Multicolor star-critical Ramsey numbers and Ramsey-good graphs

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Abstract

This paper seeks to develop the multicolor version of star-critical Ramsey numbers, which serve as a measure of the strength of the corresponding Ramsey numbers. We offer several general theorems, some of which focus on Ramsey-good cases (i.e., cases in which the corresponding Ramsey number is equal to a general lower bound). We also prove some specific cases for small graphs, and conclude with a table of known multicolor star-critical Ramsey numbers.

Keywords: deleted edge number, Ramsey minimal, size Ramsey number
Mathematics Subject Classification: 05C55, 05D10, 05C35
DOI: 10.5614/ejgta.2022.10.1.4

1. Introduction

First defined by Hook in [27], star-critical Ramsey numbers seek to measure the strength of the Ramsey number for a given pair of graphs. We focus our investigation on the multicolor analogue of star-critical Ramsey numbers. As is standard, $K_n$ will denote a complete graph of order $n$ and $K_{1,n}$ will denote a star of order $n + 1$ (containing exactly $n$ vertices of degree 1 and one vertex of degree $n$). The Ramsey number $r = r(G_1, G_2, \ldots, G_t)$ is defined to be the least natural number such that every $t$-coloring of the edges of $K_r$ contains a monochromatic copy of $G_i$ in color $i$, for some $1 \leq i \leq t$. The star-critical Ramsey number $r^*(G_1, G_2, \ldots, G_t)$ is defined to be the least
natural number \( k \) such that every \( t \)-coloring of the edges of \( K_{r-1} \cup K_{1,k} \) contains a monochromatic \( G_i \) in some color \( i \). Here, \( K_{r-1} \cup K_{1,k} \) is the graph formed by taking the disjoint union of \( K_{r-1} \) with a vertex \( v \), then adding edges between \( v \) and exactly \( k \) of the vertices in the \( K_{r-1} \). Star-critical Ramsey numbers in the case where \( t = 2 \) have been extensively studied (e.g., see [24], [27], [28], [29], [33], [34], [36], [42], and [44]).

While star-critical Ramsey numbers determine the minimum number of edges that must be introduced between a vertex and \( K_{r-1} \) to establish the Ramsey property, the deleted edge number was introduced in [5] to determine how many edges of a star \( K_{1,k} \) must be removed from \( K_r \) in order to destroy the Ramsey property. To be precise, first define the \( k \)-deleted edge number by 
\[
D_k = \min \{ |E(G) - E(K_{1,k})| : G \text{ is a subgraph of } K_r \}.
\]

It follows from these definitions that 
\[
D_0 = \min \{ 0, |E(G) - E(K_{1,k})| : G \text{ is a subgraph of } K_r \}.
\]

for all \( k \). The deleted edge number \( de(G) \) is then defined to be the unique value of \( k \) such that 
\[
D_{k-1}(G) < D_k(G) < D_{k+1}(G).
\]

It follows from these definitions that 
\[
r^*_c(G) + de(G) = r(G).
\]

When \( G_1 = G_2 = \cdots = G_t \), we denote the \( t \)-colored Ramsey number by \( r^t(G) \), the \( t \)-colored deleted edge number by \( de^t(G) \), and the \( t \)-colored star-critical Ramsey number by \( r^t_*(G) \). We extend the definition of the Ramsey number to the 1-color case (and leave off the superscript) by setting 
\[
r(G) = |V(G)|.
\]

This extension follows from the observation that \( G \) is a subgraph of \( K_{|V(G)|} \) and \( |V(G)| \) vertices are needed to have a monochromatic \( G \). When \( G \) is assumed to be connected, the 1-color deleted edge number can then be defined by 
\[
de(G) = |V(G)| - \delta(G),
\]

where \( \delta(G) \) is the minimum degree among the vertices of \( G \). It follows from Equation (1) that 
\[
r_*(G) = \delta(G).
\]

A well-known lower bound for the Ramsey number \( r(G) \) was proved by Burr [7]:
\[
r(G_1, G_2) \geq (c(G_1) - 1)(\chi(G_2) - 1) + s(G_2) \quad \text{whenever } c(G_1) \geq s(G_2),
\]

where \( c(G_1) \) is the order of a maximal connected component in \( G_1 \), \( \chi(G_2) \) is the chromatic number for \( G_2 \), and \( s(G_2) \) is the chromatic surplus of \( G_2 \) (the least cardinality of a color class among all vertex colorings of \( G_2 \) using \( \chi(G_2) \) colors). When equality holds, one says that \( G_1 \) is \( G_2 \)-good. This concept was introduced by Burr and Erdős in [8, 9] and was investigated from the perspective of star-critical Ramsey numbers by Zhang, Broersma, and Chen in [44].

