Chromatic number of super vertex local antimagic total labelings of graphs

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Abstract

Let $G(V, E)$ be a simple graph and $f$ be a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}$ where $f(V) = \{1, 2, \ldots, |V|\}$. For a vertex $x \in V$, define its weight $w(x)$ as the sum of labels of all edges incident with $x$ and the vertex label itself. Then $f$ is called a super vertex local antimagic total (SLAT) labeling if for every two adjacent vertices their weights are different. The super vertex local antimagic total chromatic number $\chi_{\text{slat}}(G)$ is the minimum number of colors taken over all colorings induced by super vertex local antimagic total labelings of $G$. We classify all trees $T$ that have $\chi_{\text{slat}}(T) = 2$, present a class of trees that have $\chi_{\text{slat}}(T) = 3$, and show that for any positive integer $n \geq 2$ there is a tree $T$ with $\chi_{\text{slat}}(T) = n$.

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1. Introduction

All graphs defined in this paper are simple and connected. Introduced by Arumugam et al. [1], a vertex local antimagic labeling is a bijective function \( f : E(G) \to \{1, 2, \ldots, |E(G)|\} \) such that \( w(u) \neq w(v) \) for any adjacent vertices \( u \) and \( v \), where the weight \( w(x) \) of a vertex \( x \in V \) is the sum of labels of all edges incident with \( x \). The minimum number of distinct weights needed for a graph \( G \) to have a vertex local antimagic labeling is denoted by \( \chi_{la}(G) \). They conjectured that every connected graph other than \( K_2 \) is a vertex local antimagic graph, which was confirmed by Haslegrave using probabilistic method [4].

Putri et al. [7] introduced a new variant of vertex local antimagic labeling, called vertex local antimagic total labeling. A vertex local antimagic total labeling is a bijective map \( f : \sum_{xy \in E(G)} f(xy) \) where \( f \) assigns distinct weights to the edges of a graph \( G \). The minimum number of distinct weights needed for a graph \( G \) to have a vertex local antimagic total labeling is denoted by \( \chi_{lat}(G) \). The minimum number of distinct weights needed for a graph \( G \) to have a vertex local antimagic labeling is denoted by \( \chi_{la}(G) \). Lau [5] adopts a result from Haslegrave [4] to show that every connected graph is a vertex local antimagic total graph. For more information on local antimagic or antimagic labelings, we refer the reader to Gallian’s survey [3].

Furthermore, Slamin et al. [8] introduced a new variant of the labeling. A super vertex local antimagic total labeling is a bijective map \( f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\} \) where \( f(V(G)) = \{1, 2, \ldots, |V(G)|\} \) such that \( w(u) \neq w(v) \) for any two adjacent vertices \( u \) and \( v \), where \( w(x) = f(x) + \sum_{xy \in E(G)} f(xy) \). The minimum number of distinct weights needed for a graph \( G \) to have a super vertex local antimagic labeling is denoted by \( \chi_{slat}(G) \). From the definition, we can perceive the super vertex local antimagic labeling as a vertex coloring of a graph with some additional conditions. An easy observation then follows.

Observation 1.1. For any graph \( G \), \( \chi_{slat}(G) \geq \chi(G) \).

We limit our current research to some classes of trees; in particular, stars \( S_n \), paths \( P_n \), caterpillars \( S_{n_1,n_2,\ldots,n_k} \) and shrubs \( \tilde{S}(n_1,n_2,\ldots,n_k) \). A shrub \( \tilde{S}(n_1,n_2,\ldots,n_k) \) is defined as a tree constructed from a star \( S_m \), every leaf of which is adjacent to some number of isolated vertices (see [6]).

Slamin et al. [8] proved the following. If \( T \) is a tree on \( n \geq 2 \) vertices with \( k \) leaves, then \( \chi_{slat}(T) \leq n - k + 1 \). For a star \( S_n \) and a double star \( S_{k,n-k} \), we have \( \chi_{slat}(S_{n+1}) = 2 \) and \( \chi_{slat}(S_{k,n-k}) = 3 \). In addition, if \( P_n \) is a path, \( \chi_{slat}(P_n) = 3 \) if \( n \) is odd and \( n \geq 5 \), or \( 3 \leq \chi_{slat}(P_n) \leq 4 \) if \( n \) is even and \( n \geq 6 \).

