Lower and upper bounds on independent double Roman domination in trees

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Abstract

For a graph $G = (V, E)$, a double Roman dominating function (DRDF) $f : V \to \{0, 1, 2, 3\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a neighbor $u \in N(v)$, with $f(u) = 3$, or at least two neighbors $x, y \in N(v)$ having $f(x) = f(y) = 2$, and every vertex with value 1 under $f$ has at least a neighbor with value 2 or 3. The weight of a DRDF is the sum $f(V) = \sum_{v \in V} f(v)$. A DRDF $f$ is an independent double Roman dominating function (IDRDF) if the vertices with weight at least two form an independent set. The independent double Roman domination number $i_{dR}(G)$ is the minimum weight of an IDRDF on $G$. In this paper, we show that for every tree $T$ with diameter at least three, $i(T) + i_R(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i(T) + i_R(T) + s(T) - 2$, where $i(T)$, $i_R(T)$ and $s(T)$ are the independent domination number, the independent Roman domination number and the number of support vertex of $T$, respectively.

Keywords: double Roman domination, independent double Roman dominating function, independent double Roman domination number

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1. Introduction

In a graph \( G = (V, E) \), the open neighborhood of a vertex \( v \in V \) is \( N(v) = \{u \in V \mid uv \in E\} \), and the closed neighborhood is \( N(v) \cup \{v\} \). The degree of a vertex \( v \) denoted by \( \deg_G(v) \) is the cardinality of its open neighborhood. The maximum degree of a graph \( G \) is denoted by \( \Delta = \Delta(G) \). A leaf of a tree \( T \) is a vertex of degree one, while a support vertex of \( T \) is a vertex adjacent to a leaf. A strong support vertex is a support vertex adjacent to at least two leaves. We denote the set of leaves and support of \( G \) by \( L(G) \) and \( S(G) \), respectively. The distance between two vertices \( u \) and \( v \) in a connected graph \( G \) is the length of a shortest \( uv \)-path in \( G \). The diameter of \( G \), denoted by \( \text{diam}(G) \), is the maximum value among minimum distances between all pairs of vertices of \( G \).

For a vertex \( v \) in a rooted tree \( T \), let \( C(v) \) and \( D(v) \) denote the set of children and descendants of \( v \), respectively and let \( D[v] = D(v) \cup \{v\} \). Also, the depth of \( v \), \( \text{depth}(v) \), is the largest distance from \( v \) to a vertex in \( D(v) \). The maximal subtree \( T_v \) at \( v \) is the subtree of \( T \) induced by \( D[v] \). A double star \( DS_{p,q} \) is a tree containing exactly two vertices that are not leaves, where one of which is adjacent to \( p \) leaves and the other is adjacent to \( q \) leaves. A healthy spider is a tree obtained from the star \( K_{1,k} \) for \( k \geq 2 \) by subdividing each edge once, while a wounded spider \( S_{k,t} \) is obtained from a star \( K_{1,k} \) by subdividing \( t \) edges exactly once, where \( 1 \leq t \leq k - 1 \).

A set \( S \subseteq V \) is a dominating set of \( G \) if every vertex \( V - S \) has a neighbor in \( S \). The independent domination number \( i(G) \) is the minimum cardinality of a set that is both independent and dominating.

A function \( f : V(G) \to \{0, 1, 2\} \) is a Roman dominating function (RDF) on \( G \) if every vertex \( u \in V \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) with \( f(v) = 2 \). The weight of an RDF \( f \) is \( f(V(G)) = \sum_{u \in V(G)} f(u) \). Roman domination was introduced by Cockayne et al. in [14], and has been intensively studied in recent years [2, 3, 6, 11, 15, 19].

An independent Roman dominating function (IRDF) on \( G \) is an RDF such that the set \( \{u \in V(G) \mid f(u) \geq 1\} \) is independent set. The independent Roman domination number \( i_{R}(G) \) is the minimum weight of an IRDF on \( G \). The concept of independent Roman dominating function was first defined in [14] and studied by several authors, see [12, 13].

In [10], Beeler et al. introduced double Roman domination defined as follows. A double Roman dominating function (DRDF) on \( G \) is a function \( f : V \to \{0, 1, 2, 3\} \) having the property that if \( f(v) = 0 \), then vertex \( v \) has at least two neighbors assigned 2 under \( f \) or one neighbor \( w \) with \( f(w) = 3 \), and if \( f(v) = 1 \), then vertex \( v \) has at least one neighbor \( w \) with \( f(w) \geq 2 \). The double Roman domination number \( \gamma_{dR}(G) \) is the minimum weight of a DRDF on \( G \). For a DRDF \( f \), let \( V_i = \{v \in V \mid f(v) = i\} \) for \( i = 0, 1, 2, 3 \). Since these four sets determine \( f \), we can equivalently write \( f = (V_0, V_1, V_2, V_3) \) (or \( f = (V_0^f, V_1^f, V_2^f, V_3^f) \) to refer \( f \)). We note that \( \omega(f) = |V_0| + |V_2| + 3|V_3| \). Double Roman domination is studied for example in [1, 4, 5, 8, 9, 16, 18, 21, 22, 23], and elsewhere.

A DRDF \( f = (V_0, V_1, V_2, V_3) \) is an independent double Roman dominating function (IDRDF) if \( V_2 \cup V_3 \) is an independent set. The independent double Roman domination number \( i_{dR}(G) \) is the minimum weight of an IDRDF on \( G \). Clearly, for all \( G \) we have the following,

\[
\gamma_{dR}(G) \leq i_{dR}(G).
\]
In this paper, we prove that for any tree \( T \) with diameter at least three,
\[
i(T) + i_R(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i(T) + i_R(T) + s(T) - 2.
\]

We make use of the following results in this paper.

**Proposition A** ([17]). Let \( G \) be a graph. There exists an \( i_{dR} \)-function \( f = (V_0, V_1, V_2, V_3) \) such that \( V_1 = \emptyset \).

By Proposition A, we assume no vertex needs to be assigned the value 1 for any \( i_{dR}(G) \)-function \( f \).