In the multicolor setting, assuming that \( r(G_1, G_2, \ldots, G_t-1) \geq s(G_t) \), it can be shown that 
\[
r(G_1, G_2, \ldots, G_t) \geq (r(G_1, G_2, \ldots, G_{t-1}) - 1)(\chi(G_t) - 1) + s(G_t).
\]

The proof of this inequality follows that of the analogous statement for Gallai-Ramsey hypergraph numbers proved in Theorem 5 of [6]. When equality holds, we say that the multiset
\( \mathcal{H} = \{G_1, G_2, \ldots, G_{t-1}\} \) is \( G_t \)-good. Section 2 focuses on general multicolor star-critical Ramsey number theorems. In Theorem 2.1, we prove an inequality involving multicolor deleted edge numbers that depends on a multiset being \( G_t \)-good. Theorem 2.2 gives bounds for certain star-critical Ramsey numbers involving multiple complete graphs. Theorem 2.3 generalizes the tree-complete graph star-critical Ramsey number evaluated in [29] to multiple complete graphs, and Theorem 2.4 considers certain star-critical Ramsey numbers for multiple stars. While we find it easier to prove (and state) many of these results as deleted edge number results, we provide their analogues in terms of star-critical Ramsey numbers for the sake of being comprehensive.

In Section 3, we prove several explicit values of multicolor star-critical Ramsey numbers for small graphs, including several new 2-color cases. We conclude with a table of known multicolor star-critical Ramsey numbers.

2. General Multicolor Results

The following theorem can be viewed as a generalization of a weakened version of Theorem 3 in [44].

**Theorem 2.1.** Suppose that \( G_1, G_2, \ldots, G_t \) are connected graphs such that \( \mathcal{H} = \{G_1, G_2, \ldots, G_{t-1}\} \) is a \( G_t \)-good multiset satisfying \( r(G_1, G_2, \ldots, G_{t-1}) \geq s(G_t) \). Then

\[
de(G_1, G_2, \ldots, G_{t}) \leq de(G_1, G_2, \ldots, G_{t-1}).
\]

**Proof.** Let \( m = r(G_1, G_2, \ldots, G_{t-1}) \). We will construct a \( t \)-coloring of

\[
K_{(m-1)(\chi(G_t)-1)+s(G_t)} - E(K_{1,de(G_1,G_2,...,G_{t-1})})
\]

that lacks a monochromatic copy of \( G_i \) in color \( i \), for all \( 1 \leq i \leq t \). Start with a \((t-1)\)-coloring of \( K_m - E(K_{1,de(G_1,G_2,...,G_{t-1})}) \) that lacks a monochromatic copy of \( G_i \) in color \( i \) for all \( 1 \leq i \leq t-1 \), and call this graph \( A_1 \). Let \( a \) be the center vertex for the missing star. Let \( A_2, A_3, \ldots, A_{\chi(G_t)-1} \) be copies of \( A_1 - \{a\} \). Finally, let \( A_{\chi(G_t)} \) be formed by taking another copy of \( A_1 - \{a\} \) and removing \( m - s(G_t) \) vertices. Hence, \( A_{\chi(G_t)} \) is a \((t-1)\)-colored \( K_{s(G_t)-1} \) that lacks a monochromatic copy of \( G_i \) in color \( i \) for all \( 1 \leq i \leq t-1 \). Form the union

\[
\bigcup_{1 \leq j \leq \chi(G_t)} A_j,
\]

and interconnect the \( A_j \) with edges in color \( t \). By construction, the resulting

\[
K_{(m-1)(\chi(G_t)-1)+s(G_t)} - E(K_{1,de(G_1,G_2,...,G_{t-1})})
\]

lacks a monochromatic copy of \( G_i \) in color \( i \) for all \( 1 \leq i \leq t-1 \). To see that it also lacks a monochromatic copy of \( G_t \) in color \( t \), we consider two cases. First, if \( s(G_t) = 1 \), then any subgraph in color \( t \) can be vertex colored using \( \chi(G_t) - 1 \) colors (by assigning colors according to the vertex sets of \( A_j \)). Hence, \( G_t \) cannot be such a subgraph. Second, if \( s(G_t) > 1 \), then coloring any \( t \)-colored subgraph using \( \chi(G_t) \) colors results in a chromatic surplus of at most \( s(G_t) - 1 \). Once again, we find that \( G_t \) cannot be such a subgraph. Thus, the theorem follows. \( \square \)
When the hypotheses of Theorem 2.1 are met, it follows from Equation (1) that
\[ r_s(G_1, G_2, \ldots, G_t) - r(G_1, G_2, \ldots, G_t) \geq r_s(G_1, G_2, \ldots, G_{t-1}) - r(G_1, G_2, \ldots, G_{t-1}). \]

In the case where \( t = 2 \), the implication becomes
\[ de(G_1, G_2) \leq |V(G_1)| - \delta(G_1) \]
(equivalently, \( r_s(G_1, G_2) \geq r(G_1, G_2) - |V(G_1)| + \delta(G_1) \)).