In this paper, we characterize trees \( T \) with \( \chi_{slat}(T) = 2 \), show existence of trees with \( \chi_{slat}(T) = 3 \), and construct trees \( T \) that have \( \chi_{slat}(T) = n \) for any positive integer \( n \geq 2 \).

2. Characterization of Trees with \( \chi_{slat}(T) = 2 \)

We start by determining the lower bound of \( \chi_{slat}(T) \). The following Lemma 2.1 shows sufficient condition for vertices having different weights based on their degrees.
Lemma 2.1. Let $T$ be a tree graph which has SLAT-labeling $f$ and $v_1, v_2 \in V(T)$. If $2 \deg(v_1) + 1 \leq \deg(v_2)$, then $w(v_1) < w(v_2)$.

Proof. Let $\deg(v_1) = d$ and $|V| = n$, so that $\deg(v_2) \geq 2d + 1$ and $|E| = n - 1$. By assigning $v_1$ and edges incident with $v_1$ labels such that the weight of $v_1$ is as large as possible, we have

$$w(v_1) \leq (d + 1)|V| + d|E| - \sum_{i=1}^{d} (i - 1)$$

$$w(v_1) \leq (d + 1)n + d(n - 1) - \frac{(d - 1)d}{2}$$

$$w(v_1) \leq 2dn + n - \frac{d^2 + d}{2}.$$ 

Then, by assigning $v_2$ and edges incident with $v_2$ labels such that the weight of $v_2$ is as small as possible, we have

$$w(v_2) \geq (2d + 1)|V| + \sum_{i=1}^{2d+1} (i + 1)$$

$$w(v_2) \geq (2d + 1)n + \frac{(2d + 1)(2d + 2)}{2} + 1$$

$$w(v_2) \geq 2dn + n + 2d^2 + 3d + 1.$$ 

It can be seen that $w(v_1) < w(v_2)$. \qed

The following special case where $v_1$ is a leaf will be useful.

Corollary 2.1. For an arbitrary tree, if $v_1$ is a leaf vertex and $v_2$ is a vertex with $\deg(v_2) \geq 3$, then $w(v_1) \neq w(v_2)$.

Based on [8], $\chi_{slat}(S_n) = 2$. We will show that stars are the only trees with $\chi_{slat}(T) = 2$. In our proof, we provide a labeling different from the one in [8].

Theorem 2.1. Suppose $T$ is a tree graph, then $\chi_{slat}(T) = 2$ if and only if $T \cong S_n$ for $n \in \mathbb{N}$.

Proof. Let $T \cong S_n$ for $n \in \mathbb{N}$, we will show that $\chi_{slat}(T) = 2$. By the fact that $\chi(T) = 2$ and Observation 1.1 we conclude that $\chi_{slat}(T) \geq 2$. To show $\chi_{slat}(T) \leq 2$, define $f : V(T) \cup E(T) \rightarrow \{1, 2, \ldots, |V(T)| + |E(T)|\}$ as follows:

$$f(c) = n + 1,$$

$$f(v_i) = i, 1 \leq i \leq n,$$

$$f(cv_i) = 2n + 2 - i, i \leq i \leq n.$$ 

From here, we get

$$w(v_i) = 2n + 2, 1 \leq i \leq n,$$

$$w(c) = \frac{3}{2}n^2 + \frac{5}{2}n + 1.$$ 

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Therefore, $\chi_{slat}(T) \leq 2$. We conclude that if $T \cong S_n$, then $\chi_{slat}(T) = 2$.

Now let $\chi_{slat}(T) = 2$, we will show that $T \cong S_n$.

Let the partition of $V(T)$ be $V_1, V_2$. Without loss of generality, let $x_0 \in V_1$ and $P = x_0, y_1, x_1, \ldots$ be a diametrical path. Then $x_0$ is of degree one. By Corollary 2.1, all vertices in $V_1$ are of degree at most two and therefore all vertices of $V_2$ belong to $P$. Denote by $p$ the number of leaves in $V_1$ and by $q$ the number of vertices of degree two. We want to show that $q = 0$.

Using this notation, we can see that $p = |V_1| - 2q - 1$.

Denote by $V_i$ the set of vertices of degree $i$ in $V_1$. Then we have $|V_1^1| = p$ and $|V_1^2| = q$. Denote $|V| = m$.