**Proposition B** ([17]). Let \( T \) be a tree of order \( n \geq 3 \). Then

1. \( T \) has an \( i_{dR}(T) \)-function \( f = (V_0, \emptyset, V_2, V_3) \) such that \( L(T) \cap V_3 = \emptyset \).
2. For any IDRDF \( f = (V_0, \emptyset, V_2, V_3) \) of \( T \), \( V_2 \cap S(T) = \emptyset \).

**Proposition C** ([20]). Let \( T \) be a tree of order at least three. Then

1. \( T \) has an \( i_R(T) \)-function \( f = (V_0, V_1, V_2) \) such that \( L(T) \cap V_2 = \emptyset \).
2. For any IRDF \( f = (V_0, V_1, V_2) \) of \( T \), \( V_1 \cap S(T) = \emptyset \).

**Proposition D.** Let \( G \) be a graph of order \( n \geq 4 \). Then \( i_R(G) = 3 \) if and only if (a) \( \Delta(G) = n - 2 \) or (b) \( n = 3 \) and \( \Delta(G) \leq 1 \).

**Proposition E** ([7]). For any graph \( G \), \( i(G) \leq i_R(G) \leq 2i(G) \), with equality in lower bound if and only if \( G = \overline{K_n} \).

The next result is easy to establish, and so we omit the proof.

**Proposition 1.1.** For any graph \( G \), \( i_R(G) \leq i_{dR}(G) \).

### 2. Trees

In this section, we present bounds on independent double Roman domination of a tree in terms of the sum its independent domination and independent Roman domination numbers. We start with the following lemmas.

**Lemma 2.1.** Let \( r, s, t, \ell \) be non-negative integers and let \( T \) be a tree and \( T' \) a subtree of \( T \).

1. If \( i_{dR}(T) \leq i_{dR}(T') + 3s + 2t - \ell, i_R(T') + 2s + t - \ell \leq i_R(T), i(T') + s + t - r \leq i(T), s(T') \leq s(T) - r, \) and \( i_{dR}(T') - i_R(T') - s(T') + 2 \leq i(T') \), then \( i_{dR}(T) - i_R(T) - s(T) + 2 \leq i(T) \).
2. If \( i_{dR}(T) \geq i_{dR}(T') + 3s + 2t - \ell, i_R(T') \geq i_R(T) - 2s - t + \ell, i(T') \geq i(T) - s - t - r, s(T') \leq s(T) - 2r, \) and \( i(T') \leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} - 1 \), then \( i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1 \).
Proof. (1) By the assumptions we have
\[ i(T) \geq i(T') + s + t - r \]
\[ \geq i_{dR}(T') - i_R(T') - s(T') + 2 + s + t - r \]
\[ \geq (i_{dR}(T) - 3s - 2t + \ell) - (i_R(T) - 2s - t + \ell) - (s(T) - r) + 2 + s + t - r \]
\[ \geq i_{dR}(T) - i_R(T) - s(T) + 2. \]

(2) By the assumptions we obtain
\[ i(T) \leq i(T') + s + t + r \]
\[ \leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + s + t + r - 1 \]
\[ \leq (i_{dR}(T) - 3s - 2t + \ell) - (i_R(T) - 2s - t + \ell) + \frac{s(T) - 2r}{2} + s + t + r - 1 \]
\[ < i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1. \]

Lemma 2.2. Let \( T \) be a tree. Then
(i) \( i_{dR}(T) = i_R(T) + 1 \) if and only if \( T \) is a star.
(ii) \( i_{dR}(T) = i_R(T) + 2 \) if and only if \( T \) is a wounded spider with only one foot or \( T \) is a tree obtained from a double star by subdividing its central edge once or twice.

Proof. (i) If \( T \) is a star, then clearly \( i_{dR}(T) = 3 \) and \( i_R(T) = 2 \) and we are done. Let \( i_{dR}(T) = i_R(T) + 1 \). We show that \( T \) is a star. Let \( f = (V_0, \emptyset, V_2, V_3) \) be an \( i_{dR} \)-function of \( T \) such that \( |V_3| \) is as large as possible. We consider two cases.

Case 1. \( V_3 \neq \emptyset \).
Let \( v \in V_3 \). If \( T = N_T[v] \), then \( T \) is a star and we are done. Suppose \( T \neq N_T[v] \) and let \( T' = T - N_T[v] \). Assume \( T_1, T_2, \ldots, T_q \) \((q \geq 1)\) are the components of \( T' \). Clearly, the function \( f \), restricted to \( T' \) is an IDRDF of \( T' \) and hence
\[ i_{dR}(T') = i_{dR}(T_1) + i_{dR}(T_2) + \cdots + i_{dR}(T_q) \leq i_{dR}(T) - 3. \] (2)

On the other hand, any \( i_{dR} \)-function of \( T' \) can be extended to an IDRDF of \( T \) by assigning a 3 to \( v \) and a 0 to vertices in \( N_T(v) \) and so
\[ i_{dR}(T) \leq i_{dR}(T') + 3 = i_{dR}(T_1) + i_{dR}(T_2) + \cdots + i_{dR}(T_q) + 3. \] (3)

By (2) and (3), we have
\[ i_{dR}(T) = i_{dR}(T_1) + i_{dR}(T_2) + \cdots + i_{dR}(T_q) + 3. \] (4)

Similarly, we have
\[ i_R(T) = i_R(T_1) + i_R(T_2) + \cdots + i_R(T_q) + 2 \] (5)
We deduce from the assumption

\[ i(T) = i(T_1) + i(T_2) + \ldots + i(T_q) + 1 = i(T') + 1. \]  

(6)

By (4), (5) and Proposition 1.1, we obtain

\[ i_{dR}(T) - i_R(T) \geq \sum_{i=1}^{q} (i_{dR}(T_i) - i_R(T_i)) + 1 \geq q + 1 \]

which contradicts the assumption \( i_{dR}(T) = i_R(T) + 1. \)

**Case 2.** \( V_3 = \emptyset. \)

Then all leaves of \( T \) are assigned 2 under \( f \). Since \( V_3 = \emptyset \), \( \text{diam}(T) = 3 \) is impossible. So, let \( \text{diam}(T) \geq 4 \) and \( u, v \) be two leaves at distance \( \text{diam}(T) \), then the function \( g: V(T) \to \{0, 1, 2\} \)

defined by \( g(u) = g(v) = 1 \) and \( g(x) = f(x) \) for \( x \in V(T) - \{u, v\} \), is an IRDF of \( T \) of weight at most \( i_{dR}(T) - 2 \) which is a contradiction. Therefore \( \text{diam}(T) \leq 2 \) and so \( T \) is a star.

