A bound on connectivity of iterated line graphs

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Abstract

For simple connected graphs that are neither paths nor cycles, we define $l(G) = \max\{m : G$ has a divalent path of length $m$ that is not both of length 2 and in a $K_3\}$, where a divalent path is a path whose internal vertices have degree two in $G$. Let $G$ be a graph and $L^n(G)$ be its $n$-th iterated line graph of $G$. We use $\kappa'_e(G)$ and $\kappa(G)$ for the essential edge connectivity and vertex connectivity of $G$, respectively. Let $G$ be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l \geq 1$. We prove that (i) for integers $s \geq 1$, $\kappa'_e(L^{l+s}(G)) \geq 2^s + 2$; (ii) for integers $s \geq 2$, $\kappa(L^{l+s}(G)) \geq 2^{s-1} + 2$. The bounds are best possible.

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1. Introduction

We use [1] for terminology and notation not defined here, and consider finite and simple graphs only. Let $A, B$ be two vertex sets in a graph $G$, and $[A, B]$ denote the set of edges with one end in $A$ and the other in $B$. In particular, $\kappa(G)$ and $\kappa'(G)$ represent the connectivity and edge-connectivity of a graph $G$, respectively. A graph is trivial if it contains no edges. An edge cut $Y$ of $G$ is essential if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a graph $G$ is essentially $k$-edge-connected if $G$ does not have an essential edge cut $Y$ with $|Y| < k$. We use $\kappa'_e(G)$ to

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denote the essential edge connectivity of a graph $G$. A degree sum property follows immediately from the definition of essential edge connectivity.

**Proposition 1.1.** (Shao, Proposition 2.1 of [4]) Let $n \geq 1$ be an integer and $G$ be a graph which is not $K_{1,n-1}$ or $K_3$. Then the degree sum of any two adjacent vertices is at least $\kappa'_e(G) + 2$.

The line graph of a graph $G$, denoted by $L(G)$ or $L^1(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding two edges in $G$ have a common vertex. Iteratively, $L^n(G) = L(L^{n-1}(G))$ and $L^0(G) = G$ for integers $n \geq 1$. The following connectivity properties can be derived from the definitions.

**Proposition 1.2.** (Shao, Proposition 1.2 of [4]) Let $n \geq 1$ be an integer and $G$ be a graph which is not $K_3$ or $K_{1,n-1}$. Then each of the following holds.

(i) $\kappa'_e(G) \geq \kappa'(G)$.
(ii) $\kappa'_e(G) = \kappa(L(G))$.
(iii) $\kappa'_e(L(G)) \geq \kappa'_e(G)$.
(iv) $\kappa(L(G)) \geq \kappa(G)$.

Proposition 1.2(ii) indicates that it is useful to investigate properties of essential edge connectivity in order to obtain properties of vertex connectivity in line graphs. Here is a result on how essential edge connectivity grows for iterated line graphs.

**Theorem 1.1.** (Shao, Theorem 1.3 of [4]) Let $n \geq 1$ be an integer and $G$ be a graph which is not $K_{1,n-1}$ or $K_3$. If $G$ does not contain degree two vertices, then $\kappa'_e(L(G)) \geq 2\kappa'_e(G) - 2$.

For simple connected graphs that are neither paths nor cycles, we define $l(G) = \max\{m : G$ has a divalent path of length $m$ that is not both of length 2 and in a $K_3\}$, where a divalent path is a path whose internal vertices have degree two in $G$. We call $l(G)$ the divalent length of $G$.

For the graph $G$ depicted in Figure 1, $\kappa'_e(L(G)) = \kappa'_e(G) = 3$; the one in Figure 1, $\kappa'_e(L(G)) = \kappa'_e(G) = 1$. In many cases as these graphs depicted in Figures 0 and 1, when $G$ has a divalent path with internal vertices of degree two, the connectivity of the line graphs stays the same. Since each iteration will reduce the lengths of all divalent paths by one, after $l(G) - 1$ times of iterations, the internal vertices of degree 2 either disappear or lie in a triangle, and the connectivity is likely to increase.

![Figure 1](image.png)

Figure 1. An example for $\kappa'_e(L(G)) = \kappa'_e(G)$.

A former result gives a bound on the essential edge connectivity of the $(l + 1)$-th iterated line graphs.
**Proposition 1.3.** (Zhang et al., Lemma 2.5 of [6]) Let $G$ be a connected graph with $l(G) = l$. Then $\kappa'_e(L^{l+1}(G)) \geq 4$.