We know that all vertices in $V_1$ have the same weight, call it $w^*$. We first look at the $p$ vertices of degree one, observing that

$$\sum_{x_i \in V_1^1} w(x_i) = pw^*. \quad (1)$$

We also know that

$$\sum_{x_i \in V_1^1} w(x_i) = \sum_{x_i \in V_1^1} f(x_i) + \sum_{x_i \in V_1^1, x_i y_j \in E} f(x_i y_j) \leq \sum_{s=m-p+1}^m s + \sum_{t=2m-p}^{2m-1} t = \frac{(2m-p+1)p}{2} + \frac{(4m-p-1)p}{2} \quad (2)$$

Combining (1) and (2), we obtain

$$pw^* = \sum_{x_i \in V_1^1} w(x_i) \leq \frac{(2m-p+1)p}{2} + \frac{(4m-p-1)p}{2}, \quad (3)$$

which yields

$$w^* \leq \frac{(2m-p+1)}{2} + \frac{(4m-p-1)}{2} = 3m - p, \quad (4)$$

Now we look at the $q$ vertices of degree two, observing that

$$\sum_{x_i \in V_1^2} w(x_i) = qw^*. \quad (5)$$
We also know that
\[
\sum_{x_i \in V_1^2} w(x_i) = \sum_{x_i \in V_1^2} f(x_i) + \sum_{x_i \in V_1^2, x_i y_j \in E} f(x_i y_j) \\
\geq \sum_{s=1}^{q} s + \sum_{t=m+1}^{m+2q} t \\
= \frac{(q + 1)q}{2} + \frac{(2m + 2q + 1)(2q)}{2}
\]
Combining (5) and (6), we obtain
\[
qw^* = \sum_{x_i \in V_1^2} w(x_i) \geq \frac{(q + 1)q}{2} + \frac{(2m + 2q + 1)(2q)}{2},
\]
which for \(q > 0\) yields
\[
w^* \geq \frac{q + 1}{2} + (2m + 2q + 1) = 2m + \frac{5q + 3}{2}.
\]
We noted above that
\[
p = |V| - 2q - 1 = m - 2q - 1.
\]
Substituting (9) into (4), we have
\[
w^* \leq 3m - p = 3m - (m - 2q - 1) = 2m + 2q + 1.
\]
Now comparing (8) and (10), we get
\[
2m + \frac{5q + 3}{2} \leq w^* \leq 2m + 2q + 1,
\]
which is impossible for \(q > 0\). Hence, \(q = 0\). We already noticed that \(|V_2| = q + 1 = 1\), which implies that \(T\) must be the star \(S_p\).

In Figure 1, we give an example of SLAT labeling on \(S_8\).

Corollary 2.2. Suppose \(T\) is a non-trivial tree graph and \(S_n\) is a star graph. If \(T\) is not isomorphic to \(S_n\), then \(\chi_{slat}(T) \geq 3\).
3. Existence of Trees with $\chi_{slat}(T) = 3$

Slam et al. in [8] investigated paths $P_n$ and proved that $\chi_{slat}(T_n) = 3$ when $n$ is odd, and $3 \leq \chi_{slat}(T_n) \leq 4$ when $n$ is even. In Theorem 3.1, we present a more straightforward proof.

**Theorem 3.1.** Let $P_n$ be a path on $n$ vertices, $n \geq 4$. Then $\chi_{slat}(P_n) = 3$ when $n$ is odd or $n \in \{4, 6, 8, 10\}$ and $3 \leq \chi_{slat}(P_n) \leq 4$ when $n$ is even and $n \geq 12$.

**Proof.** Let $V(P_n) = \{v_i|1 \leq i \leq n\}$ and $E(P_n) = \{v_iv_{i+1}|1 \leq i \leq n-1\}$ with $n \in \mathbb{N}$. According to Corollary 2.2, graphs that are not isomorphic to a star have $\chi_{slat}(P_n) \geq 3$. To show the upper bound, the problem is divided into two cases, according to the parity of $n$.