(ii) Let \( i_{dR}(T) = i_R(T) + 2 \). Assume that \( f = (V_0, \emptyset, V_2, V_3) \) is an \( i_{dR} \)-function of \( T \) such that \( |V_3| \) is as large as possible. First let \( V_3 \neq \emptyset \). As above, we have

\[ i_{dR}(T) - i_R(T) \geq \sum_{i=1}^{q} (i_{dR}(T_i) - i_R(T_i)) + 1 \geq q + 1. \]

We deduce from the assumption \( i_{dR}(T) - i_R(T) = 2 \) that \( q = 1 \) and \( i_{dR}(T') - i_R(T') = 1 \), that is \( T' \) is a star (by (i)). Using (6) we obtain

\[ 2 = i_{dR}(T) - i_R(T) = i_{dR}(T') - i_R(T') + 1 = i(T') + 1 = i(T). \]

It follows from Proposition E that \( 3 \leq i_R(T) \leq 4 \). If \( i_R(T) = 3 \), then by Proposition D, we have \( \Delta(G) = n - 2 \) and so \( T \) is a wounded spider with only one foot. Assume that \( i_R(T) = 4 \). Then

\[ i_R(T) = 2i(T) \]

and using the constructive characterization given by Chellali and Jafari Rad [13] we can see that the only trees satisfying \( i_{dR}(T) - i_R(T) = 2 \) are trees obtained from a double star by subdividing its central edge once or twice.

**Theorem 2.1.** Let \( T \) be a tree with \( s(T) \geq 2 \) support vertices. Then

\[ i_R(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_R(T) + i(T) + s(T) - 2. \]

**Proof.** It is enough to prove \( i_{dR}(T) - i_R(T) - s(T) + 2 \leq i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1 \). The proof is by induction on \( t = i_{dR}(T) - i_R(T) \). Since \( T \) is not a star, we have \( t > 1 \) by Lemma 2.2 (item (i)). If \( t = 2 \), then the result holds by Lemma 2.2 (item (ii)). Assume that \( t \geq 3 \) and statement holds for each tree \( T' \) with \( i_{dR}(T') - i_R(T') < t \). Let \( T \) be a tree with \( t = i_{dR}(T) - i_R(T) \). It follows from Lemma 2.2 (item (i)) that \( \text{diam}(T) \geq 3 \). If \( \text{diam}(T) = 3 \), then \( T = DS_{p,q} \) \( (q \geq p \geq 1) \) and hence \( i_{dR}(T) = 3 + 2p \), \( i_R(T) = 2 + p \) and \( i(T) = 1 + p \) and clearly the inequalities hold. Assume that \( \text{diam}(T) \geq 4 \) and \( v_1v_2\ldots v_k \) \( (k \geq 5) \) is a diametral path in \( T \) such that \( \text{deg}(v_2) \) is as large as possible. We consider the following cases.

**Case 1.** \( \text{deg}(v_2) \geq 3 \) and \( v_3 \) is not a support vertex and has a child \( a \) with depth 1 and degree 2.

Let \( v_3aa' \) be a path in \( T \) and let \( T' = T - \{a, a', v_1\} \). First we show that \( i_{dR}(T) - 4 \leq i_{dR}(T') \leq i_{dR}(T) - 3 \). To proved the left side, suppose that \( f = (V_0, \emptyset, V_2, V_3) \) is an \( i_{dR}(T') \)-function such
that $V_3 \cap L(T') = \emptyset$. By Lemma B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then $f(v_3) = 0$ and the function $g : V(T') \rightarrow \{0, 1, 2, 3\}$ defined by $g(a) = 3$, $g(x) = 0$ for $x \in \{v_1, a'\}$ and $g(x) = f(x)$ for $x \in V(T')$, is an IDRDF of $T$ yielding $i_{dR}(T) \leq i_{dR}(T') + 3$. If $f(v_2) = 0$, then $f(v_3) \geq 2$ and the function $g : V(T) \rightarrow \{0, 1, 2, 3\}$ defined by $g(v_1) = g(a') = 2$, $g(a) = 0$ and $g(x) = f(x)$ for $x \in V(T')$, is an IDRDF of $T$ and we have $i_{dR}(T) \leq i_{dR}(T') + 4$. To proved the right side, suppose that $f = (V_0, \emptyset, V_2, V_3)$ is an $i_{dR}(T)$-function such that $V_3 \cap L(T) = \emptyset$. By Lemma B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then $f(v_3) = 0$ and $f(a) + f(a') = 3$ and the function $f$ restricted to $T'$ is an IDRDF of $T$ and we have $i_{dR}(T) \geq i_{dR}(T') + 3$. If $f(v_2) = 0$, then $f(v_3) \geq 2$ and $f(v_1) = f(a') = 2$ and the function $f$ restricted to $T'$ is an IDRDF of $T$ and we have $i_{dR}(T) \geq i_{dR}(T') + 4$.

