$$
$$H[0, 1, 6, 9, 14, 2, 12, 7, 11], H[0, 2, 8, 7, 13, 4, 12, 1], H[0, 3, 7, 8, 12, 5, 14, 9, 13],$$
$$H[0, 2, 12, 7, 8, 10, 1, 3, 11] \},$$

$$B_2 = \{ H[0, 5, 10, 1, 2, 6, 7, 11, 12], H[1, 6, 11, 2, 3, 7, 8, 12, 13], H[2, 7, 12, 3, 4, 8, 9, 13, 14],$$
$$H[3, 8, 13, 4, 5, 9, 10, 14, 0], H[4, 9, 14, 5, 6, 10, 11, 0, 1], H[0, 2, 5, 13, 3, 12, 14, 7, 10],$$
$$H[4, 6, 9, 14, 1, 8, 11, 12, 7], H[8, 10, 13, 3, 5, 12, 0, 11, 1] \}.$$

Then a maximum $H$-packing of $K_{15}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $j \mapsto j + 1 \pmod{15}$ along with the $H$-blocks in $B_2$ and a leave consisting of the edges $\{1, 3, 6\}$, $\{2, 4, 7\}$, and $\{9, 11, 14\}$, which are vertex-disjoint.