**Case 1.** $n$ is odd

Define $f : V(P_n) \cup E(P_n) \rightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ as follows

$$f(v_i) = \begin{cases} 
2i - 1, & \text{if } i \in \{1, 2\}, \\
2, & \text{if } i = n, \\
n - i + 2, & \text{if } 3 \leq i \leq n - 2, i \text{ is odd}, \\
n - i + 4, & \text{if } 4 \leq i \leq n - 1, i \text{ is even}.
\end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 
2n - 1, & \text{if } i = 1, \\
n + \frac{i-1}{2}, & \text{if } 3 \leq i \leq n - 2, i \text{ is odd}, \\
\frac{3}{2}(n - 1) + \frac{i}{2}, & \text{if } 2 \leq i \leq n - 1, i \text{ is even}.
\end{cases}$$

Then we have the weights as follows.

$$w(v_i) = \begin{cases} 
2n, & \text{if } i \in \{1, n\}, \\
\frac{7}{2}n - \frac{1}{2}, & \text{if } 3 \leq i \leq n - 2, i \text{ is odd}, \\
\frac{7}{2}n + \frac{3}{2}, & \text{if } 4 \leq i \leq n - 1, i \text{ is even}.
\end{cases}$$

Therefore, $\chi_{slat}(P_n) \leq 3$.

**Case 2.** $n$ is even

Define $f : V(P_n) \cup E(P_n) \rightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ as follows.

$$f(v_i) = \begin{cases} 
n, & \text{if } i = 1, \\
n - 1, & \text{if } i = n, \\
n - i - 1, & \text{if } 2 \leq i \leq n - 2, i \text{ is even}, \\
n - i + 1, & \text{if } 3 \leq i \leq n - 1, i \text{ is odd}.
\end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 
n + \frac{i}{2}, & \text{if } 2 \leq i \leq n - 2, i \text{ is even}, \\
\frac{3}{2}n + \frac{i}{2}, & \text{if } 1 \leq i \leq n - 1, i \text{ is odd}.
\end{cases}$$

Then we have the weights as follows.

$$w(v_i) = \begin{cases} 
\frac{5}{2}n, & \text{if } i = 1, \\
3n - 2, & \text{if } i = n, \\
\frac{7}{2}n, & \text{if } 3 \leq i \leq n - 2, i \text{ is odd}, \\
\frac{7}{2}n - 2, & \text{if } 2 \leq i \leq n - 1, i \text{ is even}.
\end{cases}$$
We conclude that $\chi_{\text{slat}}(P_n) = 3$ when $n$ is odd or $n \{4, 6, 8, 10\}$, and $3 \leq \chi_{\text{slat}}(P_n) \leq 4$ when $n$ is even and $n \geq 12$.

In Figure 2 we present SLAT labelings of $P_4$, $P_6$, $P_8$, $P_{10}$ and also $P_7$ as an example for $n$ odd. The labeling is not unique. Here are some labelings for $P_6$, $P_8$, $P_{10}$. First bracket is vertex labels, second edges, third weights. The labelings for $P_6$, $P_8$, and $P_{10}$ were found by Branson [2].

\[
P_6\]
\[\{5, 3, 4, 1, 2, 6\}\{11, 9, 7, 8, 10\}\{16, 23, 20, 16, 20, 16\}\]

\[
P_8\]
\[\{8, 4, 1, 6, 5, 3, 2, 7\}\{14, 12, 13, 11, 10, 9, 15\}\{22, 30, 26, 30, 26, 22, 26, 22\}\]

\[
P_{10}\]
\[\{8, 1, 7, 3, 2, 5, 4, 9, 6, 10\}\{19, 14, 15, 16, 18, 11, 12, 13, 17\}\{27, 34, 36, 34, 36, 34, 27, 34, 36, 27\}\]

\[
\begin{align*}
P_6 & \quad \text{(a), } \chi_{\text{slat}}(P_6) = 3 \\
& \quad \text{(b), } \chi_{\text{slat}}(P_6) = 3 \\
& \quad \text{(c), } \chi_{\text{slat}}(P_6) = 4
\end{align*}
\]

Figure 2: SLAT labeling on $P_7$, $P_4$ and $P_6$.

Based on the labelings of the short even paths above, we state the following.

**Conjecture.** For any even $n \geq 4$, $\chi_{\text{slat}}(P_n) = 3$.

As a common generalization of caterpillars and shrubs, we introduce a new class of trees called shrubs. A shrub $\hat{S}(m, n, p)$ is defined by its vertex and edge set as follows.

\[
\begin{align*}
V(\hat{S}(m, n, p)) & = \{c, v_i, v_i^j, u_k | 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\} \\
E(\hat{S}(m, n, p)) & = \{cv_i, v_i v_i^j, cu_k | 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\}
\end{align*}
\]
When \( p = 0 \), then \( \hat{S}(m, n, p) \) is a regular shrub (all \( u_k \) vertices and \( cu_k \) edges are omitted). Else, if \( m \leq 2 \), then \( \hat{S}(m, n, p) \) is a caterpillar. However, when \( m = 0, n = 0 \), or \( m + p = 1 \), then \( \hat{S}(m, n, p) \) is a star. Since we already know that \( \chi_{\text{slat}}(T) = 2 \) for \( T \cong S_n \), the case of graph which is isomorphic to a star is omitted.