Using Proposition C and a similar argument we can see that $i_{R}(T') = i_{R}(T) - 2$. Now we show that $i(T) \leq i(T') + 1$. To show $i(T') + 1 \geq i(T)$, let $S$ be an $i(T)$-set. If $v_3 \not\in S$, then we may assume $v_2 \in S$ and clearly $S \cup \{a'\}$ is an IDS of $T$ and so $i(T) \leq i(T') + 1$. Assume that $v_3 \in S$. If $N_{T'}(v_4) \cap S \neq \{v_3\}$, then $(S - N_{T'}(v_2)) \cup \{v_3\}$ is an independent dominating set of $T'$ smaller than $S$ which is a contradiction. Hence, $N_{T'}(v_4) \cap S = \{v_3\}$. Now $(S - N_{T'}(v_2)) \cup \{v_2, v_4, a\}$ is an independent dominating set of $T$ which implies that $i(T) \leq i(T') + 1$. To prove $i(T) \geq i(T') + 1$, let $S$ be an $i(T)$-set. Clearly $|S \cap \{a, a'\}| = 1$ and either $v_2 \in S$ or $L_{v_2} \subseteq S$. In both cases, $(S - (\{a, a'\} \cup L_{v_3})) \cup \{v_2\}$ is an IDS of $T'$ and so $i(T) \geq i(T') + 1$. Thus $i(T) = i(T') + 1$. Therefore

$$i_{dR}(T') - i_{R}(T') \leq i_{dR}(T) - 3 - (i_{R}(T) - 2) = i_{dR}(T) - i_{R}(T) - 1 \leq t - 1.$$ 

Using the induction hypothesis on $T'$ and setting $s = t = r = \ell = 1$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_{R}(T) - s(T) + 2$ and using the induction hypothesis on $T'$ and setting $s = 1, t = r = \ell = 0$, Proposition 2.1 leads to $i(T) \leq i_{dR}(T) - i_{R}(T) + \frac{s(T)}{2} - 1$.

**Case 2.** $\deg(v_2) \geq 3$ and $v_3$ is not a support vertex and any child of $v_3$ has degree at least 3. Let $T' = T - T_{v_3}$. Clearly, $s(T') \leq s(T)$ and any $i_{dR}(T')$-function (resp. $i_{R}(T)$-function) can be extended to an IDRDF (resp. IRDF) of $T$ by assigning a 3 (resp. a 2) to each child of $v_3$ and a 0 to remaining vertices and hence $i_{dR}(T) \leq i_{dR}(T') + 3|C(v_3)|$ and $i_{R}(T) \leq i_{R}(T') + 2|C(v_3)|$. Likewise we have $i(T) \leq i(T') + |C(v_3)|$. Now we show that $i_{dR}(T) \geq i_{dR}(T') + 3|C(v_3)|$. Let $f$ be an $i_{dR}(T')$-function. By Proposition B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then $f(v_3) = 0$ and $f$ must assign 2 to each child of $v_3$ and the function $f$ restricted to $T'$ is an IDRDF of $T'$ implying that $i_{dR}(T) \geq i_{dR}(T') + 3|C(v_3)|$. If $f(v_2) = 0$, then $f(v_3) \geq 2$ and $f$ assigns 2 to each leaf of $T_{v_3}$. If $N(v_4) \cap ((V_2 \cup V_3) - \{v_3\}) \neq \emptyset$ and $z \in N(v_4) \cap ((V_2 \cup V_3) - \{v_3\})$, then the function $g : V(T') \rightarrow \{0, 1, 2, 3\}$ defined by $g(z) = 3$ and $g(x) = f(x)$ otherwise, is an IDRDF of $T'$ implying that $i_{dR}(T) \geq i_{dR}(T') + 1 + 4|C(v_3)|$ and if $N(v_4) \cap ((V_2 \cup V_3) - \{v_3\}) = \emptyset$, then the function $g : V(T') \rightarrow \{0, 1, 2, 3\}$ defined by $g(v_4) = 3$ and $g(x) = f(x)$ otherwise, is an IDRDF of $T'$ yielding $i_{dR}(T) \geq i_{dR}(T') + 4|C(v_3)|$. Thus $i_{dR}(T) = i_{dR}(T') + 3|C(v_3)|$. Similarly we can see that $i_{R}(T) = i_{R}(T') + 2|C(v_3)|$ and $i(T) = i(T') + |C(v_3)|$. It follows that

$$i_{dR}(T') - i_{R}(T') \leq i_{dR}(T) - 3|C(v_3)| - i_{R}(T) + 2|C(v_3)| = i_{dR}(T) - i_{R}(T) - |C(v_3)| \leq t - 1.$$ 

Applying the induction hypothesis on $T'$ and setting $s = 1$ and $t = r = \ell = 0$, Proposition 2.1 leads to $i_{dR}(T) - i_{R}(T) - s(T) + 2 \leq i(T) \leq i_{dR}(T) - i_{R}(T) + \frac{s(T)}{2} - 1$. 

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Case 3. $\deg(v_2) \geq 3$ and $v_3$ is a support vertex.  
Let $v' \in L_{v_3}$. We distinguish the following subcases.

Subcase 3.1. $|L_{v_3}| \geq 2$.

Let $T' = T - \{v_1, v'\}$. Obviously $s(T) = s(T')$. Now we show that $i_{dR}(T') = i_{dR}(T) - 2$. Let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T')$-function such that $L(T') \cap V_3 = \emptyset$. By Proposition B, $f(v_2) = 3$ or $f(v_3) = 0$. If $f(v_2) = 3$ then $f$ can be extended to an IDRDF of $T$ by assigning a 2 to $v'$ and a 0 to $v_1$, and if $f(v_3) = 0$ then to double Roman dominate $v_2$ and the leaf adjacent to $v_2$ and nothing that $f$ is a $i_{dR}(T')$-function, we must have $f(v_3) = 3$, and $f$ can be extended to an IDRDF of $T$ by assigning a 2 to $v_1$ and a 0 to $v'$, and hence $i_{dR}(T) \leq i_{dR}(T') + 2$. To prove the inverse inequality, let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T)$-function such that $L(T) \cap V_3 = \emptyset$.

