Power Graphs: A Survey

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Abstract. This article gives a survey of all results on the power graphs of groups and semigroups obtained in the literature. Various conjectures due to other authors, questions and open problems are also included.

1 Introduction

Many interesting results on the power graphs have been obtained recently. This article gives a survey of the current state of knowledge on this research direction by presenting all results and open questions recorded in the literature dealing with power graphs. Various conjectures formulated by other authors and several new open problems are also included. This is the first survey devoted to the power graphs.

2 Definition of power graphs

Let us begin with the definition of the directed and undirected power graphs, which are the main concepts considered in this survey. In the literature, the power graphs have been defined and considered for groups and semigroups. Section 4 contains concise prerequisites on the terminology of the group and semigroup theories used in the results (cf. [27,74]). It is well known that the operation in every group or semigroup is power-associative. Therefore the following definition applies to all groups and semigroups.

Definition 1. Let $S = (S, \cdot)$ be a set with a power-associative operation $\cdot$. The *directed power graph* $\overrightarrow{\mathscr{G}}(S)$ of S is a directed graph with the set S of vertices and with all edges (u, v) such that $u \neq v$ and v is a power of u. The *undirected power graph* $\mathscr{G}(S)$ of S is the underlying undirected graph of $\overrightarrow{\mathscr{G}}(S)$. This means that the set of vertices of $\mathscr{G}(S)$ is equal to S and two distinct vertices are adjacent if and only if one of them is a power of the other.

The directed power graph could also be called the *power digraph*, although the latter term has not been used in the literature yet.

Definition 2. If G is a group with identity e and $\overrightarrow{\mathscr{G}}(G)$, $\mathscr{G}(G)$ are the directed and undirected power graphs of G, respectively, then the graphs obtained by deleting the vertex e from $\overrightarrow{\mathscr{G}}(G)$ and $\mathscr{G}(G)$ are denoted by $\overrightarrow{\mathscr{G}}^*(G) = \overrightarrow{\mathscr{G}}(G) - \{e\}$ and $\mathscr{G}^*(G) = \mathscr{G}(G) - \{e\}$, respectively.

For clarity, formulating all results in this article we use only complete terms 'directed power graph' and 'undirected power graph'. The shorter expression 'power graph' is used here only when we refer to both of these concepts simultaneously and in general discussions that may apply to both of these notions.

3   Historical comments

Historically, the concept of a directed power graph was first introduced and considered in [39] in the case of groups. For semigroups, it was first investigated in [43], and then in [40, 42]. All of these papers used only the brief term 'power graph' to refer to the directed power graph, with the understanding that the undirected power graph is the underlying undirected graph of the directed power graph.

The undirected power graphs became the main focus of study in [23] and in the subsequent papers by P.J. Cameron et al. [21, 22], which introduced the use of the brief term 'power graph' in the second meaning of an undirected power graph. For a group G, the digraph $\mathscr{G}^*(G)$ was considered in [71] as the main object of study.

Thus, the readers may need to be aware that in the literature the brief term 'power graph' can mean either a 'directed power graph' or an 'undirected power graph' depending on which type of graphs is the main subject of study in each particular paper. For clarity, the present article states all results using only the complete terms 'directed power graph' and 'undirected power graph'.

4   Preliminaries and background information

We use standard terminology and refer the readers to the monographs [24,27,33,34,73,74] for more details, further bibliography and explanations.

Throughout $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all integers and all positive integers, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ stands for the set of all natural numbers. The *Euler totient function* of a positive integer n is denoted by $\phi(n)$.

All graphs considered in this paper are assumed to be simple. In particular, they do not have loops. If $D = (V, E)$ is a directed graph, then the *underlying undirected graph* of D is the undirected graph with the same set V of vertices, where two distinct vertices u and v are adjacent if and only if $(u, v) \in E$ or $(v, u) \in E$.

Subsequent sections of this survey article are organised according to the graph theoretical properties being considered, and the definitions of these properties are reviewed in the following sections. This rest of this section contains concise preliminaries concerning groups and semigroups.

