On the construction of super edge-magic total graphs

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Abstract

Suppose $G = (V, E)$ be a simple graph with $p$ vertices and $q$ edges. An edge-magic total labeling of $G$ is a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ where there exists a constant $r$ for every edge $xy$ in $G$ such that $f(x) + f(y) + f(xy) = r$. An edge-magic total labeling $f$ is called a super edge-magic total labeling if for every vertex $v \in V(G)$, $f(v) \leq p$. The super edge-magic total graph is a graph which admits a super edge-magic total labeling. In this paper, we consider some families of super edge-magic total graph $G$. We construct several graphs from $G$ by adding some vertices and edges such that the new graphs are also super edge-magic total graphs.

Keywords: edge-magic total labeling, super edge-magic total graph, super edge-magic total labeling

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1. Introduction

We assume that all graphs in this paper are simple and finite. Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges. Let $f$ be a bijection function defined as $f : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$. Ringel and Llado \cite{16} provided the definition that the function $f$ is called an edge-magic total labeling if there exists a constant $r$ for every edge $xy$ in $G$ such that the weight of the edge $f(x) + f(y) + f(xy) = r$. We can say the constant $r$ as a magic constant of $f$. Wallis \cite{18} then called a graph $G$ admitting an edge-magic labeling as an edge-magic total graph.
The edge-magic total concept was introduced by Kotzig and Rosa [10, 11]. They proved that complete bipartite graphs $K_{m,n}$ ($m, n \geq 1$) and cycles $C_n$ ($n \geq 3$) are edge-magic total graphs. They also proved that a complete graph $K_n$ is edge-magic total graph if and only if $n = 1, 2, 3, 4, 5$ or 6; and the disjoint union of $n$ copies of $P_2$ has an edge-magic total labeling if and only if $n$ is odd. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [1, 8, 14, 16, 17].

In this paper, we consider an edge-magic total labeling of $G$ where the $p$ smallest labels are given to $V(G)$. Enomoto et al. [3] defined this version of edge-magic total labeling as a super edge-magic total labeling. If there exists a super edge-magic total labeling in a graph $G$, then $G$ is called as a super edge-magic total graph.

Enomoto et al. [3] proved that caterpillars are super edge-magic total. They also determined that a complete graph $K_n$ is super edge-magic total if and only if $n = 1, 2, 3, 4, 5$ or 6; and a complete bipartite graph $K_{m,n}$ is super edge-magic total if and only if $m = 1$ or $n = 1$. Enomoto et al. also proved that odd cycles are super edge-magic total. Some other results on super edge-magic total graph can be seen in [4, 5, 6, 7, 8, 9, 15].

The following properties are useful to show whether a graph $G$ is super edge-magic total or not. A graph $G = (V, E)$ is super edge-magic total graph if there exists a vertex labeling that causes a consecutive labeling.

**Lemma 1.1.** [2, 6] A graph $G$ is super edge-magic total if and only if there is a vertex labeling $f$ such that $f(V(G))$ and $\{f(u) + f(v) \mid uv \in E(G)\}$ are both consecutive.

In this case, in order to show that graph $G$ is super edge-magic total graph, it is simply indicated by taking a bijection of vertex labeling $f : V \rightarrow \{1, 2, \ldots, p\}$ where $\{f(u) + f(v) \mid uv \in E(G)\}$ is consecutive. The vertex labeling $f$ can be extended to be a total labeling by defining $f(uv) = p + q + \min\{f(u) + f(v) \mid uv \in E(G)\} - f(u) - f(v)$ for every edge $uv \in E(G)$. So that, the total labeling $f$ is a super edge-magic total labeling of $G$.

In this paper, we will construct some families of super edge-magic total graph which obtained from a known super edge-magic total graph. We obtain four results. First theorem is related to a path $P_n$. Lee and Lee [12] have provided a construction on a path $P_2$ such that a new graph is super edge-magic total. In this paper, we generalized such construction on a path $P_{2n}$ ($n \geq 1$).

The second result is related to disjoint union graph and joint product graphs. For graphs $G$ and $H$, a disjoint union graph $G \cup H$ is a graph with vertex set $V(G) \cup V(H)$ and an edge set $E(G) \cup E(H)$. A joint product graph of $G$ and $H$, denoted by $G \times H$, is a graph with $V(G \times H) = V(G) \cup V(H)$ and $E(G \times H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. For any super edge-magic total graphs $G$, we construct a new graph from $G$ by using disjoint union and joint product with some graphs, such that a new graph is also super edge-magic total.