As above $f(v_2) = 3$ and $f(v_3) = 0$ or $f(v_2) = 0$ and $f(v_3) = 3$. In each case, the function $f$ restricted to $T'$ is an IDRDF of $T'$ of weight $i_{dR}(T) - 2$ and so $i_{dR}(T) \geq i_{dR}(T') + 2$. Thus $i_{dR}(T) = i_{dR}(T') + 2$. Similarly, we can verify that $i_{R}(T) = i_{R}(T') + 1$ and $i(T) = i(T') + 1$. It follows that $i_{dR}(T') - i_{R}(T') = i_{dR}(T) - 2 - i_{R}(T) + 1 = i_{dR}(T) - i_{R}(T) - 1 = t - 1$. Applying the induction hypothesis on $T'$ and setting $t = 1$ and $s = r = \ell = 0$, Proposition 2.1 leads to $i_{dR}(T) - i_{R}(T) - s(T) + 2 \leq i(T) \leq i_{dR}(T) - i_{R}(T) + \frac{s(T)}{2} - 1$.

Subcase 3.2. $|L_{v_3}| = 1$.

Let $T' = T - \{v_1, v'\}$. Obviously, $s(T') = s(T) - 1$ and as above we can see that $i_{dR}(T') \leq i_{dR}(T) - 2$, $i_{R}(T') \leq i_{R}(T) - 1$ and $i(T') = i(T) - 1$. Next we show that $i_{dR}(T) \leq i_{dR}(T') + 3$.

Suppose that $f = (V_0, \emptyset, V_2, V_3)$ is an $i_{dR}(T)$-function such that $V_3 \cap L(T') = \emptyset$. By Lemma B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then as in Subcase 3.1, we can see that $i_{dR}(T) \leq i_{dR}(T') + 2$. If $f(v_2) = 0$, then $f(v_3) \geq 2$ and the function $g : V(T) \rightarrow \{0, 1, 2, 3\}$ define by $g(v_1) = 2$, $g(v') = 0$, $g(v_3) = 3$ and $g(x) = f(x)$ for $x \in V(T')$, is an IDRDF of $T$ and so $i_{dR}(T) \leq i_{dR}(T') + 3$. Hence $i_{dR}(T') + 2 \leq i_{dR}(T) \leq i_{dR}(T') + 3$.

Likewise, we can see that $i_{R}(T) \leq i_{R}(T') + 1$ and so $i_{R}(T) = i_{R}(T') + 1$. Hence

$$i_{dR}(T') - i_{R}(T') = i_{dR}(T) - 2 - i_{R}(T) + 1 = i_{dR}(T) - i_{R}(T) - 1 \leq t - 1.$$ Using the induction hypothesis on $T'$ and setting $s = 0, t = 2, r = \ell = 1$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_{R}(T) - s(T) + 2$ and using the induction hypothesis on $T'$ and setting $t = 1, s = r = \ell = 0$, Proposition 2.1 leads to $i(T) \leq i_{dR}(T) - i_{R}(T) + \frac{s(T)}{2} - 1$.

Considering Cases 1, 2, and 3 we may assume that $\deg(v_2) = 2$ and by the choice of diametral path any child of $v_3$ will be of degree two. We proceed with further cases.

Case 4. $\deg(v_2) = 2$.

Let $T' = T - v_2$. Clearly $s(T') \leq s(T) - 1$ and any $i_{dR}(T')$-function (resp. $i_{R}(T')$-function) can be extended to an IDRDF of $T$ by assigning a 3 (resp. a 2) to $v_2$ and a 0 to remaining vertices and so $i_{dR}(T) \leq i_{dR}(T') + 3$ and $i_{R}(T) \leq i_{R}(T') + 2$. Also any $i_{R}(T')$-set can be extended to an IDS of $T$ by adding $v_2$ and so $i(T) \leq i(T') + 1$. Now let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T)$-function. By Proposition B we have $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then the function $f$ restricted to $T'$ is an IDRDF of $T'$ yielding $i_{dR}(T') \geq i_{dR}(T') + 3$. Assume that $f(v_2) = 0$. Then $f(v_1) = 2$ and $f(v_3) \geq 2$. If $f(v_3) = 3$, then clearly $(N(v_4) - \{v_3\}) \cap (V_2 \cup V_3) = \emptyset$ and the function $g : V(T') \rightarrow \{0, 1, 2, 3\}$ defined by $g(v_4) = 2$ and $g(x) = f(x)$ is an IDRDF of $T'$ yielding $i_{dR}(T) \geq i_{dR}(T') + 3$, and if $f(v_3) = 2$, then clearly $(N(v_4) - \{v_3\}) \cap (V_2 \cup V_3) \neq \emptyset$ and the
function \( g : V(T') \to \{0, 1, 2, 3\} \) defined by \( g(z) = 3 \) for some \( z \in (N(v_4) - \{v_3\}) \cap (V_2 \cup V_3) \) and \( g(x) = f(x) \) is an IDRDF of \( T' \) implying that \( i_{dR}(T) \geq i_{dR}(T') + 3 \). Hence \( i_{dR}(T) \geq i_{dR}(T') + 3 \) and thus \( i_{dR}(T) = i_{dR}(T') + 3 \). Likewise we have \( i_R(T) = i_R(T') + 2 \) and \( i(T) = i(T') + 1 \). Hence \( i_{dR}(T') - i_R(T') = t - 1 \).

Applying the induction hypothesis on \( T' \) and setting \( s = 1 \) and \( t = r = \ell = 0 \), Proposition 2.1 leads to \( i_{dR}(T) - i_r(T) - s(T) + 2 \leq i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1 \).

**Case 5.** \( v_3 \) is a support vertex and \( v_3 \) has two children \( a \) and \( b \) with depth 1 and degree 2. Suppose \( v_3a'a' \) and \( v_3bb' \) are paths in \( T \). Let \( T' = T - \{a, a', b, b'\} \). It is easy to verify that \( s(T') = s(T) - 2, i_{dR}(T') = i_{dR}(T) - 4, i_R(T') + 2 \leq i_R(T) \leq i_R(T') + 3 \) and \( i(T') + 1 \leq i(T) \leq i(T') + 2 \). Hence \( i_{dR}(T') - i_r(T') \leq i_{dR}(T') - i_r(T') + 2 = i_{dR}(T') - i_r(T') + 2 \leq t - 1 \).

