The dispersability of the Kronecker cover of the product of complete graphs and cycles

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Abstract

The Kronecker cover of a graph $G$ is the Kronecker product of $G$ and $K_2$. The matching book embedding of a graph $G$ is an embedding of $G$ with the vertices on the spine, each edge within a single page so that the edges on each page do not intersect and the degree of vertices on each page is at most one. The matching book thickness of $G$ is the minimum number of pages in a matching book embedding of $G$ and it denoted by $mbt(G)$. A graph $G$ is dispersable if $mbt(G) = \Delta(G)$, nearly dispersable if $mbt(G) = \Delta(G) + 1$. In this paper, the dispersability of the Kronecker cover of the Cartesian product of complete graphs $K_p$ and cycles $C_q$ is determined.

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1. Introduction

The book embedding of a graph was first introduced by Bernhart and Kainen [1]. They defined an $n$-book which is composed of a line $L$ in 3-space (called spine) and $n$ distinct half-planes (called pages), where $L$ forms the common boundary of the $n$ half-planes. An $n$-book embedding is an embedding of $G$ such that each vertex of a graph $G$ is placed on the spine and each edge is placed on at most one page with no two edges on the same page intersecting. The book thickness of a graph $G$ is the smallest $n$ so that $G$ has an $n$-book embedding and it denoted by $bt(G)$.

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A matching book embedding of a graph $G$ is a book embedding where on each page the maximum degree of vertices is at most one. The matching book thickness of a graph $G$ is the smallest $k$ such that $G$ has an $k$ page matching book embedding and it denoted by $mbt(G)$. A graph $G$ is dispersable (resp. nearly dispersable) if $mbt(G) = \Delta(G)$ (resp. if $mbt(G) = \Delta(G) + 1$), where $\Delta(G)$ represents the maximum number of edges adjacent to a vertex $v$ in the graph $G$. In 1998, Overbay proved the complete bipartite graph $K_{n,n}$ ($n \geq 1$), even cycle $C_{2n}$ ($n \geq 2$), binary $n$-cube $Q(n)$ ($n \geq 0$) and tree are dispersable [5], also she got that the odd cycles and complete graphs $K_n$ ($n \geq 3$) are near-dispersable.

Kainen obtained that the the Cartesian product of two even cycles is dipersable, the product of an odd cycle and an even cycle is nearly dispersable [4]. Shao-et-al proved that for all odd integers $s$, $t$ and $s \geq t \geq 7$, $mbt(C_s \square C_t) = 5$ [8]. This solved the matching book embedding of the Cartesian product of two cycles. Shao-et-al also obtained that $K_p \square C_q$ is nearly dispersable when $p, q \geq 3$ [7].

The main goal of this paper is to prove that the matching book thickness of the Kronecker cover of the Cartesian product of complete graphs $K_p$ and cycles $C_q$ is equal to $p + 1$ when $p, q \geq 3$.

2. Preliminaries

In this section, we present some definitions and results which we needed in our work.

**Definition 1.** The matching book embedding of $G$ is an embedding of a graph $G$ with the vertices on the spine, each edge within a single page so that the edges on each page do not intersect and the degree of vertices on each page is at most one.

**Definition 2.** The matching book thickness of a graph $G$ is the minimum number of pages in a matching book embedding of $G$ and it denoted by $mbt(G)$.

**Definition 3.** [2] The Cartesian product of two arbitrary graphs $G$ and $B$ is the graph denoted by $G \square B$ whose vertex set is $V(G) \times V(B)$, the vertex $(u_1, v_1)$ and the vertex $(u_2, v_2)$ are adjacent in $G \square B$ if and only if $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $B$, or $v_1 = v_2$ and $u_1$ is adjacent to $u_2$ in $G$.

**Definition 4.** [9] The product $G_1 \land G_2$ of two graphs $G_1$ and $G_2$ (often known as their Kronecker product) has vertex set $V(G_1 \land G_2)$ equal to the Cartesian product $V(G_1) \times V(G_2)$ of the vertex sets of the given graphs, with adjacency in $G_1 \land G_2$ given by $(v_1, w_1) \sim (v_2, w_2)$ if and only if $v_1 \sim v_2$ in $G_1$ and $w_1 \sim w_2$ in $G_2$. The Kronecker cover, also known as the canonical double cover, of a graph $G$ is essentially the Kronecker product of $G$ and the complete graph $K_2$, where the projection morphism $p : G \land K_2 \to G$ on vertices is defined by

$$(v, 0) \mapsto v,$$

$$(v, 1) \mapsto v,$$

and this induces a $(2:1)$ map on edges :

$$[(v, 0), (w, 1)] \mapsto (v, w).$$
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Let $D$ denote the Kronecker cover of graph $G$. In graph $D$, both the number of vertices and the number of edges are twice that of graph $G$.

**Lemma 2.1.** [9] If $p : D \rightarrow G$ is a double cover projection, then the vertex $v$ has degree $d$ in $G$ if and only if the two associated vertices $v_1$ and $v_2$ in $D$ both have degree $d$.