An *operation* or a *binary operation* on a set S is a mapping from the direct product $S \times S$ to S. If $\circ$ is a binary operation on a set S and $x, y \in S$, then the image of the pair (x, y) under $\circ$ is denoted by $x \circ y$, which is called the *infix notation* for the operation $\circ$. In *additive* terminology and notation the operation $\circ$ is denoted by $+$ and is called *addition*, so that the result of the operation is written down as $x + y$. In *multiplicative* terminology and notation the operation $\circ$ is referred to as *multiplication* and is denoted by $\cdot$ so that the result of the operation is recorded as $xy = x \cdot y$. In this survey we use multiplicative notation to discuss sets with a binary operation. A set with a binary operation is called a *magma* [18, 26] or a *groupoid* [82].

An operation $\cdot$ on the set S is *associative* if it satisfies the following associative law: $x(yz) = (xy)z$ for all $x, y, z \in S$. A *semigroup* is a set S equipped with an associative binary operation $\cdot$. For example, the set of natural numbers or positive integers is a semigroup with respect to addition, and at the same time it is a semigroup with respect to multiplication. If we want to distinguish between these two semigroups on the same set, we can indicate the corresponding operation explicitly by using the symbols $(\mathbb{N}, +)$ or $(\mathbb{N}, \cdot)$, respectively.

Let $M = (M, \cdot)$ be a magma. A subset S of M is called a *submagma* of M if it is closed for the operation of M, i.e., if $x, y \in S$ implies $xy \in S$, for all $x, y \in M$. The intersection of all submagmas of M that contain S is the smallest submagma of M containing S. It is denoted by $\langle S \rangle$. An operation $\cdot$ on M is said to be *power-associative* if the submagma $\langle x \rangle$ is associative, for every $x \in M$. In particular, every semigroup is power-associative.

An element e of a semigroup S is called an *identity* of S if $xe = ex = x$ for all $x \in S$. It is well known and easy to verify that a semigroup cannot have two distinct identity elements. This means that if a semigroup has an identity, then the identity is unique.

Let S be a semigroup with identity e, and let $x \in S$. An element $y \in S$ is called an *inverse* of x if $xy = yx = e$. It is also well known and easy to prove that if an element x has an inverse, then the inverse is unique. The inverse of x is denoted by x^{-1}, as no ambiguity can arise. A *group* is a semigroup with identity where every element has an inverse.

Let G be a group. The cardinality $|G|$ is also called the *order* of G. For $n \in \mathbb{N}$, the *cyclic group* of order n can be defined as the group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of residues modulo n.

A *subgroup* of G is a subset of G that is a group with the same identity and with respect to the same operation. In particular, every subgroup H of G is *closed* for the operation of the group, which means that $xy \in H$ for all $x, y \in H$. Besides, if H is a subgroup of G, then $x^{-1} \in H$ for all $x \in H$. A subgroup H of G is said to be *normal* if $g^{-1}hg \in H$ for all $h \in H$, $g \in G$. If H is a subgroup of G, then we write $H < G$. If H is a normal subgroup of G, then we write $H \lhd G$. If H is a maximal subgroup of G, then we write $H <_{\max} G$.

For any subset T of G, the intersection of all subgroups of G that contain T is the smallest subgroup of G containing T. It is denoted by $\langle T \rangle$ and is called the *subgroup generated by* T in G. In particular, the set $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$ is the cyclic group generated by g in G.

The *order* of the element x in a group or a semigroup is defined as the cardinality of $\langle x \rangle$. The order of x is denoted by $o(x)$. A group is said to be *torsion* if all of its elements have finite order.

The set of all elements of G of order k is denoted by $\Omega_k(G)$. The set of the orders of all elements of G is denoted by $\pi(G)$ and is called the *spectrum* of G. The *exponent* of a finite group G is the least common multiple of the orders of all elements of G. It is denoted by $\mathrm{Exp}(G)$.

Let p be a prime. Then all elements with orders equal to a power of p form a subgroup of G called a *primary component* of G, or a *p-primary component* of G. The p-primary component of a group G is denoted by G_p. A group G is called a *p-group* if $G = G_p$, i.e., if the order of every element of G is a power of p.

A group G is said to be an *EPO-group*, if the orders of all nontrivial elements of G are prime. An *EPPO-group* is a group where the order of every nontrivial element is a power of a prime.

The *dihedral* group D_{2n} is defined by the following presentation

$$D_{2n} = \langle x, y \mid x^n = y^2 = e, y^{-1}xy = x^{-1} \rangle.$$

The *semi-dihedral* group SD_{2^n} is presented by

$$SD_{2^n} = \langle x, y \mid x^{2^n-1} = y^2 = e, yxy = x^{2^{n-2}+1} \rangle.$$

The *symmetric group* $\mathbb{S}_n$ on a finite set of n letters is the group consisting of all permutations of these n letters with respect to the operation of composition of permutations, regarded as functions.