For the third result, we define graph $G(+P_m(+H))$ where $m \geq 2$ as a graph obtained by taking one copy of the graphs $G$ and $H$ and a path $P_m$, then connect an end point of $P_m$ to all vertices of $G$ and the other end point of $P_m$ to all vertices of $H$. For any super edge-magic total graphs $G$, we provide some graphs $H$ such that $G(+P_m(+H))$ is also super edge-magic total. The last result is a construction of a super edge-magic total graph from a super edge-magic total graph by considering a super edge-magic labeling of the origin graph.
2. Main Results

In this section, we provide some constructions to obtain a new super edge-magic total graph which obtained from a super edge-magic total graph.

First, we consider a path $P_n$ ($n \geq 2$). López et al. [13] have proved that paths are super edge-magic total. Now, we define a graph $(P_n \cup hK_1)(+2K_1)$ ($h \geq 1$), which is a graph obtained by taking one copy of a path $P_n$, $h$ copies of $K_1$, and two isolated vertices ($2K_1$), then connect all end points of $P_n$ and all vertices of $h$ copies of $K_1$ to both two vertices of $2K_1$. We can say that $V((P_n \cup hK_1)(+2K_1)) = V(P_n) \cup V(hK_1) \cup V(2K_1)$ and $E((P_n \cup hK_1)(+2K_1)) = E(P_n) \cup \{uv \mid u \in V(hK_1) \text{ or } u \text{ is an end point of } P_n; v \in V(2K_1)\}$. In [12], Lee and Lee have proved that $(P_2 \cup hK_1)(+2K_1)$ ($h \geq 1$) are super edge-magic total. In the following theorem, we generalize Lee and Lee construction on a path $P_{2n}$ ($n \geq 1$).

**Theorem 2.1.** For integers $h, n \geq 1$, graphs $(P_{2n} \cup hK_1)(+2K_1)$ are super edge-magic total.

![Figure 1. Graph $(P_{2n} \cup hK_1)(+2K_1)$.](image)

**Proof of Theorem 2.1.** Let $V(hK_1) = \{x_i \mid 1 \leq i \leq h\}$, $V(2K_1) = \{y_1, y_2\}$, $V(P_{2n}) = \{z_i \mid 1 \leq i \leq 2n\}$, and $E(P_{2n}) = \{z_i z_{i+1} \mid 1 \leq i \leq 2n - 1\}$. It is easy to verify that $(P_{2n} \cup hK_1)(+2K_1)$ has $2n + h + 2$ vertices and $2n + 2h + 3$ edges.

Now, we define a vertex labeling $f : V((P_{2n} \cup hK_1)(+2K_1)) \to \{1, 2, \ldots, 2n + h + 2\}$ where for $v \in V((P_{2n} \cup hK_1)(+2K_1))$,

$$f(v) = \begin{cases} 
1, & \text{if } v = y_1, \\
2n + h + 2, & \text{if } v = y_2, \\
1 + i, & \text{if } v = z_{2i} \text{ with } 1 \leq i \leq n, \\
n + 1 + h + i, & \text{if } v = z_{2i-1} \text{ with } 1 \leq i \leq n, \\
n + 1 + i, & \text{if } v = x_i \text{ with } 1 \leq i \leq h.
\end{cases}$$

By the labeling above, we obtain that for $uv \in E((P_{2n} \cup hK_1)(+2K_1))$:

- If $u = y_1$, since $v$ is an end point of $P_{2n}$ or an element of $V(hK_1)$ then $\{f(u) + f(v)\} = \{1 + f(v)\} = \{n + 2, n + 3, \ldots, n + h + 3\}$.
• If \( u, v \in V(P_{2n}) \), then \( \{f(u) + f(v)\} = \{f(z_{2i-1}) + f(z_{2i}) \mid 1 \leq i \leq n\} \cup \{f(z_{2i}) + f(z_{2i+1}) \mid 1 \leq i \leq n-1\} = \{n+h+4, n+h+6, \ldots, 3n+h+2\} \cup \{n+h+5, n+h+7, \ldots, 3n+h+1\} = \{n+h+4, n+h+5, \ldots, 3n+h+2\}.

• If \( u = y_2 \), since \( v \) is an end point of \( P_{2n} \) or an element of \( V(hK_1) \) then \( \{f(u) + f(v)\} = \{2n + h + 2 + f(v)\} = \{3n + h + 3, 3n + h + 4, \ldots, 3n + 2h + 4\}. \)

Therefore, \( \{f(u) + f(v) \mid uv \in E((P_{2n} \cup hK_1)(+2K_1))\} \) is a consecutive sequence. By Lemma 1.1, the graph \((P_{2n} \cup hK_1)(+2K_1)\) is a super edge-magic total graph.