**Lemma 2.2.** [5] For any simple graph $G$, $\Delta(G) \leq \chi'(G) \leq mbt(G)$, where $\chi'(G)$ is the chromatic index of $G$.

**Lemma 2.3.** [5] For a regular graph $G$, $G$ is dispersable only if $G$ is bipartite.

3. The dispersability of $D(K_p \square C_q)$

In this section, we study the matching book embedding and dispersability of some standard graphs and their Cartesian product.

**Lemma 3.1.** The Kronecker cover $D$ of cycle $C_n$ is dispersable.

**Proof.** According to Definition 4, it is easy to see that the Kronecker cover of a cycle $C_n$ is an even cycle $C_{2n}$, regardless of whether $n$ is odd or even. Therefore, the Kronecker cover of a cycle is dispersable.

**Lemma 3.2.** The Kronecker cover $D$ of complete graph $K_n$ is dispersable.

**Proof.** Let $V(K_n) = \{1, 2, \ldots, n\}$. Assuming that in the Kronecker cover of the complete graph, both $i_1$ and $i_2$ correspond to the vertex $i$ in the complete graph, where $1 \leq i \leq n$. It is clear that $mbt(D) \geq \Delta(D) = n - 1$ by Lemma 2.1. According to the parity of $n$, we need to consider two cases to compute the matching book thickness of $D$.

**Case 1.** $n$ is odd

For the Kronecker cover of the complete graph, we assign the vertex ordering as $1_1, 2_2, 3_1, 4_2, \ldots, (n-2)_1, (n-1)_2, n_1, (n-1)_1, (n-2)_2, (n-3)_1, (n-4)_2, \ldots, 2_1, 1_2$. The matching book embedding of $D$ in this case is given as follows:

Page 1: edges $\{(i_1, j_2) | i - j = 2 ; 3 \leq i \leq n, i \text{ is odd}\}$, edges $\{(i_2, j_1) | i - j = 2 ; 4 \leq i \leq n - 1, i \text{ is even}\}$, and edges $\{(1_1, 2_2), (n_2, (n-1)_1)\}$.

Page 2: edges $\{(i_1, j_2) | i - j = 2 ; 3 \leq j \leq n, j \text{ is odd}\}$, edges $\{(i_2, j_1) | j - i = 2 ; 4 \leq j \leq n - 1, j \text{ is even}\}$, and edges $\{(2_1, 1_2), ((n-1)_2, n_1)\}$.

Page 3: edges $\{(i_1, j_2) | i - j = 4 ; 5 \leq i \leq n, i \text{ is odd}\}$, edges $\{(i_2, j_1) | i - j = 4 ; 6 \leq i \leq n - 1, i \text{ is even}\}$, and edges $\{(1_1, 4_2), (2_2, 3_1), ((n-1)_1, (n-2)_2), (n_2, (n-3)_1)\}$.

Page 4: edges $\{(i_1, j_2) | j - i = 4 ; 5 \leq j \leq n, i \text{ is odd}\}$, edges $\{(i_2, j_1) | j - i = 4 ; 6 \leq j \leq n - 1, i \text{ is even}\}$, and edges $\{(4_1, 1_2), (3_2, 2_1), ((n-2)_1, (n-1)_2), ((n-3)_2, n_1)\}$.

... Page $n-2$: edges $\{(1_1, (n-1)_2), (2_2, (n-2)_1), (3_1, (n-3)_2), \ldots, ((n-1)_s, (n-2)_t) | s = 1 \text{ and } t = 2 \text{ when } \frac{n-1}{2} \text{ is odd}\}$, and edges $\{(n_1, 1_2), (n_2, 2_1), ((n-1)_1, 3_2), ((n-2)_2, 4_1), \ldots, ((n+3)_s, (n+1)_t) | s = 1 \text{ and } t = 2 \text{ when } \frac{n+3}{2} \text{ is even}\}$.
Page $n-1$: edges $\{(n-1)_1, 1_2\}, \{(n-2)_2, 2_1\}, \{(n-3)_1, 3_2\}, \ldots, ((\frac{n+1}{2})_s, (\frac{n-1}{2})_t)|s = 1$ and $t = 2$ when $\frac{n+1}{2}$ is even, and edges $\{(1_1, n_2), (2_2, n_1), (3_1, (n-1)_2), (4_2, (n-2)_1)\ldots, ((\frac{n+1}{2})_s, (\frac{n+3}{2})_t)|s = 1$ and $t = 2$ when $\frac{n+3}{2}$ is odd).

It is clear that $mbt(D) \leq n - 1$. Therefore, the Kronecker cover of complete graph is dispersable when $n$ is odd(see Fig.1 for the case $n = 5$).

Case 2. $n$ is even

Let the vertex ordering on spine be as $1_1, 2_2, 3_1, 4_2, \ldots,(n-1)_1, n_2, n_1, (n-1)_2, (n-2)_1, (n-3)_2, \ldots, 1_2$. The dispersability of the Kronecker cover of the product of complete graphs and cycles when $n$ is even, and edges $\{(1_1, 2_2), (n_1, (n-1)_2)\}$.

