Graceful labeling construction for some special
tree graph using adjacency matrix

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Abstract

In 1967, Rosa introduced $\beta$ – labeling which was then popularized by Golomb under the name graceful. Graceful labeling on a graph $G$ is an injective function $f : V(G) \rightarrow \{0, 1, 2, \ldots, |E(G)|\}$ such that, when each edge $uv \in E(G)$ is assigned the label $|f(u) - f(v)|$ the resulting edge labels are distinct. If graph $G$ has graceful labeling then $G$ is called a graceful graph. Rosa also introduced $\alpha$ – labeling on graph $G$ which is a graceful labeling $f$ with an additional condition that there is $\lambda \in \{1, 2, \ldots, |E(G)|\}$ so that for every edge $uv \in E(G)$ where $f(u) < f(v)$ then $f(u) \leq \lambda < f(v)$. This paper gives a new approach to showing a graph is admitted $\alpha$ – labeling using an adjacency matrix. Then this construction will be used to construct graceful labeling for the superstar graph. Moreover, we give a graceful labeling construction for a super-rooted tree graph.

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1. Introduction

A graph $G$ consists of a finite nonempty set $V(G)$ of objects called vertices and a set $E(G)$ of objects called edges. The order of $G$ is the number of vertices of graph $G$. The size of $G$ is the number of edges of graph $G$ [3]. In 1967, Rosa introduced $\beta$-labeling which is more well known as graceful labeling after Golomb in 1972. Graceful labeling is an injective function $f : V(G) \to \{0, 1, 2, 3, \ldots, |E(G)|\}$ where $|E(G)|$ is the number of edges in the graph $G$ such that, when each edge $uv \in E(G)$ is assigned the label $|f(u) - f(v)|$ the resulting edge labels are distinct. If $f$ is a graceful labeling of a graph $G$ of size $m$, then $g(v) = m - f(v)$ is graceful [3]. In 1967, Rosa also introduced labeling called $\alpha$-labeling. The $\alpha$-labeling is a graceful labeling with an addition that there exists $\lambda \in \{1, 2, \ldots, |E(G)|\}$ such that for an arbitrary edge $uv \in E(G)$ where $f(u) < f(v)$ then $f(u) \leq \lambda < f(v)$ [8]. The value of $\lambda$ is called the boundary value of the $\alpha$-labeling $f$ [4]. If graph $G$ has $\alpha$-labeling then $G$ is called an $\alpha$-labeling graph.

There have been many studies conducted on graceful labeling, where one of the famous conjectures regarding graceful labeling is the Ringel-Kotzig conjecture which says all tree graphs are graceful [4]. People try several constructions and methods to prove the conjecture, for example using computer search as in [1]. The new tree graph which is constructed from the known graph can be found in [9], and [10, 11]. In 2009, Cavalier [2] introduces a new way to show that a graph is graceful using the adjacency matrix. Ghosh [5] also shows several classes of lobster are graceful using an adjacency matrix. For more results in graceful labeling, the reader can see in [4].

Let $G$ be a graph with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ then the matrix $A_G = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E(G), \\ 0, & \text{otherwise,} \end{cases}$$

is called the adjacency matrix of $G$. Let $G$ be a graph with $|E(G)| = m$ and a valuation $f : V(G) \to \{0, 1, 2, \ldots, m\}$. Then the $(m+1) \times (m+1)$ matrix $A_G = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } xy \in E(G) \text{ for } f(x) = i \text{ and } f(y) = j, \\ 0, & \text{otherwise,} \end{cases}$$

is called the generalized adjacency matrix of $G$ induced by $f$. The generalized adjacency matrix is an adjacency matrix where the vertex labels are used as the indices.

The graph used in this paper is simple and undirected, then its adjacency matrix is symmetric and entries on the main diagonal are 0. Let $A$ be an $n \times n$ matrix, then the $k$th diagonal line of $A$ is the collection of entries $D_k = \{a_{ij} | j - i = k\}$.