The fact that \( de(K_{n_1}, K_{n_2}) = 1 \) was first proved by Erdős and Faudree [18], and it was observed by Cowen [17] that this fact easily extends to the more general multicolor result
\[ de(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = 1. \]

Suppose that \( G_1, G_2, \ldots, G_s \) are connected graphs such that \( r = r(G_1, G_2, \ldots, G_s) \) and \( r' = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \). If
\[ r(G_1, G_2, \ldots, G_s, K_\ell) = (r - 1)(\ell - 1) + 1 \]
for all \( \ell \geq 2 \), then Omidi and Raeisi (see Theorem 2.1 of [38]) proved that
\[ r(G_1, G_2, \ldots, G_s, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = (r - 1)(r' - 1) + 1. \]  \( \text{(2)} \)

This result motivates the following theorem.

**Theorem 2.2.** Let \( G_1, G_2, \ldots, G_s \) be connected graphs, \( r = r(G_1, G_2, \ldots, G_s) \), and \( r' = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \). If \( r(G_1, G_2, \ldots, G_s, K_\ell) = (r - 1)(\ell - 1) + 1 \) for all \( \ell \geq 2 \), then
\[ de(G_1, G_2, \ldots, G_s, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq de(G_1, G_2, \ldots, G_s). \]

**Proof.** Consider a \( t \)-coloring of \( K_{r'} - e \) that lacks a monochromatic copy of \( K_{n_j} \) in color \( s + j \) for all \( 1 \leq j \leq t \). The existence of such a coloring follows from
\[ de(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = 1. \]
Let \( a \) and \( b \) be the vertices of the missing edge. Replace each vertex other than \( a \) with a copy of \( K_{r-1} \) that uses the first \( s \) colors and lacks a monochromatic copy of \( G_i \) in color \( i \), for all \( 1 \leq i \leq s \). Edges interconnecting the copies of \( K_{r-1} \) with each other and with \( a \) are colored according to the edges between the vertices that were replaced. The missing edge between \( a \) and \( b \) is now a missing \( E(K_{1,r-1}) \). None of these edges can be given colors \( s + 1, s + 2, \ldots, s + t \) without producing a copy of \( K_{n_j} \) in color \( s + j \) for some \( 1 \leq j \leq t \). It is possible to color such edges using colors \( 1, 2, \ldots, s \), with only \( de(G_1, G_2, \ldots, G_s) \) edges having to remain missing. Thus, we have formed an \( (s + t) \)-colored
\[ K_{(r-1)(r'-1)+1} - E(K_{1,de(G_1,G_2,\ldots,G_s)}) \]
that lacks \( G_i \) in color \( i \) for all \( 1 \leq i \leq s \) and \( K_{n_j} \) in color \( s + j \), for all \( 1 \leq j \leq t \). It follows that
\[ de(G_1, G_2, \ldots, G_s, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq de(G_1, G_2, \ldots, G_s), \]
completing the proof.
Assuming the hypotheses stated in Theorem 2.2 and applying Equations (1) and (2), the above implication can be restated as

\[ r_*(G_1, G_2, \ldots, G_s, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \geq r_*(G_1, G_2, \ldots, G_s) + (r - 1)(r' - 2). \]

Next, we consider the case of a tree versus complete graphs. In [29], it was shown that when \( T_m \) is a tree of order \( m \),

\[ r_*(T_m, K_n) = (m - 1)(n - 2) + 1. \]

The equivalent result for deleted edge numbers was considered in Theorem 2.1 of [5], but the proof given there contains a mistake in the inductive step. At the present time, the only correct proof that we know of is the proof of Theorem 2.5 in Hook and Isaak’s paper [29]. In the next theorem, we extend this result to the multicolor case involving a single tree and multiple complete graphs.

**Theorem 2.3.** Let \( n_i \geq 3 \) for all \( 1 \leq i \leq t \), where \( t \geq 1 \). Assume that \( T_m \) is a tree of order \( m \geq 2 \) that satisfies \( m \leq r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \). Then

\[ \text{de}(T_m, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = m - 1. \]

**Proof.** Since \( \text{de}(T_m) = m - 1 \) and \( r(T_m, K_n) = (m - 1)(n - 1) + 1 \) (see [15]), Theorem 2.2 implies that

\[ \text{de}(T_m, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq m - 1. \]

To prove the opposite inequality, let \( r = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \) and consider a \((t + 1)\)-coloring of the edges in

\[ K_{(m-1)(r-1)+1} - E(K_{1,m-2}). \]

Viewing colors \( 2, 3, \ldots, t + 1 \) as a single color and using Hook and Isaak’s 2-color result [29], we find that the resulting coloring contains a copy of \( T_m \) in the first color or a copy of \( K_r \) in the second color. In the first case, we are done, so assume that there is a copy of \( K_r \) spanned by edges in colors \( 2, 3, \ldots, t + 1 \). Since \( r = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \), there is a copy of \( K_{n_i} \) in color \( i \), for some \( 2 \leq i \leq t + 1 \). It follows that

\[ \text{de}(T_m, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \geq m - 1, \]

completing the proof of the theorem. \( \square \)