Before we continue to the next constructions, we need to show the following property of a super edge-magic total labeling.

**Lemma 2.1.** Let \( G \) be a connected graph with \( m \geq 2 \) vertices. Let \( f \) be a super edge-magic total labeling of \( G \). Then \( \max\{f(u) + f(v) \mid uv \in E(G)\} \geq m + 1 \).

*Proof.* Suppose that \( \max\{f(u) + f(v) \mid uv \in E(G)\} \leq m \). Since \( G \) is connected, a vertex \( u \) with \( f(u) = m \) will be adjacent to another vertex \( v \). So, \( f(u) + f(v) = m + f(v) \geq m + 1 \), a contradiction.

In the following theorem, we give a construction of a super edge-magic total graph obtained from any super edge-magic total graphs by applying disjoint union and joint product to an origin graph.

**Theorem 2.2.** Let \( G_m \) be a connected graph with \( m \geq 3 \) vertices. Let \( f \) be a super edge-magic total labeling of \( G_m \). If \( k = \max\{f(u) + f(v) \mid uv \in E(G_m)\} \), then \((G_m \cup (k-m-1)K_1) + K_1\) is a super edge-magic total graph.

![Graph](image_url)

Figure 2. Graph \((G_m \cup (k-m-1)K_1) + K_1\) where: (a) \( k = m + 1 \); (b) \( k \geq m + 2 \).

*Proof.* Let \( H = (G_m \cup (k-m-1)K_1) + K_1 \). By considering Lemma 2.1, we obtain \( k-m-1 \geq 0 \). In case \( k-m-1 = 0 \), we have \( H = (G_m \cup (k-m-1)K_1) + K_1 = G_m + K_1 \). We define \( V((k-m-1)K_1) = \{x_i \mid 1 \leq i \leq k-m-1\} \). Note that \((k-m-1)K_1\) is a graph
without edges. Thus, we can say that \( V(H) = V(G_m) \cup V((k - m - 1)K_1) \cup \{y\} \). Meanwhile, 
\( E(H) = E(G_m) \cup \{uy \mid u \in V(G_m \cup (k - m - 1)K_1)\} \). It is easy to see that \( |V(H)| = k \) and 
\( |E(H)| = |E(G_m)| + k - 1 \).

Let \( f \) be a super edge-magic labeling of \( G_m \) where \( k = \max\{f(u) + f(v) \mid uv \in E(G_m)\} \). 
Note that for \( v \in V(G_m) \), \( f(v) \in \{1, 2, \ldots, m\} \). Now, we define a vertex labeling \( g : V(H) \to \{1, 2, \ldots, k\} \) where for \( v \in V(H) \),

\[
g(v) = \begin{cases} 
  f(v), & \text{if } v \in V(G_m), \\
  k, & \text{if } v = y, \\
  m + i, & \text{if } v = x_i \text{ with } 1 \leq i \leq k - m - 1.
\end{cases}
\]

By the labeling above, we obtain that for \( uv \in E(H) \):

- If \( u, v \in V(G_m) \), since \( f \) is a super edge-magic labeling of \( G_m \), then \( \{g(u) + g(v)\} = \{f(u) + f(v)\} \) is a consecutive sequence, whose greatest element is \( k \).
- If \( u \in V(G_m) \) and \( v = y \), then \( \{g(u) + g(v)\} = \{g(u) + k\} = \{k + 1, k + 2, \ldots, k + m\} \).
- If \( u \in V((k - m - 1)K_1) \) and \( v = y \), then \( \{g(u) + g(v)\} = \{g(u) + k\} = \{k + m + 1, k + m + 2, \ldots, 2k - 1\} \).

Therefore, \( \{g(u) + g(v) \mid uv \in E(H)\} \) is a consecutive sequence. By Lemma 1.1, the graph \( H \) is a super edge-magic total graph.

Now, let us consider the graph \( G(+)P_m(+)H \) where \( m \geq 2 \). Let \( u \) and \( v \) be two end points of the path \( P_m \). Then we can write \( V(G(+)P_m(+)H) = V(G) \cup V(P_m) \cup V(H) \) and 
\( E(G(+)P_m(+)H) = E(G) \cup E(P_m) \cup E(H) \cup \{ux, vy \mid x \in V(G); y \in V(H)\} \). Thus, 
\( |V(G(+)P_m(+)H)| = |V(G)| + |V(P_m)| + |V(H)| \) and \( |E(G(+)P_m(+)H)| = |E(G)| + |E(P_m)| + |E(H)| + |V(G)| + |V(H)| \).