**Theorem 3.2.** Suppose \( \hat{S}(m, n, p) \) is a modified shrub. For positive \( m, n \), non-negative \( p \) and \( m + p \neq 1 \), \( \chi_{\text{slat}}(\hat{S}(m, n, p)) = 3 \).

**Proof.** Let \( \hat{S}(m, n, p) = \{c, v_i, v_i^j, u_k | 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\} \) and \( E(\hat{S}(m, n, p)) = \{cv_i, v_i^jv_i^j, cu_k | 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\} \). By Corollary 2.2, graphs other than stars have \( \chi_{\text{slat}}(\hat{S}(m, n, p)) \geq 3 \). To show the upper bound, the proof is divided into two cases.

**Case 1.** \( p + m \geq n + 1 \)

The case is divided into three subcases, according to the parity of \( n \) and \( m \).

**Subcase 1.1.** \( n \) is even

Define \( f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\} \) as follows

\[
f(u_k) = k, 1 \leq k \leq p,
\]

\[
f(v_i^j) = \begin{cases} 
  m(j - 1) + p + i, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd,} \\
  mj - i + p + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even.}
\end{cases}
\]

\[
f(v_i) = mn + p + i, 1 \leq i \leq m,
\]

\[
f(c) = m(n + 1) + p + 1,
\]

\[
f(v_iv_i^j) = \begin{cases} 
  m(2n - j + 1) + p + i + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even,} \\
  m(2n - j + 2) + p - i + 2, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd.}
\end{cases}
\]

\[
f(cv_i) = m(2n + 2) + 2p - i + 2, 1 \leq i \leq m,
\]

\[
f(cu_k) = m(2n + 1) + 2p - k + 2, 1 \leq k \leq p.
\]

When \( p = 0 \), then vertices \( v_k \) and edges \( cu_k \) are omitted.

We have

\[
w(u_k) = w(v_i^j) = m(2n + 1) + 2p + 2, 1 \leq k \leq p, 1 \leq i \leq m, 1 \leq j \leq n,
\]

\[
w(v_i) = m(n(2n + \frac{9}{2}) - \frac{n(n + 1)}{2} + 2) + p(n + 3) + \frac{3n}{2} + 2, 1 \leq i \leq m,
\]

\[
w(c) = m((2m + 1)(n + 1) + p(2n + 3) + 2) + \frac{p(3p + 5)}{2} - \frac{m(m + 1)}{2} + 1.
\]

It can be seen that these three weights are different.
Subcase 1.2. Both \( n \) and \( m \) are odd
Define \( f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\} \) as follows

\[
f(u_k) = k, 1 \leq k \leq p,
\]

\[
f(v_i^j) = \begin{cases} 
  m(j - 1) + p + i, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd,} \\
  m_j - i + p + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even.}
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
  mn + p + \frac{m+1}{2} - i + 1, & \text{if } 1 \leq i \leq \frac{m+1}{2}, \\
  m(n+1) + p + \frac{m+1}{2} - i + 1, & \text{if } \frac{m+3}{2} \leq i \leq m.
\end{cases}
\]

\[
f(c) = m(n+1) + p + 1,
\]

\[
f(v_i v_i^j) = \begin{cases} 
  m(2n - j + 1) + p + i + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even,} \\
  m(2n - j + 2) + p - i + 2, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd.}
\end{cases}
\]

\[
f(cv_i) = \begin{cases} 
  m(2n + 1) + 2p + 2i, & \text{if } 1 \leq i \leq \frac{m+1}{2}, \\
  2mn + 2p + 2i, & \text{if } \frac{m+3}{2} \leq i \leq m.
\end{cases}
\]

\[
f(cv_k) = m(2n + 1) + 2p - k + 2, 1 \leq k \leq p.
\]

When \( p = 0 \), then vertices \( v_k \) and edges \( cv_k \) are omitted.