**Observation 1.1.** Let $G$ be a labeled graph and let $A_G$ be the generalized adjacency matrix for $G$. Then $A_G$ has exactly one entry 1 in each diagonal line, except the main diagonal of zeros, if and only if the valuation $f$ on $G$ that induces $A_G$ is graceful.

For any two graphs $G$ and $H$, let $u$ and $v$ be fixed vertices of $G$ and $H$, respectively. Then, the vertex amalgamation of $G$ and $H$ is the graph obtained from $G$ and $H$ by identifying $G$ and $H$ at the vertices $u$ and $v$[6]. In this paper, we expand the construction given by Cavalier to show that
a graph is $\alpha$-labeling using the adjacency matrix. Then the construction will be used to make a bigger graph from two $\alpha$-labeling graphs by amalgamating the vertex labeled 0.

In previous research, Pakpahan et al. [7] show that a supercaterpillar constructed from several caterpillar graphs of the same size with each caterpillar having uniform pairs is graceful. So, in this paper, we generalized the result to show that a supercaterpillar constructed from several star graphs of different sizes which is then called a superstar is graceful. Finally, we give a graceful labeling construction of a super-rooted tree graph.

2. Result

In this section, we introduce a new approach by using an adjacency matrix to show that a graph admits $\alpha$-labeling by using the adjacency matrix, and then we give a graceful labeling construction for several families of trees.

$\alpha$-Labeling Using Adjacency Matrix

**Theorem 2.1.** Let $G$ be a graceful graph of size $m$ and $A_G = \begin{bmatrix} a_{ij} \end{bmatrix}, i, j = 0, 1, \ldots, m$ induces by graceful labeling $f$ be the generalized adjacency matrix of graph $G$. Then the labeling $f$ on $G$ that induces $A_G$ is $\alpha$-labeling if and only if there is $a_{\beta \lambda} = 1$ where $\beta - \lambda = 1$, $a_{ij} = 0$ for every $i < \beta, j < i$ and $a_{ij} = 0$ for every $j > \lambda, j < i$.

**Proof.** Since $G$ is a graceful graph and $A_G$ is its generalized adjacency graph, then $A_G$ is symmetric. So, we can only focus on the lower triangular part of $A_G$. That is for edges $v_iv_j$ where $j < i$.

Suppose there is $a_{\beta \lambda} = 1$ where $\beta - \lambda = 1$, $a_{ij} = 0$ for every $i < \beta$, and $a_{ij} = 0$ for every $j > \lambda$. Then entry $a_{ij} = 1$ is only possible for $i \geq \beta$ and $j \leq \lambda$. Since $a_{ij} = 1$ if and only if $f(u) = i, f(v) = j$ and $uv \in E(G)$. Then there is $\lambda = \lambda_1 \in \{1, 2, \ldots, |E(G)|\}$ for an arbitrary edge $uv \in E(G)$ where $f(u) = j, f(v) = i$ and $j < i$, where $f(u) = j \leq \lambda < \lambda + 1 = \beta_1 \leq i = f(v)$. It is concluded that $A_G$ induces $\alpha$-labeling on $G$.

Suppose $f$ is $\alpha$-labeling on $G$ then there is $\lambda \in \{1, 2, \ldots, |E(G)|\}$ such that for an arbitrary edge $uv \in E(G)$ where $f(u) < f(v)$ then $f(u) \leq \lambda < f(v)$. So, there is $\lambda_1 = \lambda$ and $\beta_1 = \lambda + 1$ for every edge $uv \in E(G)$ where $f(u) = j, f(v) = i$ and $f(u) = j \leq \lambda_1 < \lambda_1 + 1 = \beta_1 \leq i = f(v)$. It means that entry $a_{ij} = 1$ is only possible for $i \geq \beta_1$ and $j \leq \lambda_1$. It is concluded that for every $i < \beta_1, j < i$ then $a_{ij} = 0$, and for every $j > \lambda_1, j < i$ then $a_{ij} = 0$. $\square$