Let $S = (S, \cdot)$ be a semigroup. An equivalence relation ϱ on S is called a *congruence* if it is *compatible* with the operation $\cdot$ of S, that is if $(x, y) \in \varrho$ implies $(xz, yz) \in \varrho$ and $(zx, zy) \in \varrho$ for all $x, y, z \in S$. If ϱ is a congruence, then we can define an operation on the set S/ϱ of all equivalence classes of ϱ by the rule $[x][y] = [xy]$, where $[x]$ stands for the equivalence class containing x. This means that the product of two classes is the class containing the product of representatives of these classes. This definition makes sense, because it is easy to verify that the resulting class $[xy]$ depends only on the whole classes $[x]$ and $[y]$, but does not depend on the choice of particular representatives x and

y. It follows that the set S/ϱ is a semigroup too. It is called the *quotient semigroup* of S modulo ϱ.

Now let S be a semigroup, and let T be a subset of S. Then the intersection of all subsemigroups of S containing T is the smallest subsemigroup of S containing T. It is denoted by $\langle T \rangle$ and is called the *subsemigroup of S generated by T*. In particular, for any $x \in S$, the subsemigroup $\langle x \rangle$ coincides with the set $\{x^n \mid n \in \mathbb{N}\}$. The *ideal* (*left ideal* or *right ideal*) generated by T in S is the smallest subsemigroup I containing T and such that $GI \cup IG \subseteq I$ (respectively, $GI \subseteq I$, $IG \subseteq I$).

Let S be a semigroup with an ideal I. Then the *Rees quotient semigroup S/I* is the semigroup with zero θ obtained from S by identifying all elements of the ideal I with θ. If S has zero θ and $I = \{\theta\}$, then $S/I = S$.

An element of a semigroup is said to be *periodic* if it has a finite order. A semigroup entirely consisting of periodic elements is called a *periodic* semigroup.

An operation on a set S is *commutative* if it satisfies the commutative law, i.e., if $xy = yx$ for all $x, y \in S$. An *abelian group* is a group with commutative operation. A semigroup satisfying the commutative law is said to be commutative.

For a prime p, a group G is called an *elementary abelian p-group* if G is finite, abelian and every nontrivial element of G has order p. The *quasicyclic p-group* is the infinite group denoted by C_{p^∞} that has generators $g_1, g_2, \ldots$ such that $g_1^p = e$ and $g_i^p = g_{i-1}$, for all $i > 1$, where e is the identity of the group. Quasicyclic groups are the only infinite abelian groups all subgroups of which are finite.

Let G be a group with a subgroup H. A *right* (*left*) *coset* of the subgroup H is a set of the form Hg (respectively, gH), where $g \in G$. The set of left cosets has the same cardinality as the set of right cosets, and this cardinality is called the *index* of H in G. If N is a normal subgroup, then the *quotient group G/N* is the set of all right cosets of N in G with the following operations: $(Ng_1)(Ng_2) = Ng_1g_2$ and $(Ng)^{-1} = Ng^{-1}$, for all $g, g_1, g_2 \in G$.

The *center* of a semigroup S is the set $Z(S) = \{z \in S \mid \forall x \in S : xz = zx\}$ consisting of those elements that commute with all elements of S. The center of every group G is a normal subgroup of G. A group G is said to be *center-by-finite* if the quotient group $G/Z(G)$ is finite.

Rees matrix semigroups are very well known in semigroup theory and play crucial roles in describing the structure of semigroups (cf. [27]). For examples of applications of these concepts, let us also refer to [50–54]. Suppose that G is a group, and I, Λ are nonempty sets. Let $P = [p_{\lambda i}]$ be a $(\Lambda \times I)$-matrix with entries $p_{\lambda i} \in G^0$, for all $\lambda \in \Lambda$, $i \in I$. The *Rees matrix semigroup $M^0(G; I, \Lambda; P)$* over G with *sandwich-matrix* P consists of zero θ and all triples $(g; i, \lambda)$, where $g \in G^0$, $i \in I$, $\lambda \in \Lambda$, where all triples $(\theta; i, \lambda)$ are identified with θ, and where multiplication is defined by the rule $(g_1; i_1, \lambda_1)(g_2; i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2; i_1, \lambda_2)$, for all $g_1, g_2 \in G$, $i_1, i_2 \in I$, $\lambda_1, \lambda_2 \in \Lambda$.