In this note, we obtain the bounds for essential edge connectivity and connectivity of iterated line graphs.

**Theorem 1.2.** Let $G$ be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l \geq 1$. Then each of the following holds:

(i) For integers $s \geq 1$, $\kappa'_e(L^{l+s}(G)) \geq 2s + 2$. The bound is best possible.

(ii) For integers $s \geq 2$, $\kappa(L^{l+s}(G)) \geq 2s - 1 + 2$. The bound is best possible.

Note that (i) does not hold for $s = 0$ and (ii) does not hold for $s = 1$ as there exist graphs with $\kappa'_e(L^1(G)) = \kappa(L^{l+1}(G)) = 2$. As shown in Figure 2(a), $l(G) = 2$ and in Figure 2(c), $\kappa'_e(L^2(G)) = 2$. By Proposition 1.2(ii), $\kappa(L^3(G)) = \kappa'_e(L^2(G)) = 2$.

![Figure 2](image-url)

Figure 2. An example to show (i) does not hold for $s = 0$.

2. Previous Results

For a graph $G$ and a vertex $v \in V(G)$, define

$$N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$$

and

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

Let $L(G)$ be the line graph of $G$, and

$X$ be a minimal essential edge cut of $L(G)$.

By (1), $L(G) - X$ has exactly two nontrivial components, so we assume

$L_1, L_2$ are the only two nontrivial components of $L(G) - X$. (2)

By (2), $V(L(G))$ is a disjoint union of $V(L_1)$ and $V(L_2)$. By the definition of line graphs, $E(G)$ is a disjoint union of two edge sets corresponding to $V(L_1)$ and $V(L_2)$. For each edge $e \in E(G)$, let $v_e \in V(L(G))$ denote its corresponding vertex in the line graph $L(G)$. Let $f : E(G) \mapsto \{1, 2\}$ be a 2-edge-coloring of $G$ such that

$$f(e) = i \text{ if and only if } v_e \in V(L_i) \text{ for } i = 1, 2.$$
Proposition 2.1. (Shao, [3]) Let $G$ be a graph, $e_1, e_2 \in E(G)$ and $v_{e_1}, v_{e_2} \in V(L(G))$ be their corresponding vertices in the line graph $L(G)$. Then each of the following holds:

(i) For $i = 1, 2$, $|\{e \in E(G) : f(e) = i\}| = |V(L_i)| \geq 2$.

(ii) $v_{e_1}v_{e_2} \in X$ if and only if $e_1, e_2$ share a common vertex and $f(e_1) \neq f(e_2)$.

A vertex of a graph $G$ is mono-colored if all incident edges have the same color in $G$, and a vertex is bi-colored if the incident edges have two different colors.

Proposition 2.2. (Shao, [3]) Let $G$ be a graph, $V_{12}$ the set of all bi-colored vertices of $G$, or equivalently,

$$V_{12} = \{v \in V(G) : \text{there exist } e_1, e_2 \in E_G(v) \text{ such that } f(e_1) = 1 \text{ and } f(e_2) = 2\}. \quad (3)$$

Then each of the following holds.

(i) $d_G(v) \geq 2$ for each $v \in V_{12}$.

(ii) Each vertex of $G - V_{12}$ is mono-colored in $G$, and moreover, for each component $H$ of $G - V_{12}$, all edges with at least one end in $H$ have the same color as the edges of $H$.

(iii) If $1 \leq |V_{12}| \leq 3$ and $V_{12}$ is not a vertex cut, then the subgraph of $G$ induced by $V_{12}$, $G[V_{12}]$, is connected.

(iv) If $|V_{12}| = 1$ or 2, then $V_{12}$ is a vertex cut of $G$.

Let $u \in V(G)$ and

$$E_i(u) = \{e \in E_G(u) : f(e) = i\} \text{ for } i = 1, 2. \quad (4)$$

Proposition 2.3. (Shao, Proposition 2.4 of [4]) Each of the following holds:

(i) $|X| = \sum_{u \in V_{12}}|E_1(u)| \cdot |E_2(u)|$.

(ii) For each $u \in V_{12}$ and $i = 1, 2$, $|E_1(u)| \cdot |E_2(u)| \geq d_G(u) - 1$.

(iii) If for each $u \in V_{12}$ and $i = 1, 2$, $|E_i(u)| \geq 2$, then $|E_1(u)| \cdot |E_2(u)| \geq 2(d_G(u) - 2)$.