**Theorem 2.3.** Let \( G_m \) be a connected graph with \( m \geq 3 \) vertices. Let \( f \) be a super edge-magic total labeling of \( G_m \) and \( m + k = \max\{f(u) + f(v) \mid uv \in E(G_m)\} \). Then for \( k \geq 2 \) and \( n \geq 1 \), the graph \( G_m(+)P_{2k-2}(+)nK_1 \) is a super edge-magic total graph.

![Figure 3. Graph $G_m(+)P_{2k-2}(+)nK_1$.](image-url)
Proof of Theorem 2.3. Let $H = G_m (+) P_{2k−2}(+) nK_1$ where $n \geq 1$. It is easy to see that $|V(H)| = m + n + 2k − 2$ and $|E(H)| = |E(G_m)| + m + n + 2k − 3$. We define $V(nK_1) = \{x_i \mid 1 \leq i \leq n\}$. Note that $nK_1$ is a graph without edges. Let $V(P_{2k−2}) = \{z_i \mid 1 \leq i \leq 2k − 2\}$ and $E(P_{2k−2}) = \{z_i z_{i+1} \mid 1 \leq i \leq 2k − 3\}$. We assume that $z_1$ and $z_{2k−2}$ is adjacent to all vertices of $G_m$ and $nK_1$, respectively.

Let $f$ be a super edge-magic labeling of $G_m$. By considering Lemma 2.1, we have $\max\{f(u) + f(v) \mid uv \in E(G_m)\} \geq m + 1$. Now, we assume that $\max\{f(u) + f(v) \mid uv \in E(G_m)\} = m + k \geq m + 2$. Note that for $v \in V(G_m)$, $f(v) \in \{1, 2, \ldots, m\}$. Define a vertex labeling $g : V(H) \rightarrow \{1, 2, \ldots, m + n + 2k − 2\}$ where for $v \in V(H)$,

$$g(v) = \begin{cases} f(v), & \text{if } v \in V(G_m), \\ m + i, & \text{if } v = z_{2i} \text{ where } 1 \leq i \leq k - 1, \\ m + k + i, & \text{if } v = z_{2i+1} \text{ where } 0 \leq i \leq k - 2, \\ m + 2k - 2 + i, & \text{if } v = x_i \text{ with } 1 \leq i \leq n. \end{cases}$$

By the labeling above, we obtain that for $uv \in E(H)$:

- If $u, v \in V(G_m)$, since $f$ is a super edge-magic labeling of $G_m$, then $\{g(u) + g(v)\} = \{f(u) + f(v)\}$ is a consecutive sequence, whose greatest element is $m + k$.
- if $u \in V(G_m)$ and $v = z_1$, then $\{g(u) + g(v)\} = \{g(u) + (m + k)\} = \{m + k + 1, m + k + 2, \ldots, 2m + k\}$.
- if $u, v \in P_{2k−2}$, then $\{g(u) + g(v)\} = \{2m + k + 1, 2m + k + 2, \ldots, 2m + 3k - 3\}$.
- if $u \in V(nK_1)$ and $v = z_{2k−2}$, then $\{g(u) + g(v)\} = \{g(u) + (m + k - 1)\} = \{2m + 3k - 2, 2m + 2k - 1, \ldots, 2m + 3k - 3 + n\}$.

Therefore, $\{g(u) + g(v) \mid uv \in E(H)\}$ is a consecutive sequence. By Lemma 1.1, the graph $H$ is a super edge-magic total graph.

In the last theorem below, we will construct a super edge-magic total graph from a super edge-magic total graph by considering a super edge-magic labeling of the origin graph.

**Theorem 2.4.** Let $G_m$ be a connected graph with $m \geq 3$ vertices. Let $f$ be a super edge-magic total labeling of $G_m$. Let $F = \{f(u) + f(v) \mid uv \in E(G_m)\}$. For $ab \in E(G_m)$, let $f(a) + f(b) = \min(F)$ where $f(a) < f(b)$, $\max(F) = m + k$, and for $c \in V(G_m)$, $f(c) = k$.

1. For $f(a) = 1$, let $G^*_m$ be a graph obtained by taking one copies of $G_m$ and $nK_1$ where $n \geq 1$, then connect all vertices of $nK_1$ to $b$. Then $G^*_m$ is a super edge-magic total graph.
2. Let $G^{**}_m$ be a graph obtained by taking one copies of $G_m$ and $nK_1$ where $n \geq 1$, then connect all vertices of $nK_1$ to $c$. Then $G^{**}_m$ is a super edge-magic total graph.