Combining Equations (1) and (2) with Theorem 2.3, it follows that

\[ r_*(T_m, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = (m - 1)(r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) - 2) + 1. \]

Now we consider the case of multiple stars. Let \( S = \{m_1, m_2, \ldots, m_t\} \), where each \( m_i \geq 2 \). Define \( N = \sum_{1 \leq i \leq t} m_i \) and denote by \( k \) the number of elements in \( S \) that are even. Burr and Roberts [10] proved

\[ r(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = \begin{cases} N - t + 1, & \text{if } k \geq 2 \text{ is even}, \\ N - t + 2, & \text{otherwise}. \end{cases} \]

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Theorem 2.4. If \( k = 0 \) or \( k \) is odd, then

\[
 r^*(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = 1.
\]

Proof. In the case where \( k = 0 \) or \( k \) is odd,

\[
 r(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = N - t + 2.
\]

Consider a \( t \)-colored \( K_{N-t+2} \) and observe that removing \( N - t \) edges incident with a fixed vertex still leaves one vertex having degree \( N - t + 1 \). By the pigeonhole principle, this vertex must be incident with at least \( m_i \) edges in color \( i \) for some \( i \). Hence, all \( N - t + 1 \) edges incident with a fixed vertex must be removed in order to destroy the Ramsey property. It follows that

\[
 de(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = N - t + 1,
\]

and applying (1) and (3) completes the proof. \( \square \)

In 1974, Harary and Prins [26] defined the Ramsey multiplicity \( R(G_1, G_2, \ldots, G_t) \) to be the smallest possible total number of \( G_1 \) in color 1, \( G_2 \) in color 2, \ldots, \( G_t \) in color \( t \) in any \( t \)-coloring of \( K_r \), where \( r = r(G_1, G_2, \ldots, G_t) \). In the case where \( G_1, G_2, \ldots, G_t \) are all stars, Jacobson [31] proved that when all \( m_i \geq 2 \) are integers,

\[
 R(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = \begin{cases} 
 \frac{k}{2}, & \text{if } k \geq 2 \text{ is even}, \\
 N - t + 2, & \text{otherwise}.
\end{cases}
\]

In particular, when \( k = 2 \), we find that there exists a \( t \)-coloring of \( K_r \) that contains a single monochromatic \( K_{1,m_i} \) in some color \( i \), and which does not contain a \( K_{1,m_i+1} \) in color \( i \). It follows that a single edge in color \( i \) can be removed to produce a \( K_r - e \) that lacks a monochromatic \( K_{1,m_i} \) in color \( i \), for all \( 1 \leq i \leq t \). Hence, we obtain the following theorem.

Theorem 2.5. If \( m_1, m_2, \ldots, m_t \) are integers greater than 1, exactly two of which are even, then

\[
 r^*(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = N - t.
\]

In general, we conjecture the following evaluation of \( r^*(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) \).

Conjecture 1. If \( m_1, m_2, \ldots, m_t \) are integers greater than 1, exactly \( k \) of which are even, then

\[
 r^*(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = \begin{cases} 
 N - t, & \text{if } k \geq 2 \text{ is even}, \\
 1, & \text{otherwise}.
\end{cases}
\]

When trying to prove the case where \( k > 2 \) is even, we must construct a \( t \)-coloring of \( K_{N-t+1} - e \) that lacks a copy of \( K_{1,m_i} \) in color \( i \) for all \( 1 \leq i \leq t \). We can use Theorem 9.1 of Harary’s book [25], which states that \( K_{2n} \) contains a 1-factorization. Start with \( K_{N-t} \), where \( N - t \) is necessarily even. We can color 1-factors, but must switch the colors of some of the edges, as was done in Theorem 3.1 of [5]. This is not very simple to handle in general.
3. Some Small Multicolor Star-Critical Ramsey Numbers

In this section, we give some specific multicolor star-critical Ramsey numbers, including a few new 2-color numbers, when the graphs being considered are small. We denote by $K_n - e$ the graph formed by removing a single edge from $K_n$ and we denote by $P_n$ a path on $n$ vertices. The graph $K_{1,3} + e$ is formed by adding a single edge connecting two of the leaves in $K_{1,3}$.

**Theorem 3.1.** $r^*(K_4 - e, K_3) = 5$.

**Proof.** Since $r(K_4 - e, K_3) = 7$ (see [16]), it follows that $K_4 - e$ is $K_3$-good. Theorem 2.1 then implies $de(K_4 - e, K_3) \leq 2$ (also, see Figure 1). To obtain the inequality $de(K_4 - e, K_3) \geq 2$, we must show that every red/blue coloring of $K_7 - E(K_{1,2})$ that lacks a red $K_4 - e$ and a blue $K_3$ (and hence, a blue $K_{1,3} + e$).