A graph $G$ is $k$-triangular if each edge of $G$ lies in at least $k$ triangles and $G$ is triangular if it is 1-triangular. We summarize some properties of iterated line graphs of $G$ in terms of $l(G)$ as follows.

Proposition 2.4. (Zhang, Eschen, Lai and Shao, Lemma 3.2 of [5]) Let $G$ be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l \geq 1$. Then each of the following holds:

(i) For an integer $m \geq 0$,

$$l(L^m(G)) = \begin{cases} 
  l - m, & \text{if } 0 \leq m < l, \\
  1, & \text{if } m \geq l.
\end{cases}$$

(ii) For integers $k \geq 0$,

$$\delta(L^{l+k}(G)) \geq \begin{cases} 
  2, & \text{if } k = 0 \text{ or } k = 1, \\
  2k - 2 + 2, & \text{if } k \geq 2.
\end{cases}$$

(iii) $L^1(G)$, $L^{l+1}(G)$ and $L^{l+2}(G)$ are triangular. Moreover, $L^{l+k}(G)$ is $2k-3$-triangular when $k \geq 3$.

(iv) For an integer $k \geq 0$, $\kappa(L^{l+k}(G)) \geq k + 1$. 

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3. Proof of Theorem 1.2

In this section, we first prove Proposition 3.1, then apply induction on \( s \) to prove Theorem 1.2. Following the definitions of \( X, V_{12} \) and \( E_i(u) \) (see (1), (3) and (4)) in Section 2, let \( X \) be a minimal essential edge cut of \( L^{l+2}(G) \) and \( V_{12} \) the set of bi-colored vertices in \( L^{l+1}(G) \). And \( E_i(u) \) denotes the set of edges incident with \( u \) in \( L^{l+1}(G) \) with color \( i \). We use \( d(v) \) for \( d_{L^{l+1}(G)}(v) \). Now we prove Theorem 1.2(i) holds for \( s = 2 \).

**Proposition 3.1.** Let \( G \) be a simple connected graph that is not a path, a cycle or \( K_{1,3} \), with \( l(G) = l \). Then \( \kappa'_e(L^{l+2}(G)) \geq 6 \).

**Proof.** If \( \delta(L^{l+1}(G)) \geq 3 \), then by Theorem 1.1 and Proposition 1.3, \( \kappa'_e(L^{l+2}(G)) \geq 2\kappa'_e(L^{l+1}(G)) - 2 = 2 \cdot 4 - 2 = 6 \). Then Proposition 3.1 is proved.

Also, by Proposition 2.4(ii),

\[
\delta(L^{l+1}(G)) \geq 2. \tag{5}
\]

So we may assume that \( \delta(L^{l+1}(G)) = 2 \).

By Proposition 2.4(iv), \( \kappa(L^{l+1}(G)) \geq 2 \). By Proposition 2.2(iv), if \( |V_{12}| = 1 \), then \( V_{12} \) is a vertex cut of size one in \( L^{l+1}(G) \), a contradiction. So \( |V_{12}| \geq 2 \).

**Claim 1.** If \( |V_{12}| = 2 \), then \( |X| \geq 6 \).

**Proof of Claim 1.** Let \( V_{12} = \{v_1, v_2\} \). If \( v_1v_2 \notin E(L^{l+1}(G)) \), then \( L^{l+1}(G) \) is triangular (Proposition 2.4(iii)), and \( v_i \) is bi-colored, \( |E_i(v_i)| \geq 2 \) for \( i = 1, 2 \) by Proposition 2.2(ii). By Proposition 2.3(i) and (iii), \( |X| \geq \sum_{v \in V_{12}} (d(v) - 2) \geq 2 \cdot 2 + 2 \cdot 2 = 8 \).

So we may assume \( v_1v_2 \in E(L^{l+1}(G)) \). By Proposition 2.2(iv), \( V_{12} \) is a vertex cut of \( L^{l+1}(G) \). So we will finish the proof of Claim 1 by the following three cases.