**Theorem 2.2.** Let $G$ be an $\alpha$-labeling graph of size $m$. Let $A_G = [a_{ij}], i, j = 0, 1, \ldots, m$ be the generalized adjacency matrix induced by $\alpha$-labeling $f$ on $G$ with the boundary value is $\lambda$. Then the generalized adjacency matrix $A_G^* = [a^*_{m-x \ y}]$ where

$$a^*_{m-x \ y} = \begin{cases} 0, & \text{if } m-x < \lambda+1 \text{ and } y \leq \lambda, \\ 0, & \text{if } m-x \geq \lambda+1 \text{ and } y > \lambda, \\ a_{(\lambda+1+x) \mod(m+1) \ (\lambda-y) \mod(m+1)+1}, & \text{others}, \end{cases}$$

induces a new $\alpha$-labeling $f^*$ on graph $G$. 

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Proof. Let \( G \) be an \( \alpha \)-labeling graph of size \( m \). \( \alpha \)-labeling \( f \) on \( G \) induces \( A_G \). Labeling \( f \) is graceful since \( f \) is \( \alpha \)-labeling. It means there will be exactly one entry 1 on each diagonal \( D_k, k = 1, 2, \ldots, m \). Let \( a^*_{m-x,y} = a_{ij} \) then \( m-x = i \rightarrow x = m-i \) and \( y = j \). Therefore

\[
(m-x) - y = ((\lambda + 1 + x) \mod (m+1)) - ((\lambda - y) \mod (m+1))
\]

\[
= (\lambda + 1 + x - \lambda + y) \mod (m+1)
\]

\[
= ((\lambda + 1 - \lambda) + (x + y)) \mod (m+1)
\]

\[
= (1 + (m-i+j)) \mod (m+1)
\]

\[
= (1 + m - (i-j)) \mod (m+1)
\]

\[
= -(i-j) \mod (m+1).
\]

Notice that if \( i - j = k \) then \(-(i-j) \mod (m+1) = m-(k-1)\). That means entry 1 on each diagonal \( D_k \) on \( A_G \) mapped to \( D_{m-(k-1)} \) on \( A^* \). Therefore, for each diagonal \( D_l, l = (m-x) - y = m-(k-1) = 1, 2, \ldots, m \) on \( A^* \) has exactly one entry 1. Then we can conclude that the generalized adjacency matrix \( A^*_G \) induces graceful labeling \( f^* \) on graph \( G^* \). The operation from matrix \( A_G \) to matrix \( A^*_G \) is a permutation \((0, 1, 2, \ldots, m) \mapsto (\lambda, \lambda-1, \ldots, 0, m, m-1, \ldots, \lambda+1)\). Permuting the indices of a graph produces an isomorphic graph, then graph \( G^* \) is isomorphic to \( G \). Therefore \( f^* \) is graceful labeling on graph \( G \).

Since \( f \) is \( \alpha \)-labeling then there is \( a_{\beta,\lambda} = 1 \) where \( \beta - \lambda = 1 \) and \( a_{ij} = 0 \) for every \( i < \beta, j \leq \lambda, j < i \) and \( a_{ij} = 0 \) for every \( j > \lambda, i \geq \beta, j < i \). Let \( \lambda^* = \lambda \) and \( \beta^* = \lambda + 1 = \lambda + 1 = \beta \). If \( \beta^* = m-x \) then \( m-x = \lambda+1 \). If \( \lambda^* = y \) then \( y = \lambda \). So, \( a^*_{\beta^*,\lambda^*} = a_{(\lambda+1+x) \mod (m+1)} (\lambda-y) \mod (m+1) \cdot \)