![Figure 1](image)

Figure 1. A 2-coloring of $K_7 - E(K_{1,2})$ that lacks a red $K_4 - e$ and a blue $K_3$ (and hence, a blue $K_{1,3} + e$).

must show that every red/blue coloring of $K_7 - e$ contains a red $K_4 - e$ or a blue $K_3$. Consider an arbitrary 2-coloring of $K_7 - e$ and let $a$ and $b$ be the vertices of the missing edge. If we remove vertex $b$, we have a 2-coloring of $K_6$, which must contain a red $K_3$ or a blue $K_3$ since $r(K_3, K_3) = 6$ (see [21]). In the latter case, we are done, so suppose there is a red $K_3$. We must now consider two cases, based on whether or not $a$ is one of the vertices in the red $K_3$.

**Case 1.** Suppose that $a$ is not in the red $K_3$. Label the vertices in the red $K_3$ by $x, y, z$ and the other vertices $a, b, c, d$. If any of $a, b, c, d$ is adjacent via 2 or more red edges to $x, y, z$, then a red $K_4 - e$ is formed. Otherwise, each of $a, b, c, d$ is adjacent via at least 2 blue edges to $\{x, y, z\}$. If the subgraph induced by $\{a, b, c, d\}$ does not contain a red $K_4 - e$, then at least one edge is blue. Without loss of generality, suppose that $ac$ is blue. Then $a$ and $c$ are each adjacent via at least two blue edges to $\{x, y, z\}$. By the pigeonhole principle, there is a vertex, say $x$, in which $ax$ and $cx$ are both blue, forcing $\{a, c, x\}$ to form a blue $K_3$.

**Case 2.** Suppose that the red $K_3$ consists of vertices $a, x, y$ and the other vertices are labelled $b, c, d, e$. If any edge in the subgraph induced by $\{c, d, e\}$ is blue, then we can use an argument similar to the previous case to force the existence of a blue $K_3$. So, suppose this subgraph is a red $K_3$, then at least two of the edges $bc, bd$, and $be$ must be blue (otherwise the subgraph induced by $\{b, c, d, e\}$ is a red $K_4 - e$). Without loss of generality, suppose that $bc$ and $bd$ are blue. If $b$ is
adjacent via red edges to both of $x$ and $y$, then a red $K_4 - e$ is formed. So, assume one such edge, say $bx$ is blue. If either $cx$ or $dx$ is blue, when including $b$, we obtain a blue $K_3$. If they are both red, then the subgraph induced by $\{x, c, d, e\}$ contains a red $K_4 - e$.

In both cases, we find that $K_7 - e$ contains a red $K_4 - e$ or a blue $K_3$, completing the proof.

In the case of the pair of graphs, $K_4 - e$ and $K_{1,3} + e$, an interesting phenomenon occurs. Namely, $r(K_4 - e, K_{1,3} + e) = 7$ [16], and it is easily confirmed that $K_4 - e$ is $(K_{1,3} + e)$-good and $K_{1,3} + e$ is $(K_4 - e)$-good. By Theorem 2.1, it follows that

$$de(K_4 - e, K_{1,3} + e) \leq \min\{4 - \delta(K_4 - e), 4 - \delta(K_{1,3} + e)\} = 2.$$  

The following theorem shows that this bound is tight.

**Theorem 3.2.** $r_*(K_4 - e, K_{1,3} + e) = 5$.

**Proof.** As mentioned above, Theorem 2.1 implies that $de(K_4 - e, K_{1,3} + e) \leq 2$ (also, see Figure 1). Proving the opposite inequality requires showing that every red/blue coloring of $K_7 - e$ contains a red $K_4 - e$ or a blue $K_{1,3} + e$. Consider such a coloring and observe that Theorem 3.1 implies that there is a red $K_4 - e$ or a blue $K_3$. In the first case, we are done, so assume the latter condition. We obtain two cases, based on whether or not the missing edge is incident with a vertex in the blue $K_3$. Let $a$ and $b$ be the vertices of the missing edge.

**Case 1.** Suppose that neither $a$ nor $b$ are contained in the blue $K_3$. Label the vertices in the blue $K_3$ by $x, y, z$ and the other vertices by $a, b, c, d$. If any edge connecting $\{x, y, z\}$ to $\{a, b, c, d\}$ is blue, then a blue $K_{1,3} + e$ is formed. So, assume that all such edges are red. Other than the missing edge, if any edge in the subgraph induced by $\{a, b, c, d\}$ is red, then a red $K_4 - e$ is formed. All such edges must then be blue, forcing a blue $K_{1,3} + e$ as a subgraph.