**Case 1 of Claim 1.** \( L^{l+1}(G) - V_{12} \) has at least two nontrivial components.

**Proof of Case 1.** Let \( G_1, G_2 \) be two nontrivial components of \( L^{l+1}(G) - V_{12} \). By Proposition 1.3, \( \kappa'_e(L^{l+1}(G)) \geq 4 \). Then \( ||V(G_i), V_{12}|| \geq 4 \) for \( i = 1, 2 \), which implies \( d(v_1) + d(v_2) \geq 4 + 4 + 2 = 10 \). By Proposition 2.3(i) and (ii), \( |X| \geq (d(v_1) - 1) + (d(v_2) - 1) = d(v_1) + d(v_2) - 2 \geq 10 - 2 = 8 \).

**Case 2 of Claim 1.** \( L^{l+1}(G) - V_{12} \) has at most one nontrivial component.

**Proof of Case 2.** Let \( G_1 \) be the nontrivial component and \( G_2 \) be a single vertex component (say \( u \)) of \( L^{l+1}(G) - V_{12} \). Since \( \kappa'_e(L^{l+1}(G)) \geq 4 \) (Proposition 1.3), \( ||V(G_1), V_{12}|| \geq 4 \). Since \( u \) is a single vertex component of \( L^{l+1}(G) - V_{12} \), by (5), we have \( u v_1 \in E(L^{l+1}(G)) \) and \( u v_2 \in E(L^{l+1}(G)) \). So \( d(v_1) + d(v_2) \geq 4 + 4 + 2 = 8 \). By Proposition 2.3(i) and (ii), \( |X| \geq (d(v_1) - 1) + (d(v_2) - 1) = d(v_1) + d(v_2) - 2 \geq 8 - 2 = 6 \).

**Case 3 of Claim 1.** Every component of \( L^{l+1}(G) - V_{12} \) is trivial.

Then \( L^{l+1}(G) \) is isomorphic to \( K_{2n-2} \) with an edge joining the two vertices of degree two, where \( n \) is the number of vertices of \( L^{l+1}(G) \). If \( n = 4 \), then it has an essential 3-edge-cut, contrary to Proposition 1.3; if \( n \geq 4 \), then it contains an induced claw, contrary to the fact that line graphs are claw-free (see [2]).

**Claim 2.** If \( |V_{12}| \geq 6 \), then \( |X| \geq 6 \).
Proof of Claim 2. If \( |V_{12}| \geq 6 \), then by (5) and Proposition 2.3(i) and (ii), \( |X| \geq \sum_{i=1}^{3} |d(v_i) - 1| \geq (2 - 1)|V_{12}| = 6. \)

By Claims 1 and 2, we may assume that

\[
3 \leq |V_{12}| \leq 5. \tag{6}
\]

Claim 3. If the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) contains at least two independent edges, then \( |X| \geq 8. \)

Proof of Claim 3. We assume that \( v_1, v_2, v_3, v_4 \in V_{12} \) such that \( v_1v_2 \in E(L^{i+1}(G)) \) and \( v_3v_4 \in E(L^{i+1}(G)). \) By Proposition 1.1, \( d(v_1) + d(v_2) \geq \kappa_e(L^{i+1}(G)) + 2 \) and \( d(v_3) + d(v_4) \geq \kappa_e(L^{i+1}(G)) + 2. \) By Proposition 2.3(i), (ii) and Proposition 1.3, \( |X| \geq \sum_{i=1}^{4} |d(v_i) - 1| = \sum_{i=1}^{4} d(v_i) - 4 \geq 2(\kappa_e(L^{i+1}(G)) + 2) - 4 = 2 \cdot (4 + 2) - 4 = 8. \) This completes the proof of Claim 3.

By Claim 3, we may assume that

the matching number of the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) is at most one. \tag{7}

Claim 4. If the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) has an isolated vertex, then \( |X| \geq 6. \)

Proof of Claim 4. Without loss of generality, we assume \( v_1 \) is an isolated vertex in the induced graph of \( V_{12} \) in \( L^{i+1}(G). \) Since \( L^{i+1}(G) \) is triangular (Proposition 2.4(iii)), and \( v_1 \) is bi-colored, \( |E_i(v_1)| \geq 2 \) for \( i = 1, 2 \) by Proposition 2.2(ii). By (6), \( |V_{12}| \geq 3. \) Together with Proposition 2.3(i), (ii), (iii) and (5), we have \( |X| = \sum_{u \in V_{12}} |E_1(u)| \cdot |E_2(u)| \geq 2(d(v_1) - 2) + (2 - 1) \cdot (|V_{12}| - 1) \geq 2 \cdot 2 + (3 - 1) = 6. \)

By Claim 4, we may assume

the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) has no isolated vertices. \tag{8}