**Proof.** Let $f$ be a super edge-magic labeling of $G_m$. Let $F = \{f(u) + f(v) \mid uv \in E(G_m)\}$. For $ab \in E(G_m)$, let $f(a) + f(b) = \min(F)$ where $f(a) < f(b)$. By considering Lemma 2.1, let $\max(F) = m + k$ and for $c \in V(G_m)$, $f(c) = k$.

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We define \( V(nK_1) = \{x_i \mid 1 \leq i \leq n\} \). Note that \( nK_1 \) is a graph without edges. Let \( H \in \{G_m^*, G_m^{**}\} \). So, \( V(H) = V(G_m) \cup V(nK_1) \). It is easy to see that \( |V(H)| = m + n \). In the other hand, \( E(G_m^*) = E(G_m) \cup \{bu \mid u \in V(nK_1)\} \) and \( E(G_m^{**}) = E(G_m) \cup \{cu \mid u \in V(nK_1)\} \). Thus, we can verify that \( |E(H)| = |E(G_m)| + n \). We distinguish two cases.

**Case 1.** \( H = G_m^* \)

So, \( f(a) = 1 \). Now, we define a vertex labeling \( g: V(H) \rightarrow \{1, 2, \ldots, m + n\} \) where for \( v \in V(H) \),

\[
g(v) = \begin{cases} 
  i, & \text{if } v = x_i, \\
  f(v) + n, & \text{if } v \in V(G_m).
\end{cases}
\]

By the labeling above, we obtain that for \( uv \in E(H) \):

- If \( u \in V(nK_1) \) and \( v = b \), then \( \{g(u) + g(v)\} = \{g(u) + (f(b) + n)\} = \{f(b) + n + 1, f(b) + n + 2, \ldots, f(b) + 2n\} \).

- If \( u, v \in V(G_m) \), since \( f \) is a super edge-magic labeling of \( G_m \), then \( \{g(u) + g(v)\} = \{(f(u) + n) + (f(v) + n)\} = \{f(u) + f(v) + 2n\} \) is a consecutive sequence, whose least element is \( f(b) + 2n + 1 \).

Therefore, \( \{g(u) + g(v) \mid uv \in E(H)\} \) is a consecutive sequence. By Lemma 1.1, the graph \( H \) is a super edge-magic total graph.

**Case 2.** \( H = G_m^{**} \)

Now, we define a vertex labeling \( h: V(H) \rightarrow \{1, 2, \ldots, m + n\} \) where for \( v \in V(H) \),

\[
h(v) = \begin{cases} 
  f(v), & \text{if } v \in V(G_m), \\
  m + i, & \text{if } v = x_i.
\end{cases}
\]

By the labeling above, we obtain that for \( uv \in E(H) \):

- If \( u, v \in V(G_m) \), since \( f \) is a super edge-magic labeling of \( G_m \), then \( \{g(u) + g(v)\} = \{f(u) + f(v)\} \) is a consecutive sequence, whose greatest element is \( m + k \).

- If \( u \in V(nK_1) \) and \( v = c \), then \( \{g(u) + g(v)\} = \{g(u) + k\} = \{m + k + 1, m + k + 2, \ldots, m + k + n\} \).

Therefore, \( \{g(u) + g(v) \mid uv \in E(H)\} \) is a consecutive sequence. By Lemma 1.1, the graph \( H \) is a super edge-magic total graph.

An illustration of graphs \( G_m^* \) and \( G_m^{**} \) of a super edge-magic total graph \( G_m \) with \( m \geq 3 \) vertices can seen in Figure 4 below. Let \( G_m \) be a super edge-magic total graph with \( m \geq 3 \) vertices where \( V(G_m) = \{z_i \mid 1 \leq i \leq m\} \) and \( f \) be a super edge-magic labeling of \( G_m \). Let \( F = \{f(z_p) + f(z_q) \mid uv \in E(G_m)\} \). In figure below, we assume that \( f(z_p) + f(z_q) = \min(F) \) where \( f(z_p) < f(z_q) \). Thus, it is clear that \( a = z_p \) and \( b = z_q \). Let \( \max(F) = m + k \) and \( f(z_r) = k \). Therefore, we have \( c = z_r \). Note that it is possible to have either vertex \( c = a \) or \( c = b \), where \( k = 1 \) or \( k = f(z_q) \), respectively.
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Figure 4. Graphs $G_m^*$ (left) and $G_m^{**}$ (right).

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