**Case 2.** Without loss of generality, suppose that the vertices of the blue $K_3$ are given by $a, x, y$ and the other vertices are given by $b, c, d, e$. Similar to the previous case, with the exception of the missing edge, if any edge joining $\{a, x, y\}$ to $\{b, c, d, e\}$ is blue, then a blue $K_{1,3} + e$ is formed. Assume that all such edges are red. Avoiding a red $K_4 - e$ forces the subgraph induced by $\{b, c, d, e\}$ to contain a blue $K_{1,3} + e$.

In both cases, the $K_7 - e$ contains a red $K_4 - e$ or a blue $K_{1,3} + e$, completing the proof of the theorem.

**Theorem 3.3.** $r_*(P_4, K_4 - e) = 4$.

**Proof.** Since $r(P_4, K_4 - e) = 7$ (see [16]), it follows that $P_4$ is $(K_4 - e)$-good. So, Theorem 2.1 implies that

$$de(P_4, K_4 - e) \leq 3$$

(also, see Figure 2). To prove the other direction, consider a red/blue $K_7 - E(K_{1,2})$. Let vertex $a$ be the center of the missing star, and let vertices $b$ and $g$ be its leaves. Removing vertices $a$ and $g$ results in a red/blue coloring of $K_5$. Since $r(P_3, K_4 - e) = 5$, this coloring contains a red $P_3$ or a blue $K_4 - e$. In the latter case, we are done, so assume that there is a red $P_3$. The location of this $P_3$ in the original coloring relative to the missing $K_{1,2}$ produces three cases, as illustrated in Figure 3.
Figure 2. A 2-coloring of $K_7 - E(K_{1,3})$ that lacks a red $P_4$, and a blue $K_3$ (and hence, a red $K_{1,3} + e$ and a blue $K_4 - e$).

Figure 3. Three cases describing the location of a monochromatic $P_3$ relative to a missing $K_{1,2}$ in a 2-coloring of $K_7 - E(K_{1,2})$.

**Case 1.** Consider the case given by the first image in Figure 3. Avoiding a red $P_4$ forces edges $be$, $bf$, $bg$, $de$, $df$, and $dg$ to be blue. If no blue $K_4 - e$ exists, then $bd$, $ef$, $eg$, and $fg$ must be red. If any one of $ac$, $ad$, $af$, or $cf$ are red, then a red $P_4$ is formed. Otherwise, all four edges are blue and the subgraph induced by $\{a, c, d, f\}$ contains a blue $K_4 - e$.

**Case 2.** Consider the case given by the second image in Figure 3. Avoiding a red $P_4$ forces edges $ce$, $cf$, $de$, and $df$ to be blue. If no blue $K_4 - e$ exists, then edge $cd$ must be red. At this point, we have reduced this case to the situation that occurs in Case 1.

**Case 3.** Consider the case given by the third image in Figure 3. Avoiding a red $P_4$ forces edges $ac$, $ae$, $bc$, $be$, $cf$, $cg$, $ef$, and $eg$ to be blue. If no blue $K_4 - e$ exists, then $af$, $ce$, and $fg$ must be red. If $bg$ is red, then a red $P_4$ is formed. Otherwise, the subgraph induced by $\{b, c, e, g\}$ contains a blue $K_4 - e$.

In all three cases, we find that there is either a red $P_4$ or a blue $K_4 - e$.

**Theorem 3.4.** $r_*(K_{1,3} + e, K_3) = 4$. 

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Proof. In [16], it was shown that \( r(K_{1,3} + e, K_3) = 7 \), from which it follows that \( K_{1,3} + e \) is \( K_3 \)-good. By Theorem 2.1, it follows that

\[
de(K_{1,3} + e, K_3) \leq 3
\]

(also, see Figure 2). It remains to be shown that every 2-coloring of \( K_7 - E(K_{1,2}) \) contains a red \( K_{1,3} + e \) or a blue \( K_3 \). Consider an arbitrary 2-coloring of \( K_7 - E(K_{1,2}) \) and let \( a \) be the vertex that is incident with the 2 missing edges. Removing vertex \( a \) produces a 2-coloring of \( K_6 \), which necessarily contains a red \( K_3 \) or a blue \( K_3 \). Assume the former case, and denote the vertices in the red \( K_3 \) by \( b, c, d \). Label the remaining three vertices \( e, f, g \). If any edges connecting \( \{b, c, d\} \) with \( \{a, e, f, g\} \) are red, then a red \( K_{1,3} + e \) is formed. So suppose that all such edges are blue, resulting in three cases (see Figure 4).

Figure 4. Three cases describing the location of a monochromatic \( K_3 \) relative to a missing \( K_{1,2} \) in a 2-coloring of \( K_7 - E(K_{1,2}) \).