Claim 5. The induced graph of \( V_{12} \) in \( L^{i+1}(G) \) is isomorphic to \( K_3 \) or \( K_{1,2}. \)

Proof of Claim 5. By (7) and (8), the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) is isomorphic to \( K_3 \) or \( K_{1,|V_{12}| - 1}. \) If \( 4 \leq |V_{12}| \leq 5, \) then \( K_{1,|V_{12}| - 1} \) contains a claw, contrary to the fact that \( L^{i+1}(G) \) is claw-free. So by (6), \( |V_{12}| = 3 \) and by (7) and (8), the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) is isomorphic to \( K_3 \) or \( K_{1,2}. \)

Claim 6. If each vertex in the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) has degree at least three, then \( |X| \geq 6. \)

Proof of Claim 6. Let \( V_{12} = \{v_1, v_2, v_3\} \) and assume \( d(v_i) \geq 3 \) for \( i = 1, 2, 3. \) By Proposition 2.3(i), (ii) and (6), \( |X| \geq \sum_{i=1}^{3} (d(v_i) - 1) \geq (3 - 1)|V_{12}| \geq 6. \)

By Claim 6, we assume

at least one vertex in the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) has degree two. \tag{9}

Now we will finish the proof of Theorem 1.2 by the following two cases.

Case 1. \( L^{i+1}(G) - V_{12} \) is connected.

Proof of Case 1. Let \( G' \) be the graph induced by \( L^{i+1}(G) - V_{12}. \) By Proposition 2.2(ii), we assume all edges in \( [V(G'), V_{12}] \) have color 1. As \( v_1, v_2, v_3 \) are bi-colored, if the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) is \( K_{1,2}, \) then both edges of \( K_{1,2} \) must have color 2; if the induced graph of \( V_{12} \) in \( L^{i+1}(G) \) is \( K_3, \) then by Proposition 2.1(i), at least two edges of \( K_3 \) have color 2. In
either case, without loss of generality we may assume that both \(v_1v_2\) and \(v_2v_3\) have color 2, which means \(|E_2(v_2)| \geq 2\). Since \(v_2\) is bi-colored, there is an edge in \([V(G'), \{v_2\}]\) with color 1, which implies that the degree of \(v_2\) is at least three. By (9), we may assume \(d(v_1) = 2\), then \(d(v_2) \geq 4\) by Propositions 1.1 and 1.3. Then \(|V(G'), \{v_2\}| \geq 2\) and so \(|E_1(v_2)| \geq 2\). Together with \(|E_2(v_2)| \geq 2\), by Proposition 2.3(i), (ii) and (iii), \(|X| \geq 2(d(v_2) - 2) + (d(v_1) - 1) + (d(v_3) - 1) \geq 2 \cdot 2 + (2 - 1) + (2 - 1) = 6\). \(\Box\)

**Case 2.** \(L^{l+1}(G) - V_{12}\) is disconnected.

By Claim 5, we only need to prove Case 2 by the following two cases.

**Case 2.1.** The induced graph of \(V_{12}\) in \(L^{l+1}(G)\) is isomorphic to \(K_3\).

**Proof of Case 2.1.** By (9), without loss of generality, we assume that \(d(v_2) = 2\), then by Propositions 1.1 and 1.3, \(d(v_1) \geq 4\) and \(d(v_3) \geq 4\). By Proposition 2.3(i) and (ii), \(|X| \geq (d(v_1) - 1) + (d(v_2) - 1) + (d(v_3) - 1) \geq (4 - 1) + (2 - 1) + (4 - 1) = 7\). \(\Box\)

**Case 2.2.** The induced graph of \(V_{12}\) in \(L^{l+1}(G)\) is isomorphic to \(K_{1,2}\).

**Proof of Case 2.2.** Assume \(v_1v_2 \in E(L^{l+1}(G))\) and \(v_2v_3 \in E(L^{l+1}(G))\). If \(d(v_2) = 2\), then by Propositions 1.1 and 1.3, \(d(v_1) \geq 4\) and \(d(v_3) \geq 4\). Using a similar argument to Case 2.1, we have \(|X| \geq 7\).

By (9), we may assume that \(d(v_1) = 2\).

Then by Propositions 1.1 and 1.3 again, \(d(v_2) \geq 4\). If \(d(v_3) \geq 3\), then \(|X| \geq (d(v_1) - 1) + (d(v_2) - 1) + (d(v_3) - 1) \geq (2 - 1) + (4 - 1) + (3 - 1) = 6\). So we assume \(d(v_3) = 2\).