We have

\[
w(u_k) = w(v_i^j) = m(2n + 1) + 2p + 2, 1 \leq k \leq p, 1 \leq i \leq m, 1 \leq j \leq n,
\]

\[
w(v_i) = m(2n(n + 2) + (1 - n)\frac{n+1}{2} + 1) + p(n + 3) + \frac{m+3n}{2} + 2, 1 \leq i \leq m,
\]

\[
w(c) = m((2m + 1)(n + 1) + p(2n + 3) + 2) + \frac{p(3p + 5)}{2} - \frac{m(m + 1)}{2} + 1.
\]

It can be seen that these three weights are different.

Subcase 1.3. \( n \) is odd and \( m \) is even
Define \( f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\} \) as follows

\[
f(u_k) = k, 1 \leq k \leq p,
\]

\[
f(v_i^j) = \begin{cases} 
  m(j - 1) + p + i, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd,} \\
  m_j - i + p + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even.}
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
  mn + p + \frac{i+1}{2} + 1, & \text{if } 1 \leq i \leq m, i \text{ is odd,} \\
  mn + p + \frac{m+i}{2} + 1, & \text{if } 1 \leq i \leq m, i \text{ is even.}
\end{cases}
\]

\[
f(c) = mn + p + \frac{m}{2} + 1,
\]

\[
f(v_i v_i^j) = \begin{cases} 
  m(2n - j + 1) + p + i + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even,} \\
  m(2n - j + 2) + p - i + 2, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd.}
\end{cases}
\]
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\[ f(cv_i) = \begin{cases} 
  m(2n + 1) + 2p + \frac{i}{2} + 1, & \text{if } 1 \leq i \leq m, \text{ } i \text{ is even,} \\
  m(2n + 1) + 2p + \frac{m+i+1}{2} + 1, & \text{if } 1 \leq i \leq m, \text{ } i \text{ is odd.}
\end{cases} \]

\[ f(cu_k) = m(2n + 1) + 2p - k + 2, 1 \leq k \leq p. \]

When \( p = 0 \), then vertices \( v_k \) and edges \( cv_k \) are omitted.

We have

\[ w(u_k) = w(v^j_i) = m(2n + 1) + 2p + 2, 1 \leq k \leq p, 1 \leq i \leq m, 1 \leq j \leq n, \]

\[ w(v_i) = m(2n(n+2) + (1-n)\frac{n+1}{2} + 1) + p(n+3) + \frac{m+3n+1}{2} + 2, 1 \leq i \leq m, \]

\[ w(c) = m((2m+1)(n+1) + p(2n+3) + \frac{3}{2}) + \frac{p(3p+5)}{2} - \frac{m(m+1)}{2} + 1. \]

It can be seen that these three weights are different.

**Case 2.** \( p + m < n + 1 \)

The case is divided into three subcases according to the parity of \( n \) and \( m \).

**Subcase 2.1.** \( n \) is even

Define \( f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\} \) as follows.

\[ f(u_k) = mn + k, 1 \leq k \leq p, \]

\[ f(v^j_i) = \begin{cases} 
  m(j - 1) + i, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, \text{ } j \text{ is odd,} \\
  mj - i + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, \text{ } j \text{ is even.}
\end{cases} \]

\[ f(v_i) = mn + p + i + 1, 1 \leq i \leq m, \]

\[ f(c) = mn + p + 1, \]

\[ f(v_i v^j_i) = \begin{cases} 
  m(2n - j + 2) + 2p + i + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, \text{ } j \text{ is even,} \\
  m(2n - j + 3) + 2p - i + 2, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, \text{ } j \text{ is odd.}
\end{cases} \]

\[ f(cv_i) = m(n + 2) + p - i + 2, 1 \leq i \leq m, \]

\[ f(cu_k) = m(n + 2) + 2p - k + 2, 1 \leq k \leq p. \]

When \( p = 0 \), then vertices \( v_k \) and edges \( cv_k \) are omitted.

We have

\[ w(u_k) = w(v^j_i) = m(2n + 2) + 2p + 2, 1 \leq k \leq p, 1 \leq i \leq m, 1 \leq j \leq n, \]

\[ w(v_i) = m(n(2n + \frac{9}{2}) - \frac{n(n+1)}{2} + 2) + 2p(n+1) + \frac{3n}{2} + 3, 1 \leq i \leq m, \]

\[ w(c) = m((m + 2)(n + 2) + p) + \frac{p(3p+5)}{2} - \frac{m(m+1)}{2} + 1. \]

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It can be seen that these three weights are different.