Using the induction hypothesis on \( T' \) and setting \( s = \ell = 0, t = 2, r = 1 \), Proposition 2.1 leads to \( i(T) \geq i_{dR}(T) - i_r(T) - s(T) + 2 \) and using the induction hypothesis on \( T' \) and setting \( s = 1, t = r = \ell = 0 \), Proposition 2.1 leads to \( i(T) \leq i_{dR}(T) - i_r(T) + \frac{s(T)}{2} - 1 \).

**Case 6.** \( v_3 \) is a support vertex and \( v_3 \) has exactly one child with depth 1 and degree 2. First let \( \text{deg}(v_4) = 2 \). Suppose \( T' = T - T_{v_4} \). If \( T' \) is a star, then the result can be seen easily. Let \( T' \) is not a star. Clearly \( s(T') \leq s(T) - 1 \) and as above we can see that \( i_{dR}(T) = i_{dR}(T') + 5, i_R(T) = i_R(T') + 3, i(T) = i(T') + 2 \). Hence \( i_{dR}(T') - i_r(T') = t - 1 \). Using the induction hypothesis on \( T' \) and setting \( s = t = r = 1, t = 0 \), Proposition 2.1 leads to \( i(T) \geq i_{dR}(T) - i_r(T) - s(T) + 2 \) and using the induction hypothesis on \( T' \) and setting \( s = t = 1, r = \ell = 0 \), Proposition 2.1 leads to \( i(T) \leq i_{dR}(T) - i_r(T) + \frac{s(T)}{2} - 1 \).

Assume now that \( \text{deg}(v_4) \geq 3 \) and \( v' \in L_{v_3} \). Consider the following subcases.

**Subcase 6.1.** \( v_4 \) has a child \( a \) with depth 1 and degree 2. Suppose \( v_4a'a' \) is a path in \( T \) and let \( T' = T - \{v_4, v_2, a, a'\} \). Clearly, \( s(T) = s(T') - 2 \) and it is easy to verify that \( i_{dR}(T) = i_{dR}(T') + 5, i_R(T) = i_R(T') + 3, i(T) = i(T') + 2 \). Hence \( i_{dR}(T') - i_r(T') \leq t - 1 \) and using the induction hypothesis on \( T' \) and setting \( s = t = 1, r = t = 0 \), Proposition 2.1 leads to \( i(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_r(T) + i(T) + s(T) - 2 \).

**Subcase 6.2.** \( v_4 \) is a strong support vertex. First let \( |L_{v_3}| \geq 2 \). Suppose that \( w \in L_{v_3} \). Suppose that \( T' = T - \{v', w\} \). Clearly, \( s(T) = s(T') \) and one can easily see that \( i_{dR}(T') = i_{dR}(T) + 2, i_R(T) = i_R(T') + 1, i(T) = i(T') + 1 \). Hence \( i_{dR}(T') - i_r(T') \leq t - 1 \) and using the induction hypothesis on \( T' \) and setting \( s = t = 1, r = \ell = 0 \), Proposition 2.1 leads to \( i(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_R(T) + i(T) + s(T) - 2 \).

Now, let \( |L_{v_3}| = 1 \). Assume that \( T' = T - T_{v_3} \). Clearly \( s(T') = s(T) - 2 \) and any \( i_{dR}(T') \)-function (resp. \( i_R(T') \)-function) can be extended to an IDRDF of \( T \) by assigning a 1 (resp. a 2) to \( v_3 \), a 2 (resp. a 1) to \( v_1 \) and a 0 to remaining vertices and so \( i_{dR}(T) \leq i_{dR}(T') + 5 \) and \( i_R(T) \leq i_R(T') + 2 \). Also any \( i(T') \)-set can be extended to an IDS of \( T \) by adding \( v_2, v' \) and so \( i(T) \leq i(T') + 2 \). Now let \( f = (V_0, \emptyset, V_2, V_3) \) be an \( i_R(T) \)-function such that \( L(T) \cap V_3 = \emptyset \). By Proposition B, we have \( f(v_2) = 3 \) or \( f(v_2) = 0 \). If \( f(v_2) = 3 \), then \( f(v') = 2 \) and the function \( f \) restricted to \( T' \) is an IDRDF of \( T' \) yielding \( i_{dR}(T) \geq i_{dR}(T') + 5 \). Assume that \( f(v_2) = 0 \). Then \( f(v_1) = 2 \) and \( f(v_3) = 3 \) since \( v_3 \) is a support vertex and so \( f(x) = 2 \) for each \( x \in L_{v_3} \). Hence the function \( f \) restricted to \( T' \) is an IDRDF of \( T' \) yielding \( i_{dR}(T) \geq i_{dR}(T') + 5 \). Thus \( i_{dR}(T) = i_{dR}(T') + 5 \). Likewise we have \( i_R(T) = i_R(T') + 3 \) and \( i(T) = i(T') + 2 \). It follows that \( i_{dR}(T') - i_r(T') = t - 1 \) and using the induction hypothesis on \( T' \) and setting \( s = t = 1, r = \ell = 0 \),
Proposition 2.1 leads to \( i_R(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_R(T) + i(T) + s(T) - 2 \).

**Subcase 6.3.** \( v_4 \) is adjacent to at most one leaf, any child of \( v_4 \) with depth 1 is of degree at least 3 and for any child \( y \) of \( v_4 \) with depth 2 we have \( T_y = DS_{1,\deg(y) - 1} \) where \( \deg(y) \geq 3 \) or \( T_y \) is a healthy spider. We consider the following.

- \( |L_{v_3}| = 1 \).
  Let \( T' = T - T_{v_3} \). Clearly, \( s(T') = s(T) - 2 \), \( i_{dR}(T') + 4 \leq i_{dR}(T) \leq i_{dR}(T') + 5 \), \( i_R(T') + 2 \leq i_R(T) \leq i_R(T') + 3 \) and \( i(T') + 1 \leq i(T) \leq i(T') + 2 \).