If \(d(v_2) \geq 5\), then \(|X| \geq (d(v_1) - 1) + (d(v_2) - 1) + (d(v_3) - 1) \geq (2 - 1) + (5 - 1) + (2 - 1) = 6\).

So we assume \(d(v_2) = 4\).

Now we have \(d(v_1) = d(v_3) = 2\) and \(d(v_2) = 4\). Since \(L^{l+1}(G)\) is triangular, we assume \(v_1v_2, v_2v_3\) lie in triangles \(T_1, T_2\), respectively. If \(T_1, T_2\) share a common edge, then \(L^{l+1}(G)\) must be the graph depicted in Figure 3(a), where \(H\) represents a connected subgraph containing the edge \(u_1u_2\); if \(T_1, T_2\) do not share a common edge, then \(L^{l+1}(G)\) must be the graphs depicted in Figure 3(b) and 3(c), where \(H\) in 2(b) represents a connected subgraph containing the vertices \(u_1, u_2\) and two circles in 2(c) represent either two isolated vertices \(u_1, u_2\) or two disjoint connected subgraphs containing \(u_1, u_2\), respectively. In Figure 3(a) and 3(b), since \(H\) is a connected subgraph, \(L^{l+1}(G) - V_{12}\) is connected, contrary to the assumption of Case 2; in Figure 3(c), \(L^{l+1}(G)\) has an essential 2-edge-cut, contrary to Proposition 1.3.

\[
\text{Figure 3. Iterated Line graph } L^{l+1}(G).
\]

This completes Case 2.2. \(\Box\)
Hence, Proposition 3.1 is established.

Next we will use Proposition 3.1 to prove Theorem 1.2.

**Proof of Theorem 1.2(i).** The case $s = 1$ is implied by Proposition 1.3 and the case $s = 2$ is implied by Proposition 3.1. We assume $s \geq 3$.

By Proposition 2.4(ii), for $s \geq 3$, $\delta(L^{l+s-1}(G)) \geq 3$. Now we prove (i) by induction. Assume that $\kappa'_e(L^{l+s-1}(G)) \geq 2^{s-1} + 2$. By Theorem 1.1, $\kappa'_e(L^{l+s}(G)) \geq 2\kappa'_e(L^{l+s-1}(G)) - 2 \geq 2^{s-1} + 2$. Then (i) is proved.

**Proof of Theorem 1.2(ii).** By (i) and Proposition 1.2(ii), for $s \geq 2$, $\kappa(L^{l+s}(G)) = \kappa'_e(L^{l+s-1}(G)) \geq 2^{s-1} + 2$.

Next we give an example to show that both bounds in Theorem 1.2 are best possible. Let $G$ be the graph depicted in Figure 4(a). Then $l(G) = 2$. The first three iterated line graphs of $G$ are depicted in Figure 4(b)-(d). Note that $L^3(G)$ has a 4-cycle in which every vertex has degree equal to 3. By the definition of line graphs, a 4-cycle of a graph generates a 4-cycle in its line graph, and the degree of a vertex in a line graph is the degree sum of the end vertices of the corresponding edge in the original graph minus 2. So $L^4(G)$ has a 4-cycle in which every vertex has degree equal to $2 \cdot 3 - 2 = 4$. In general, for $s \geq 2$, $L^{l+s}(G)$ has a 4-cycle in which every vertex has degree equal to $2^{s-1} + 2$. Let $uv \in E(L^{l+s}(G))$ be an edge on the 4-cycle such that both $u$ and $v$ have degree equal to $2^{s-1} + 2$. Then $E_{L^{l+s}(G)}(u) \cup E_{L^{l+s}(G)}(v) - \{uv\}$ is an essential edge cut of size $2 \cdot (2^{s-1} + 2) - 2 = 2^s + 2$. Note that $\kappa'_e(L^{l+1}(G)) = \kappa'_e(L^{l+2}(G)) = 4$. Hence $\kappa'_e(L^{l+s}(G)) = 2^s + 2$ for $s \geq 1$. By Proposition 1.2(ii), $\kappa(L^{l+s}(G)) = \kappa'_e(L^{l+s-1}(G)) = 2^{s-1} + 2$ for $s \geq 2$.

![Figure 4. An example for the sharpness of the bounds.](image)

**References**