Page 2: edges $\{(i_1, j_2)|j - i = 2; 3 \leq j \leq n - 1, j \text{ is odd}\}$, edges $\{(i_2, j_1)|j - i = 2, 4 \leq j \leq n, j \text{ is even}\}$, and edges $\{(1_1, 2_2), (n_1, (n-1)_2)\}$.

Page 3: edges $\{(i_1, j_2)|j - i = 4; 5 \leq i \leq n - 1, i \text{ is odd}\}$, edges $\{(i_2, j_1)|j - i = 4; 6 \leq i \leq n, i \text{ is even}\}$, and edges $\{(1_1, 4_2), (2_2, 3_1), ((n-1)_2, (n-2)_1), (n_1, (n-3)_2)\}$.

Page 4: edges $\{(i_1, j_2)|j - i = 4; 5 \leq j \leq n - 1, i \text{ is odd}\}$, edges $\{(i_2, j_1)|j - i = 4; 6 \leq j \leq n, i \text{ is even}\}$, and edges $\{(4_1, 1_2), (3_2, 2_1), ((n-2)_2, (n-1)_1), (n_3)_1, n_2)\}$.

... 

Page $n-3$: edges $\{(1_1, (n-2)_2), (2_2, (n-3)_1), (3_1, (n-4)_2), \ldots, ((\frac{n-2}{2})_s, (\frac{n}{2})_t)|s = 1$ and $t = 2$ when $\frac{n}{2}$ is even, and edges $\{(n-1)_1, 1_2\}, (n_2, 2_1), (n_1, 3_2), ((n-1)_2, 4_1), ((n-2)_1, 5_2), \ldots, ((\frac{n+4}{2})_s, (\frac{n+2}{2})_t)|s = 1$ and $t = 2$ when $\frac{n+2}{2}$ is odd).

Page $n-2$: edges $\{(n-2)_1, 1_2\}, (n_2, 2_1), (n_1, 3_2), \ldots, ((\frac{n}{2})_s, (\frac{n-2}{2})_t)|s = 1$ and $t = 2$ when $\frac{n}{2}$ is even, and edges $\{(1_1, (n-1)_2), (2_2, n_1), (3_1, n_2), (4_2, (n-1)_1), (5_1, (n-2)_2), \ldots, ((\frac{n+2}{2})_s, (\frac{n+4}{2})_t)|s = 1$ and $t = 2$ when $\frac{n+4}{2}$ is odd).

Page $n-1$: edges $\{(n_1, 1_2), ((n-1)_2, 2_1), \ldots, ((\frac{n+2}{2})_s, (\frac{n}{2})_t)|s = 2$ and $t = 1$ when $\frac{n}{2}$ is even, and edges $\{(1_1, n_2), (2_2, (n-1)_1), \ldots, ((\frac{n}{2})_s, (\frac{n+2}{2})_t)|s = 1$ and $t = 2$ when $\frac{n}{2}$ is odd).

Thus, $mbt(D) \leq n - 1$, the Kronecker cover of complete graph is dispersable when $n$ is even(see Fig.2 for the case $n = 4$).

Therefore, the result is established.

**Lemma 3.3.** Let $G$ be an arbitrary graph and $H$ be a graph such that its Kronecker cover $D(H)$ is a dispersible bipartite graph, then $mbt(D(G\square H)) \leq mbt(D(G)) + \Delta(D(H))$.

**Proof.** Since $D(H)$ is dispersable, there is a $\Delta(D(H))$-edge coloring of $D(H)$ and a corresponding matching book embedding of $D(H)$ in a $\Delta(D(H))$-page book so that all edges of one color
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Theorem 3.1. For $p, q \geq 3$, $\text{mbt}(D(K_p \square C_q)) = \Delta(D(K_p \square C_q)) = p + 1$.

Proof. Since $\Delta(D(K_p \square C_q)) = p + 1$, $\text{mbt}(D(K_p \square C_q)) \geq p + 1$ by Lemma 2.2. It is known from Lemma 3.1 that the Kronecker cover of cycle is dispersable. According to Lemma 3.3, it is easy to see that $\text{mbt}(D(K_p \square C_q)) \leq \text{mbt}(D(K_p)) + \Delta(D(C_q)) = \text{mbt}(D(K_p)) + 2$. Thus, by Lemma 3.2, we have $\text{mbt}(D(K_p \square C_q)) \leq p - 1 + 2 = p + 1$. Therefore, $\text{mbt}(D(K_p \square C_q)) = \Delta(D(K_p \square C_q))$, hence $D(K_p \square C_q)$ is dispersable when $p, q \geq 3$. $\square$
4. Conclusions

In this paper, we studied the dispersability of some standard graphs and obtain that the Kronecker cover of the Cartesian product of complete graphs $K_p$ and cycles $C_q$ is dispersable when $p, q \geq 3$. By the results of Kainen and Overbay, the odd cycles and the complete graphs $K_n (n \geq 3)$ are all near-dispersability. It is interesting to find that the matching book thickness become smaller after the Kronecker cover operation as to complete graphs and cycles in this work. Naturally, we raise a question as follows.

**Question:** Is the matching book thickness nonincreasing after Kronecker cover as to general graphs?

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