If \( \beta^* = m-x \) then \( x = m - \beta^* = m - \beta = m - \lambda - 1 \). If \( \lambda^* = y \) then \( y = \lambda \). Therefore,

\[
a^*_{\beta^*,\lambda^*} = a_{(\lambda+1+x) \mod (m+1) + (\lambda-y) \mod (m+1) \cdot}
\]

\[
a = a_{m0} \mod (m+1) \cdot (\lambda - \lambda) \mod (m+1)
\]

\[
= a_{m0}
\]

\[
= 1.
\]

Since \( \lambda^* = \lambda \) and \( \beta^* = \beta \) then \( a^*_{m-x,y} = 0 \) for every \( m - x < \beta^*, y \leq \lambda^*, y < m - x \) and \( a^*_{m-x,y} = 0 \) for every \( y > \lambda^*, m - x \geq \beta^*, y < m - x \). So, by Theorem 2.1 we can conclude that \( A_G \) induces a new \( \alpha \)-labeling \( f^* \) on graph \( G \).

Consider entry 1 represented by \( a^*_{\beta^*,\lambda^*} = a_{m0} \). This operation indicates that the vertex labeled 0 on \( G \) with \( \alpha \)-labeling \( f \) turns into a vertex labeled \( \lambda \) on \( G \) with the new \( \alpha \)-labeling \( f^* \).

Example 1. Let \( T \) be a graph labeled by \( f \). Graph \( T \) and its generalized adjacency matrix is shown in Figure 1.

From Figure 1b, we can see that \( A_T \) is induced by \( \alpha \)-labeling \( f \) where \( \lambda = 1 \). Then we can construct the new adjacency matrix \( A^*_T = [a^*_{m-x,y}] \) where

\[
a^*_{m-x,y} = \begin{cases} 
0, & \text{if } m-x < 2 \text{ and } y \leq 1, \\
0, & \text{if } m-x \geq 2 \text{ and } y > 1, \\
a_{(2+x) \mod (m+1) + (1-y) \mod (m+1)}, & \text{others.}
\end{cases}
\]
We obtain the new adjacency matrix $A'_T$ and graph $T$ with a new $\alpha$-labeling $f^*$ as shown in Figure 2.

We can see that the vertex labeled 0 on $T$ with $\alpha$-labeling $f$ turns into a vertex labeled $\lambda = 1$ on $T$ with the new $\alpha$-labeling $f^*$.

**Theorem 2.3.** Let $G_1$ and $G_2$ be an $\alpha$-labeling graph of size $m_1$ and $m_2$, respectively. Then graph $G$ resulting from the amalgamation of graphs $G_1$ and $G_2$ at the vertex labeled 0 is an $\alpha$-labeling graph.

**Proof.** Let $A_1$ and $A_2$ be the generalized adjacency matrix induced by $\alpha$-labeling $f_1$ and $f_2$ on $G_1$ and $G_2$, respectively. Then from Theorem 2.2, we can get a new generalized adjacency matrix $A_1^*$ that induces a new $\alpha$-labeling $f_1^*$ on $G_1^*$ where the graph $G_1^*$ is an isomorphic graph of graph $G_1$. Let $\lambda$ be the boundary value of $\alpha$-labeling $f_1$. Then the vertex labeled 0 on $G_1$ with $\alpha$-labeling $f_1$ will turn into a vertex labeled $\lambda$ on $G_1$ with $\alpha$-labeling $f_1^*$.

Construct a new generalized adjacency matrix of graph $G$ by amalgamating the vertex labeled 0 on $G_2$ under the $\alpha$-labeling $f_2$ with the vertex labeled $\lambda$ on $G_1$ under the $\alpha$-labeling $f_1^*$ as shown in Figure 3.
Matrix $A_2$ is placed right on the main diagonal of submatrix $A_2'$ and submatrix $A_1'$ so that the first column of $A_2$ is in the column $\lambda$ of $A_0$ and the first row of $A_2$ is in the row $\lambda$ of $A_0$. Matrix $A_0$ is symmetric since $A_1'$ and $A_2$ is symmetric. Therefore we need only focus on the lower triangular part of $A_0$. That is for edges $v_iv_j$ where $j < i$.