Regardless of which case we are in, if any edges in the subgraph induced by \( \{a, e, f, g\} \) are blue, then a blue \( K_3 \) is formed. The only other possibility is that all such edges (other than those removed) are red. In all three cases, we obtain a red \( K_{1,3} + e \) as a subgraph. It follows that

\[
de(K_{1,3} + e, K_3) \geq 3,
\]

completing the proof. \( \square \)

Current literature indicates that the star-critical Ramsey number \( r^*(K_3 - e, K_n - e) \) is known and can be found in [27]. As this document is not readily available, and because the upper bounds to \( de(K_3 - e, K_n - e) \) follow from Theorem 2.1, we offer a complete proof.

**Theorem 3.5.** For all \( n \geq 4 \),

\[
r^*(K_3 - e, K_n - e) = 2n - 1.
\]

**Proof.** It is easily shown that \( \tilde{R}(K_3 - e, K_n - e) = 2n - 3 \) (see Section 3.1 of [39]), from which we see that \( K_3 - e \) is \( (K_n - e) \)-good. It follows from Theorem 2.1 that

\[
de(K_3 - e, K_n - e) \leq 2
\]
Figure 5. A 2-coloring of $K_{2n-3} - E(K_{1,2})$ that lacks a red $K_3 - e$ and a blue $K_n - e$. (also, see Figure 5). It remains to be proved that every red/blue coloring of the edges of $K_{2n-3} - e$ contains a red $K_3 - e$ or a blue $K_n - e$. Consider a two-coloring of $K_{2n-3} - e$ and let the missing edge be between vertices $a$ and $b$. Remove vertex $a$ and assume that the resulting $K_{2n-4}$ lacks a red copy of $K_3 - e$ and a blue copy of $K_n - e$ (otherwise we are done). Notice that the red edges must form a matching $M$. Let $m$ be the size of this matching. Certainly $m \leq n - 2$. A blue copy of $K_n - e$ could only be formed by taking, at most, one vertex from $m - 1$ matchings, two vertices from the remaining matching, and all vertices that are not incident with a red edge. In order to avoid this, we need

$$1 + m + (2n - 4 - 2m) < n \quad \implies \quad m > n - 3.$$ 

So, a red/blue coloring of a $K_{2n-4}$ that lacks a red copy of $K_3 - e$ and a blue copy of $K_n - e$ must contain a red matching of size $n - 2$. Now consider vertex $a$. If $a$ is incident with any red edges, a red $K_3 - e$ is formed. If all $2n - 5$ edges incident with $a$ are blue, then $a$ must be adjacent to at least one vertex in each matching, labeled $x_1, x_2, \ldots, x_{n-2}$. This only accounts for $n - 2$ edges, so there is certainly a vertex $y$ such that $ay$ is blue and $x_ky$ is red for some $1 \leq k \leq n - 2$. Then the subgraph induced by $\{a, y, x_1, x_2, \ldots, x_{n-2}\}$ forms a blue $K_n - e$. \hfill \square

Note that $K_3 - e = P_3 = K_{1,2}$.

**Theorem 3.6.** If $t \geq 1$ is odd, then

$$r_t^1(P_3) = 1.$$ 

**Proof.** It is known that when $t$ is odd, $r_t^1(P_3) = t + 2$ (see [30]). Consider a $t$-coloring of $K_{t+1}$ that lacks a monochromatic $P_3$. Such a coloring has every vertex incident with exactly one edge in
each of the \( t \) colors. Adding in an additional vertex and assigning any color to an edge joining this vertex with the \( K_{t+1} \) necessarily produces a monochromatic \( P_3 \).

Finally, we conclude this section by considering the star-critical numbers for multiple copies of \( P_3 \) versus complete graphs. Using the observation that \( P_3 \) is a star, Jacobson proved in [32] that

\[
r(P_3, P_3, \ldots, P_3, K_\ell) = (r^s(P_3) - 1)(\ell - 1) + 1
\]

for all \( \ell \geq 1 \). It follows that the multiset consisting of \( s \) copies of \( P_3 \) is \( K_\ell \)-good. Thus, Theorem 2.2 implies that

\[
de(P_3, P_3, \ldots, P_3, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq de^s(P_3).
\]

When \( s \) is odd, Theorem 3.6 implies that

\[
de(P_3, P_3, \ldots, P_3, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq s + 1.
\]

When \( s = 2 \), we have the following theorem.

**Theorem 3.7.** For all \( 1 \leq i \leq t \) and \( n_i \geq 1 \),

\[
de(P_3, P_3, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = 1.
\]

**Proof.** It is easily confirmed that \( r(P_3, P_3) = 3 \) and \( de(P_3, P_3) = 1 \), from which Theorem 2.2 implies the statement of the theorem. \( \square \)

Using Equations (1) and (2), Theorem 3.7 implies that

\[
r^s(P_3, P_3, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = r(P_3, P_3, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) - 1
\]

\[
= (r(P_3, P_3) - 1)(r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) - 1)
\]

\[
= 2(r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) - 1)
\]

\[
= 2r^s(K_{n_1}, K_{n_2}, \ldots, K_{n_t}).
\]

In particular, \( r^s(P_3, P_3, K_\ell) = 2\ell - 2 \) for all \( \ell \geq 1 \).