  It follows that \( i_{dR}(T') - i_R(T') \leq t - 1 \) and using the induction hypothesis on \( T' \) and setting \( t = 3, \ell = 1, r = 2, s = 0 \), Proposition 2.1 leads to \( i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2 \) and using the induction hypothesis on \( T' \) and setting \( s = 1, r = \ell = 0 \), Proposition 2.1 leads to \( i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1 \).

\[
i(T) \leq i(T') + 2
\leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + 1
\leq i_{dR}(T) - 4 - i_R(T) + 3 + \frac{s(T) - 2}{2} + 1
= i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1.
\]

- \( |L(v_3)| \geq 2 \)
  Let \( T' = T - T_{v_3} \). If \( T' \) is a star, then the result is immediate. Assume \( T' \) is not a star. Suppose that \( A \) is the set of children of \( v_3 \) of depth 1, \( B \) is the set of children of \( v_4 \) of depth 2 and \( C \) is the set of vertices \( x \in D(v_4) \cap L(T) \) satisfying \( d(v_4, x) = 3 \). Let \( B_1 = B \cap s(T) \) and \( B_2 = B - B_1 \). Clearly, \( s(T') \leq s(T) - 2 \), and it is not hard to see that \( i_{dR}(T') = i_{dR}(T) - 3|A| - 3|B_1| - 2|B_2| - 2|C| - 2|L_{v_4}| \), \( i_R(T') = i_R(T) - 2|A| - 2|B_1| - 2|B_2| - 2|C| - |L_{v_4}| \) and \( i(T') = i(T) - |A| - |B_1| - |C| - |L_{v_4}| \). Hence

\[
i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 3|A| - 3|B_1| - 2|B_2| - 2|C| - 2|L_{v_4}|
- (i_R(T) - 2|A| - 2|B_1| - 2|B_2| - |C| - |L_{v_4}|)
= i_{dR}(T) - i_R(T) - (|A| + |B| + |C| + |L_{v_4}|) \leq t - 1.
\]

By the induction hypothesis we have

\[
i(T) = i(T') + |A| + |B_1| + |C| + |L_{v_4}|
\geq i_{dR}(T') - i_R(T') - s(T') + 2 + |A| + |B_1| + |C| + |L_{v_4}|
> i_{dR}(T) - i_R(T) - s(T) + 2,
\]
and
\[ i(T) = i(T') + |A| + |B_1| + |C| + |L_{v_4}| \]
\[ \leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + |A| + |B_1| + |C| + |L_{v_4}| - 1 \]
\[ < i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1. \]

**Case 7.** \( \deg(v_3) \geq 3 \) and \( v_3 \) is not a support vertex.
Then \( T_{v_3} \) is a healthy spider and by that choice of diametral path and considering above cases we may assume that the maximal subtree at any child of \( v_4 \) with depth two is a healthy spider with at least two feet. We distinguish the following situations.

**Subcase 7.1.** \( \deg(v_3) \geq 4. \)
First let \( \deg(v_4) = 2 \) and let \( T' = T - T_{v_4} \). If \( T' \) is a star then the results can be verified easily. Let \( t' \) is not a star. Clearly, \( s(T') \leq s(T) - 2, i_{dR}(T') + |V(C(v_3))| \leq i_{dR}(T) \leq i_{dR}(T') + 2 + |C(v_3)| \), \( i_R(T) = i_R(T') + 2 + |C(v_3)| \) and \( i(T') + |C(v_3)| \leq i(T) \leq i(T') + |C(v_3)| + 1 \). Hence
\[ i_{dR}(T'') - i_R(T'') \leq i_{dR}(T) - 2|C(v_3)| - 2. \]
and by the induction hypothesis on \( T' \) and setting \( t = |C(v_3)|, \ell = 0, r = 1, s = 1 \), Proposition 2.1 leads to \( i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2 \). On the other hand, by the induction hypothesis on \( T' \), we obtain
\[ i(T) \leq i(T') + |C(v_3)| + 1 \]
\[ \leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + |C(v_3)| \]
\[ \leq i_{dR}(T) - 2 - 2|C(v_3)| - i_R(T) + 2 + |C(v_3)| + \frac{(s(T) - 2)}{2} + |C(v_3)| \]
\[ = i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1 \]

Now let \( \deg(v_4) \geq 3 \). Considering above cases and subcases, we may assume that any child of \( v_4 \) with depth 2, is the center of a healthy spider. Assume \( a, b \in C(v_3) - \{v_2\} \) and let \( v_3aa' \) and \( v_3bb' \) be paths in \( T \). We distinguish the following.

- \( v_4 \) has a child \( w \) with depth 1 and degree 2.
  Suppose \( v_4ww' \) is a path in \( T \). Let \( T' = T - \{v_1, a, a', w, w'\} \). Obviously, \( s(T') = s(T) - 2 \).
  We show that \( i_{dR}(T) = i_{dR}(T') + 6\). To prove \( i_{dR}(T) \leq i_{dR}(T') + 6 \), let \( f = (V_0, \emptyset, V_2, V_3) \) be an \( i_{dR}(T') \)-function such that \( L(T) \cap V_3 = \emptyset \). By Lemma B, \( f(v_3) = 3 \) or \( f(v_3) = 0 \).
  If \( f(v_3) = 3 \), then \( f(v_4) = f(v_2) = 0 \) and the function \( g : V(T) \rightarrow \{0, 1, 2, 3\} \) define by \( g(w) = 3, g(v_1) = g(v_3) = g(a') = 2, g(a) = g(w') = 0 \) and \( g(x) = f(x) \) for \( x \in V(T') \), is an IDRDF of \( T \), and so \( i_{dR}(T) \leq i_{dR}(T') + 6 \). If \( f(v_3) = 0 \), then \( f(v_2) = 2 \)
and $f(v_4) \geq 2$ and the function $g : V(T) \to \{0, 1, 2, 3\}$ define by $g(a) = 3, g(w') = 2, g(a') = g(w) = g(v_1) = 0, g(v_2) = 3$ and $g(x) = f(x)$ for $x \in V(T')$, is an IDRDF of $T$, and we have $i_{dR}(T) \leq i_{dR}(T') + 6$. To prove $i_{dR}(T) \geq i_{dR}(T') + 6$, let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T)$-function such that $L(T') \cap V_3 = \emptyset$. By Lemma B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then we may assume $f(a) = f(b) = 3$ and that $f(w) + f(w') \geq 2$ and the function $g$ defined on $T'$ by $g(v_2) = 2$ and $g(x) = f(x)$ otherwise, is an IDRDF of $T'$ of weight $i_{dR}(T) - 6$, and if $f(v_2) = 0$, then $f(v_1) = f(a') = 2, f(v_3) \geq 2, f(w) + f(w') = 3$ and the function $g$ defined on $T'$ by $g(v_3) = 2$ and $g(x) = f(x)$ otherwise, is an IDRDF of $T'$ of weight $i_{dR}(T') - 6$ and so $i_{dR}(T) \geq i_{dR}(T') + 6$. Thus $i_{dR}(T) = i_{dR}(T') + 6$.