The first column of $A_2$ is in the column $\lambda$ of $A_0$ then the vertex amalgamation occurs at the vertex labeled 0 of $G_2$ under the $\alpha$-labeling $f_2$ and the vertex labeled $\lambda$ of $G_1$ under the $\alpha$-labeling $f_1^\ast$. Placing matrix $A_2$ right on the main diagonal of $A_1^\ast$ since there is exactly one entry 1 on each diagonal $D_k, k = 1, 2, \ldots, m_1 + m_2$ and the main diagonal also other entries are 0. Therefore, we can conclude that graph $G$ is graceful. Since Matrix $A_0$ is obtained from $A_1^\ast$ and $A_2$ by permuting the indices of graph $G_1^\ast$ using permutation $(\lambda_1, \lambda_1 - 1, \ldots, 0, m_1, m_1 - 1, \ldots, \lambda_1 + 1) \mapsto (\lambda_1, \lambda - 1, \ldots, 0, m_1 + m_2, m_1 + m_2 - 1, \ldots, \lambda_1 + m_2 + 1)$ and permuting the indices of graph $G_2$ using permutation $(0, 1, \ldots, m_2) \mapsto (\lambda_1, \lambda_1 + 1, \ldots, \lambda_1 + m_2)$ then graph $G$ is obtained by vertex amalgamation from the isomorphic graph of graph $G_1$ and $G_2$ at vertex labeled 0.

Since $G_1$ is $\alpha$-labeling, then there is $a_{\beta_1\lambda_1} = 1$ where $\beta_1 - \lambda_1 = 1$ and $a_{ij} = 0$ for every $i < \beta_1, j \leq \lambda_1, i > j$ and $a_{ij} = 0$ for every $j > \lambda_1, i \geq \beta_1, i > j$ on $A_1$. Since $G_2$ is $\alpha$-labeling then there is $a_{\beta_2\lambda_2} = 1$ where $\beta_2 - \lambda_2 = 1$ and $a_{ij} = 0$ for every $i < \beta_2, j \leq \lambda_2, i > j$ and $a_{ij} = 0$ for every $j > \lambda_2, i \geq \beta_2, i > j$ on $A_2$. Therefore, there exists $a_{(\beta_1+\beta_2-1)(\lambda_1+\lambda_2)} = 1$ where $(\beta_1 + \beta_2) - (\lambda_1 + \lambda_2 - 1) = (\beta_1 - \lambda_1) + (\beta_2 - \lambda_2) - 1 = 1 + 1 - 1 = 1$ and $a_{ij} = 0$ for every $i < \beta_1 + \beta_2 - 1, j \leq \lambda_1 + \lambda_2, i > j$ and $a_{ij} = 0$ for every $j > \lambda_1 + \lambda_2, i \geq \beta_1 + \beta_2 - 1, i > j$ on $A_0$. Then it is concluded that $G$ is $\alpha$-labeling.

**Some Special Graceful Tree Graph**

**Definition 2.1.** A **superstar graph** is a rooted tree graph constructed from several star graphs by connecting the leaves from each star graph to a root vertex $r$.

An example of a superstar graph can be seen in Figure 4. This superstar graph is constructed from four stars $S_3, S_4, S_5, S_6$, and two isolated vertices ($S_0$).

![Example of superstar graph](image)

**Theorem 2.4.** **Superstar graph** constructed by star graph $S_{m_i}, i = 1, 2, \ldots, p$ and $q$ star graph $S_0$ where $|E(S_{m_i})| = m_i$ and $m_i \leq m_{i+1}$ is $\alpha$-labeling graph if $i \leq m_i, i = 1, 2, \ldots, p - 1$ for all $S_{m_i}$. 