**Subcase 2.2.** Both \( n \) and \( m \) are odd

Define \( f : V \cup E \to \{1, 2, 3, \ldots, |V| + |E| \} \) as follows.

\[
f(u_k) = mn + k, 1 \leq k \leq p,
\]

\[
f(v_i) = \begin{cases} 
  m(j - 1) + i, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}, \\
  mj - i + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}.
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
  mn + p + \frac{m+1}{2} - i + 2, & \text{if } 1 \leq i \leq \frac{m+1}{2}, \\
  m(n + 1) + p + \frac{m+1}{2} - i + 2, & \text{if } \frac{m+3}{2} \leq i \leq m.
\end{cases}
\]

\[
f(c) = mn + p + 1,
\]

\[
f(v_i v_i) = \begin{cases} 
  m(2n - j + 2) + 2p + i + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}, \\
  m(2n - j + 3) + 2p - i + 2, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}.
\end{cases}
\]

\[
f(cv_i) = \begin{cases} 
  m(n + 1) + p + 2i, & \text{if } 1 \leq i \leq \frac{m+1}{2}, \\
  mn + p + 2i, & \text{if } \frac{m+3}{2} \leq i \leq m.
\end{cases}
\]

\[
f(cv_k) = mn + p + 1,
\]

\[
f(ck) = m(2n + 2) + 2p - k + 2, 1 \leq k \leq p.
\]

When \( p = 0 \), then vertices \( v_k \) and edges \( cv_k \) are omitted.

We have

\[
w(u_k) = w(v_i^j) = m(2n + 2) + 2p + 2, 1 \leq k \leq p, 1 \leq i \leq m, 1 \leq j \leq n,
\]

\[
w(v_i) = m(2n(n + 2) + (1 - n)n + \frac{n+1}{2} + 1) + 2p(n + 1) + \frac{m + 3n}{2} + 3, 1 \leq i \leq m,
\]

\[
w(c) = m((m + 1)(n + \frac{3}{2}) + p(n + 3)) + \frac{p(3p + 5)}{2} + 1.
\]

It can be seen that these three weights are different.

**Subcase 2.3.** \( n \) is odd and \( m \) is even

Define \( f : V \cup E \to \{1, 2, 3, \ldots, |V| + |E| \} \) as follows.

\[
f(u_k) = mn + k, 1 \leq k \leq p,
\]

\[
f(v_i^j) = \begin{cases} 
  m(j - 1) + i, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}, \\
  mj - i + 1, & \text{if } 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}.
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
  mn + p + \frac{i+1}{2}, & \text{if } 1 \leq i \leq m, i \text{ is odd}, \\
  mn + p + \frac{m+i}{2} + 1, & \text{if } 1 \leq i \leq m, i \text{ is even}.
\end{cases}
\]

\[
f(c) = mn + p + \frac{m}{2} + 1,
\]
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\[ f(v_iv_j) = \begin{cases} 
  m(2n - j + 2) + 2p + i + 1, & \text{if} \ 1 \leq i \leq m, 1 \leq j \leq n, \ j \text{ is even}, \\
  m(2n - j + 3) + 2p - i + 2, & \text{if} \ 1 \leq i \leq m, 1 \leq j \leq n, \ j \text{ is odd}.
\]

\[ f(cv_i) = \begin{cases} 
  m(n + 1) + p + \frac{i}{2} + 1, & \text{if} \ 1 \leq i \leq m, \ i \text{ is even}, \\
  m(n + 1) + p + \frac{m + i + 1}{2} + 1, & \text{if} \ 1 \leq i \leq m, \ i \text{ is odd}.
\]

\[ f(cu_k) = m(2n + 1) + 2p - k + 2, \ 1 \leq k \leq p. \]

When \( p = 0 \), then vertices \( v_k \) and edges \( cv_k \) are omitted.

We have

\[ w(u_k) = w(v_i) = m(2n + 2) + 2p + 2, \ 1 \leq k \leq p, 1 \leq i \leq m, 1 \leq j \leq n, \]

\[ w(v_i) = m(2n(n + 2) + (1 - n)\frac{n + 1}{2} + 1) + 2p(n + 1) + \frac{m + 3n + 1}{2} + 2, \ 1 \leq i \leq m, \]

\[ w(c) = m((m + 1)(n + \frac{3}{2}) + p(2n + 3) + \frac{1}{2}) + \frac{p(3p + 5)}{2} + 1. \]

It can be seen that these three weights are different.