4. Conclusion

In this section, we compile the known values of \( r^s(G_1, G_2, \ldots, G_t) \). Besides the partial results contained in Theorems 2.4 and 2.5, Table 1 is intended to provide the current known multicolor star-critical Ramsey numbers, along with the corresponding Ramsey numbers and relevant citations. The only graph in Table 1 that we have not yet defined is the fan \( F_m \), defined to be the join of \( K_1 \) and \( mK_2 \), where \( mK_2 \) consists of \( m \) disjoint copies of \( K_2 \).
Multicolor Star-Critical Ramsey Numbers and Ramsey-Good Graphs  |  M. Budden and E. DeJonge

Table 1. Known star-critical Ramsey numbers, along with their corresponding Ramsey numbers.

<table>
<thead>
<tr>
<th>$r(G_1, G_2, \ldots, G_t)$</th>
<th>$r^*(G_1, G_2, \ldots, G_t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(T_m, K_n) = (m - 1)(n - 1) + 1$ [15]</td>
<td>$(m - 1)(n - 2) + 1$ [29]</td>
</tr>
<tr>
<td>$r(C_m, K_3) = 2m - 1$ for $m &gt; 3$ [11]</td>
<td>$m + 1$ [44]</td>
</tr>
<tr>
<td>$r(C_4, K_4) = 10$ [43]</td>
<td>9 [34]</td>
</tr>
<tr>
<td>$r(C_m, K_4) = 3m - 2$ for $m \geq 5$ [43]</td>
<td>$2m$ [34]</td>
</tr>
<tr>
<td>$r(C_m, K_5) = 4m - 3$ for $m \geq 5$ [2]</td>
<td>$3m - 1$ [33]</td>
</tr>
<tr>
<td>$r(F_m, K_3) = 4m + 1$ for $m \geq 2$ [35]</td>
<td>$2m + 2$ [36]</td>
</tr>
<tr>
<td>$r(F_m, K_4) = 6m + 1$ [41]</td>
<td>$4m + 2$ [24]</td>
</tr>
<tr>
<td>$r(K_4 - e, K_3) = 7$ [39]</td>
<td>5 (Theorem 3.1)</td>
</tr>
<tr>
<td>$r(K_4 - e, K_{1,3} + e) = 7$ [16]</td>
<td>5 (Theorem 3.2)</td>
</tr>
<tr>
<td>$r(P_4, K_2 - e) = 7$ [16]</td>
<td>4 (Theorem 3.3)</td>
</tr>
<tr>
<td>$r(K_{1,3} + e, K_4) = 7$ [16]</td>
<td>4 (Theorem 3.4)</td>
</tr>
<tr>
<td>$r(K_3 - e, K_{n-1} - e) = 2n - 3$ for $n \geq 4$ [43]</td>
<td>$2n - 5$ [27], (Theorem 3.5)</td>
</tr>
<tr>
<td>$r^t(P_3) = t + 2$ for $t \geq 1$ odd [30]</td>
<td>1 (Theorem 3.6)</td>
</tr>
<tr>
<td>$r(P_3, P_3, K_3) = 2\ell - 1$ [32]</td>
<td>$2\ell - 2$ (Theorem 3.7)</td>
</tr>
</tbody>
</table>

Much of the recent research on star-critical Ramsey numbers has focused on the evaluation of $r^*(C_m, K_n)$. This is due to the fact that the complete evaluation of $r(C_m, K_n)$ is a well known open problem. The fact that

$$r(C_m, K_n) \geq (m - 1)(n - 1) + 1$$

follows from Lemma 4 of [16]. In 1973, the work of Bondy and Erdős [3] led to the conjecture (see [19]) that

$$r(C_m, K_n) = (m - 1)(n - 1) + 1$$

whenever $m \geq n \geq 3$, except for the case $m = n = 3$. At the present time, this conjecture has been shown to be true when $m \geq 4n + 2$ [37], when $m > n = 3$ [11], when $m \geq n = 4$ [43], when $m \geq n = 5$ [2], when $m \geq n = 6$ [40], when $m \geq n = 7$ [13], as well as a few other special cases.

The recent paper by Wang, Li, and Li [42] introduced variations of the concept of a star-critical Ramsey number defined by removing graphs other than just stars from complete graphs. In this sense, their generalizations demonstrate that the deleted edge number is a little more natural than the star-critical Ramsey number. One direction for future inquiry that seems to follow from such generalizations is to define a deleted cycle number, where cycles of various lengths are removed to destroy the Ramsey property.

Finally, we encourage the reader to consider star-critical versions of other Ramsey-type numbers. In particular, Gallai-Ramsey numbers (see [20], [22], and [23]) and bipartite Ramsey numbers (see [1] and [14]) can also be destroyed by the removal of edges incident with a fixed vertex. We reserve such investigations for future research.
References


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