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Proof. Let $T$ be a tree graph constructed by star graph $S_{m_i}, i = 1, 2, \ldots, p$ and $q$ star graph $S_0$ where $|E(S_{m_i})| = m_i, i = 1, 2, \ldots, p$ and $m_i \leq m_{i+1}$. Label the center vertex of each star graph $S_{m_i}$ with 0 and the leaves with $1, 2, \ldots, m_i$. Let $A_i$ be the generalized adjacency matrix of each star graph $S_{m_i}$. Then construct the generalized adjacency matrix of $T$ as shown in Figure 5.

![Figure 5: Generalized adjacency matrices of graph $T$.](image)

The submatrix $A_i'$ represents the entries of matrix $A_i$ which lie on and below the main diagonal. The submatrix $A_i''$ represents the entries of matrix $A_i$ which lie on and above the main diagonal. The matrix $u^T$ and $v_i^T$ are transposes of matrix $u$ and $v_i$, respectively. Since $A_i$ is a generalized adjacency matrix, it is symmetric. Thus, $A_i'$ is the transpose of $A_i''$. We can see that the first column of $A_0$ is the transpose of the first row of $A_0$. Therefore, we can conclude that the $A_0$ matrix is symmetric. Then we only need to focus on the lower triangular part of $A_0$. Furthermore, the main diagonal of $A_0$ contains neither the entries of $A_i'$ nor $A_i''$ and crosses the main diagonal of $A_p$, then all the entries on the main diagonal are 0.

The matrix $u$ is placed in the first column since there is to be exactly one entry 1 on each diagonal $D_k, k = m_T + 1, m_T + 2, \ldots, m_T + q$. Meanwhile, placing submatrix $A_{i+1}'$ above the main diagonal of submatrix $A'_i$ since there to be exactly one entry 1 in each diagonal $D_k, k = 1, 2, 3, \ldots, m_T$, except for $k = m_p + 1, m_p + m_p - 1 + 2, m_p + m_p - 1 + m_p - 2 + 3, \ldots, m_p + m_p - 1 + m_p - 2 + \cdots + m_p - 2 + \cdots + m_p - 2 + \cdots + m_p - 1 + m_1 + p$. The remaining entry 1 on the diagonal $D_k$ for $k = m_p + 1, m_p + m_p - 1 + 2, m_p + m_p - 1 + m_p - 2 + 3, \ldots, m_p + m_p - 1 + m_p - 2 + \cdots + m_p - 2 + \cdots + m_p - 1 + m_1 + p$ is in the first column. So that all the diagonals $D_k$ except the main diagonal have exactly one entry 1. Therefore, matrix $A_0$ that constructs the graph $T$ is induced by graceful labeling.
We also have to prove graph $T$ is a superstar graph. Placing submatrix $A'_{i+1}$ in $a_{01}$ of $A'_i$. Entry 1 in $A_i$ is only on the first column of $A'_i$. Then there is no entry 1 in $A'_i$ and $A'_{i+1}$ lies in the same column. Therefore, there are no vertices in each star graph $S_{m_i}$ has the same label. Notice that entry 1 in the first column denotes the label of the vertices connected to the root vertex $r$. Entries 1 are each located on the $k$th row, where $k = m_p + 1, m_p + m_{p-1} + 2, m_p + m_{p-1} + m_{p-2} + 3, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p + 1, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p + q$. Furthermore, there are as many as $p + q$ different values of $k$.

The vertices of each $S_{m_i}$ and $S_0$ will have their respective labels in the range of values as follows.

- $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + m_1 + p + 1$ to $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + m_1 + p + q$ for as many as $S_0$
- $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + p + 1$ to $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + m_1 + p$ for $S_{m_1}$
- $m_p + m_{p-1} + m_{p-2} + \cdots + m_3 + p + 1$ to $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + m_1 + p$ for $S_{m_2}$
- $m_p + m_{p-1} + m_{p-2} + \cdots + m_4 + p + 1$ to $m_p + m_{p-1} + m_{p-2} + \cdots + m_3 + p$ for $S_{m_3}$
- $\vdots$
- $m_p + m_{p-1} + p + 1$ to $m_p + m_{p-1} + m_{p-2} + p$ for $S_{m_{p-2}}$
- $m_p + p + 14$ to $m_p + m_{p-1} + 1$ for $S_{m_{p-1}}$, and
- $p + 1$ to $m_p + p$ for $S_{m_p}$.