From the above cases, we can conclude that \( \chi_{slat}(S'(m, n, p)) \leq 3 \). Hence, \( \chi_{slat}(S'(m, n, p)) = 3 \).

In Figure 3, we have examples of two cases in the preceding theorem.

![Figure 3: SLAT labeling on \( T \), \( \chi_{slat}(T) = 3 \).](image)

Corollary 3.1. If a tree \( T \) is isomorphic to a regular shrub \( S(n, n, \ldots, n) \) or a caterpillar \( S_{n_1, n_2} \) or \( S_{n_1, n_2, n_3} \), then \( \chi_{slat}(T) = 3 \).

4. Construction of Trees \( T \) with \( \chi_{slat}(T) = n \) for any \( n \in \mathbb{N} \)

Motivated by the fact that for any tree (an in fact for any bipartite graph) the regular chromatic number \( \chi(T) = 2 \), it is natural to ask whether there exists \( k \in \mathbb{N} \) such that for every tree \( T \), \( \chi_{slat}(T) \leq k \). In the following theorem we show that no such bound exists.
Theorem 4.1. For every \( n \geq 2 \), there exists a tree \( T \) such that \( \chi_{slat}(T) = n \).

Proof. The assertion for \( n = 2 \) follows from Theorem 2.1 and for \( n = 3 \) from Theorem 3.1. Therefore, we only construct examples for \( n \geq 4 \).

We construct a tree \( T \) starting with the path \( P_{n+1} \). For every \( i = 2, 3, \ldots, n \) we define \( t_i = \lfloor \frac{i}{2} \rfloor + 1 \) and join vertex \( v_i \) to \( 2^{t_i} - 3 \) isolated vertices. From this construction, we obtain \( \deg(v_i) = 2^{t_i} - 1 \), for \( 2 \leq i \leq n \).

First, we need to show that \( \chi_{slat}(T) \geq n \). According to the definition of SLAT-labeling, adjacent vertices must have different weights, therefore \( w(v_i) \neq w(v_{i+1}) \) for \( 1 \leq i \leq n \).

By the graph construction, for any \( 1 \leq i, j \leq n \) such that \( j \geq i + 2 \) the vertices \( v_i, v_j \) are non-adjacent and satisfy the condition \( 2 \deg(v_i) + 1 \leq \deg(v_j) \). It then follows from Lemma 2.1 that \( w(v_i) \neq w(v_j) \). In addition, it follows from Corollary 2.1 that the weights of vertices of degree at least three are all greater than the weights of all leaves. Thus, the graph needs at least \( n \) distinct weights, which means \( \chi_{slat}(T) \geq n \).

To show \( \chi_{slat}(T) \leq n \), we define a labeling \( f \) as follows. For \( i = 2, 3, \ldots, n \) and \( l = 1, 2, \ldots, t_i \) we denote by \( e_{i,l} \) the pendant edges incident with vertex \( v_i \) and by \( v_{i,l} \) the leaf incident with \( e_{i,l} \). First we label edge \( v_1 v_2 \) with label \( |V| + 1 \). Then we label the remaining pendant edges starting with the lowest available edge label \( |V| + 2 \) in lexicographic order; that is, \( f(e_{i,l}) < f(e_{i,s}) \) for any \( 1 \leq l < s \leq t_i \) and \( f(e_{i,l}) < f(i, s) \) for any \( 2 \leq i < j \leq n \) and any \( l \) and \( s \). Next, label the leaf incident with an edge \( e_{i,l} \) (or \( v_1 v_2 \)) so that the sum of the edge and vertex label equals \( 2|V| - n + 2 \).

Then, label the vertices \( v_2, v_3, \ldots, v_n \) starting from \( f(v_2) = |V| - n + 2 \) consecutively in increasing order. Finally, label the remaining edges starting from \( f(v_2v_3) = 2|V| - n - 2 \) consecutively in increasing order. From this labeling, we have \( w(v_{i,l}) = 2|V| - n + 2 \) for every leaf vertex \( v_{i,l} \), while \( 2|V| - n + 2 < w(v_i) \leq w(v_j) \) for \( 2 \leq i < j \leq n \). Hence, \( \chi_{slat}(T) \leq n \).

We can conclude that \( \chi_{slat}(T) = n \).

5. Open Problems

To conclude, we state some obvious open problems.

1. Characterize trees with \( \chi_{slat}(T) = 3 \).
2. Determine \( \chi_{slat}(G) \) for other natural classes of graphs.

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References