Likewise, we can see that $i_R(T') = i_R(T) - 4$ and $i(T') = i(T) - 2$. It follows that $i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 6 - i_R(T) + 4 = i_{dR}(T) - i_R(T) - 2 \leq t - 1$. Using the induction hypothesis on $T'$ and setting $s = 2, t = \ell = r = 0$, Proposition 2.1 leads to $i_{dR}(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_{dR}(T) + i(T) + s(T) - 2$.

- $v_4$ is a strong support vertex.

Let $w \in L_{v_4}, T' = T - \{v_1, a, a', b, b', w\}$. Clearly $s(T') = s(T) - 2$, and it is easy to verify that $i_{dR}(T') = i_{dR}(T) - 7, i_R(T') + 4 \leq i_R(T) \leq i_{dR}(T') + 5, i(T) - 3 \leq i(T') \leq i(T) - 2$ and this implies that $i_{dR}(T') - i_R(T') \leq t - 1$. Using the induction hypothesis on $T'$ and setting $s = 1, t = 2, \ell = 0, r = 1$, Proposition 2.1 leads to $i_{dR}(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T)$ and also we have

$$i(T) \leq i(T') + 3$$

$$\leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + 2$$

$$\leq i_{dR}(T) - 7 - i_R(T) + 5 + \frac{(s(T) - 2)}{2} + 2$$

$$= i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$$

- $v_4$ is adjacent to at most one leaf, any child of $v_4$ with depth 1 is of degree at least 3 and for child $y$ of $v_4$ with depth 2 is the center of a healthy spider with at least two feet.

Suppose that $T' = T - v_4$. If $T'$ is a star, then the result can be seen immediately. Assume $T'$ is not a star. Let $A, B$ and $C$ be defined as in the Subcase 6.3. Clearly, $s(T') \leq s(T) - 2|B|$ and it is not hard to verify that $i_{dR}(T') = i_{dR}(T) - 3|A| - 2|B| - 2|C| - 2|L_{v_4}|, i_R(T') = i_R(T) - 2|A| - 2|B| - |C| - |L_{v_4}| - 1 \leq i(T') \leq i(T) - |A| - |C| - |L_{v_4}|$. These imply that

$$i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 3|A| - 2|B| - 2|C| - 2|L_{v_4}|$$

$$- (i_R(T) - 2|A| - 2|B| - |C| - |L_{v_4}|)$$

$$= i_{dR}(T) - i_R(T) - (|A| + |C| + |L_{v_4}|) \leq t - 1.$$
Using the induction hypothesis on $T'$ and setting $s = |A| + |B|$, $t = |C| + |L_{vu}|$, $\ell = 0$, $r = 2|B|$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$ and also we have

$$
i(T) \leq i(T') + |A| + |C| + |L_{vu}| + 1
\leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + |A| + |C| + |L_{vu}|
\leq i_{dR}(T) - i_R(T) + \frac{(s(T) - 2|B|)}{2}
\leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1.
$$

Subcase 7.2. $\deg(v_3) = 3$ and $\deg(v_4) \geq 3$.
Assume that $T' = T - T_{vu}$. If $T'$ is a star, then one can check the result easily. Suppose $T'$ is not star. Obviously, $s(T') = s(T) - 2$ and one can see that $i_{dR}(T') + 5 \leq i_{dR}(T) \leq i_{dR}(T') + 6$, $i_R(T') + 3 \leq i_R(T) \leq i_R(T') + 4$ and $i(T) = i(T') + 2$. Hence $i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 5 - i_R(T) + 3 = i_{dR}(T) - i_R(T) - 2 \leq t - 1$. Using the induction hypothesis on $T'$ and setting $s = \ell = 0$, $t = 3$, $r = 1$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$ and also we have $i(T) = i(T') + 2 \leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + 1 \leq i_{dR}(T) - 5 - i_R(T) + 4 + \frac{s(T) - 2}{2} + 1 = i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

Subcase 7.3. $\deg(v_3) = 3$ and $\deg(v_4) = 2$.
Assume that $T' = T - T_{vu}$. If $T'$ is a star, then we can check the result easily. Suppose $T'$ is not star. Obviously, $s(T') \leq s(T) - 1$ and $i_{dR}(T') + 6 \leq i_{dR}(T) \leq i_{dR}(T') + 7$, $i_R(T') = i_R(T') + 4$ and $i(T') + 2 \leq i(T) \leq i(T') + 3$. Hence $i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 6 - i_R(T) + 3 \leq t - 1$. Applying the induction hypothesis on $T'$ and setting $s = r = 1$, $t = 2$, $\ell = 0$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$. On the other hand, by the induction hypothesis we have $i(T) \leq i(T') + 3 \leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + 2 \leq i_{dR}(T) - 6 - i_R(T) + 4 + \frac{s(T) - 1}{2} + 2 = i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1/2$ and this implies $i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$ because $i(T)$ is an integer. This completes the proof.

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References