Since for each star graph $S_{m_i}, i \leq m_i$, then each different value of $k$ is exactly on each leaf vertex label range of each star graph $S_{m_i}$ and $S_0$. So, $T$ is a superstar.

Since entry 1 in each adjacency matrix $A_i, i = 1, 2, \ldots, p$ is only on the first column, so there is no $a_{ij} = 1$ on $A_0$ where $j > p + 1$ and $j < i$. Notice that entry 1 on the first column of $A_0$ is only on the row $k + 1$, where $k = m_p + 1, m_p + m_{p-1} + 2, m_p + m_{p-1} + m_{p-2} + 3, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p + 1, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p + q$. See that $k + 1 \geq m_p + 2$. So, there is no entry $a_{i0} = 1$ where $i < m_p + 2$ and $j < i$. Since $A''_{i+1}$ is on the first row of $A''_i$ and matrix $v_i$ is on the first row of $A_0$, then there is no entry $a_{ij} = 1$ on $A_0$ where $i < p + 2, j \geq 1$ and $j < i$. Note that $i \leq m_i$ then $p + 2 \leq m_p + 2$. So, there is no entry $a_{ij} = 1$ where $i < p + 2$ and $j \leq i$. Thus, there exists $a_{ij} = 1$ for every $i < p + 1, j < i$ and $a_{ij} = 0$ for every $j > p + 2, i \geq p + 1$ and $j < i$. Therefore, $T$ is $\alpha-$labeling.

Example 2. Let there be 6 star graphs as shown in Figure 6.
We have the generalized adjacency matrix of each graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$ as shown in Figure 7.

Using the construction as in Theorem 2.4, we get the generalized adjacency matrix of superstar graph $T$ shown in Figure 8, and superstar graph $T$ as shown in Figure 9.
Figure 6: Star graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$, and $2S_0$.

Figure 7: Generalized adjacency matrix of each Star graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$.

Figure 8: Generalized adjacency matrix of superstar graph constructed by star graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$ and $2S_0$.

Figure 9: Superstar graph constructed by star graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$, and $2S_0$. 
Corollary 2.1. Let $T_1$ and $T_2$ be superstar graphs constructed as in Theorem 2.4, then superstar graph $T$ constructed by amalgamating $T_1$ and $T_2$ on the root vertex is an $\alpha-$labeling graph.

Proof. Since superstar graphs $T_1$ and $T_2$ are constructed as in Theorem 2.4, then it is an $\alpha-$labeling graph. Therefore, by Theorem 2.3, graph $T$, which is constructed by amalgamating superstar graphs $T_1$ and $T_2$ on the vertex labeled 0, is an $\alpha-$labeling graph. Notice that the root vertex of each superstar graph $T_1$ and $T_2$ is labeled 0. $T$ constructed by amalgamating $T_1$ and $T_2$ on the root vertex. Then, $T$ is a superstar. \qed

Using Corollary 2.1, we can construct a superstar graph that does not satisfy Theorem 2.4.

Example 3. Construction of a superstar graph from the star graphs in Figure 10.

Since $m_4 = 3 < i = 4$, the construction on Theorem 2.4 cannot be used to construct a graceful superstar graph from star graphs $S_2, S_2, S_3, S_3, S_4,$ and $S_5$. However, we can construct superstar graph $T_1$ from star graphs $S_2, S_2,$ and $S_3$ and superstar graph $T_2$ from star graphs $S_3, S_4,$ and $S_5$ so that $T_1$ and $T_2$ satisfy Theorem 2.4. Superstar Graphs $T_1$ and $T_2$ are shown in Figure 11.

Next, we need to construct a new graceful labeling on superstar graph $T_1$ by using Theorem 2.2. Superstar graph $T_1$ with the new graceful labeling is shown in Figure 12. Then, we can construct a graceful superstar graph $T$ from star graphs $S_2, S_2, S_3, S_3, S_4,$ and $S_5$ by amalgamating superstar graphs $T_1$ and $T_2$ on the root vertex by using Theorem 2.3. Superstar graph $T$ is shown in Figure 13.

Definition 2.2. A super-rooted tree is a rooted tree constructed from several rooted trees by connecting the root vertex of each rooted tree to a root vertex $r$. 
Theorem 2.5. Let $T$ be a super rooted tree constructed by $q$ uniform rooted tree $T_1$ of size $m$. If $T_1$ is a graceful graph with the root vertex of $T_1$ labeled by 0 or $m$, then $T$ is graceful.

Proof. We start by considering that tree $T$ is constructed by $q$ uniform tree $T_1$ of size $m$. Suppose $T_1$ has graceful labeling $f$. Let the root vertex of $T_1$ be $r_1$ and $f(r_1) = 0$ or $f(r_1) = m$.

Case 1. If $f(r_1) = 0$.

Let $A_{T_1}$ be the generalized adjacency matrix of $T_1$ induced by graceful labeling $f$. Then construct the generalized adjacency matrix $T$ as shown in Figure 15.

Matrix $v^T$ is the transpose of matrix $v$, then the last row of $A_T$ is the transpose of the last column of $A_0$. Matrix $A_{T_1}$ is symmetric. Thus, matrix $A_T$ is symmetric. So, we only need to focus on the lower triangular part of $A_T$. The root vertex of tree $T$ is represented by the last row of $A_T$. Entry 1 on matrix $v^T$ is only on the first column. Since the root vertex of $T_1$ is labeled by 0, then the root vertex of each tree graph $T_1$ will connect to the root vertex of $T$. Thus $T$ is a super-rooted tree.

Since $T_1$ is graceful, then each diagonal $D_k, k = 1, 2, \ldots, m$ of $A_{T_1}$ has exactly one entry 1,
except the main diagonal and other entries are 0. Thus each diagonal $D_l, l = 1, 2, \ldots, (m+1)q$, except $l = (m+1)n, n = 1, 2, \ldots, q$ has exactly one entry 1. Matrix $v_T$ is $(m+1) \times 1$ matrix. Entry 1 in $v_T$ is only on the first column. Then, entry 1 of each $v^T$ is on the diagonal $D_l, l = (m+1)n, n = 1, 2, \ldots, q$. Thus each diagonal line of matrix $A_T$ has exactly one entry 1 except the main diagonal and the other entries are 0. Therefore, $T$ is graceful.

**Case 2.** If $f(r_1) = m$.

From [3], we know that if $f(v)$ is graceful labeling on graph $G$ of size $m$ then, $g(v) = m - f(v)$ is also graceful labeling on graph $G$. Since $f$ is graceful labeling on graph $T$ and $f(r_1) = m$, then $g = m - f$ is graceful labeling on graph $T$ and $g(r_1) = m - f(r_1) = m - m = 0$. Thus, we can use the graceful labeling $g$ with $g(r_1) = 0$ to construct a super-rooted tree graph as in **Case 1**.

**Example 4.** Let $T_1$ be the superstar graph as shown in Figure 16.

Then, we can construct the generalized adjacency matrix $A_T$ to get the super-rooted tree $T$ constructed by two superstar graphs $T_1$ as shown in Figure 17.

Since the root vertex of super-rooted tree $T$ is labeled by $m_T$. Then, following the construction from Theorem 2.5, we also can make a graceful super-rooted tree $G$ from super-rooted tree $T$ as shown in Figure 18. This process can be repeated several times to obtain a bigger tree.
Figure 17: Super-rooted tree $T$ and generalized adjacency matrix $A_T$.

Figure 18: Super-rooted tree $G$ constructed by 2 super-rooted tree $T$.

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References