A semigroup S is said to be *regular* if, for each $x \in S$, there exists $y \in S$ such that $xyx = x$. An element x of S is called an *idempotent* if $x^2 = x$.

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5 Eulerian graphs

A *walk* is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A *trail* is a walk without repeated edges. An *Eulerian circuit* of an undirected graph is a closed walk that traverses every edge of the graph exactly once. An *Eulerian* graph is a graph with an Eulerian circuit.

Theorem 1. ([25]) *The undirected power graph of any finite group of even order is not Eulerian.*

Proposition 1. ([25]) *For $n \geq 5$, the undirected power graph of the dihedral group D_{2n} is not Eulerian.*

Proposition 2. ([72]) *Let G be a finite group. Then the undirected power graph $\mathscr{G}(G)$ is Eulerian if and only if $|G|$ is odd.*

Theorem 2. ([71]) *Let G be a finite group. Then $\mathscr{G}^*(G)$ is Eulerian if and only if G is a cyclic 2-group or a generalized quaternion 2-group.*

6 Hamiltonian graphs

Proposition 3. ([72]) *Let G be a finite p-group. Then the undirected power graph $\mathscr{G}(G)$ has a Hamiltonian cycle if and only if $|G| \neq 2$ and G is cyclic.*

Proposition 4. ([72]) *Let p be an odd prime. Then the undirected power graph of a p-group is 2-connected if and only if it is Hamiltonian.*

Denote by U_n the group of units of the ring $\mathbb{Z}_n$.

Problem 1. ([23]) *Describe all positive integers n such that the undirected power graph $\mathscr{G}(U_n)$ is Hamiltonian.*

Theorem 3. ([23]) *Let $n \geq 3$. Then the graph $\mathscr{G}(U_n)$ is not Hamiltonian for $n = 2^m p_1 \cdots p_k$, where $p_1, \ldots, p_k$ are distinct Fermat primes, m and k are nonnegative integers, $m \geq 2$ for $k = 0, 1$, and $k \geq 2$ for $m = 0, 1$.*

It was conjectured in [23, p. 426] that the undirected power graph $\mathscr{G}(U_n)$ is Hamiltonian for all values $n \geq 3$ with the exception of $n = 2^m p_1 \cdots p_k$, where $p_1, \ldots, p_k$ are distinct Fermat primes, m and k are nonnegative integers, $m \geq 2$ for $k = 0, 1$ and $k \geq 2$ for $m = 0, 1$. The following collection of counterexamples to this conjecture was given in [72, Lemma 3].

Proposition 5. ([72]) *If $n = 2^v \times 3^2$, where $v \geq 3$, or $n = 2^t \times 7$, where $t \geq 2$, or $n = 2^2 3^2 p$, where p is a Fermat prime, then the undirected power graph $\mathscr{G}(U_n)$ does not have a Hamiltonian cycle.*

Theorem 4. ([23]) *If $n \geq 3$ is an integer, then $\mathscr{G}(\mathbb{Z})$ is Hamiltonian.*

7   Complete graphs

An undirected graph $D = (V, E)$ is said to be *complete* if any two distinct vertices of D are adjacent.

Theorem 5. ([23]) *Let S be a semigroup. Then the undirected power graph $\mathscr{G}(S)$ is complete if and only if the cyclic subsemigroups of S are linearly ordered with respect to the containment relation $\subseteq$.*

Theorem 6. ([23]) *A finite group has a complete undirected power graph if and only if it is cyclic and has order equal to p^m, where p is a prime and m is a nonnegative integer.*

Theorem 7. ([23]) *The undirected power graph $\mathscr{G}(U_n)$ is complete if and only if $n \in \{1, 2, 4, p, 2p\}$, where p is a Fermat prime, i.e., $p = 2^{2^m} + 1$ for $m \in \mathbb{Z}$, $m \geq 0$.*

Theorem 8. ([72]) *Let G be a finite p-group. Then the undirected power graph $\mathscr{G}(G)$ is a union of complete subgraphs which share the identity element of G if and only if G is isomorphic to a cyclic group, a p-group of exponent p, or a dihedral group.*

8   Edge number

Theorem 9. ([23]) *For each finite group G, the number of edges of the undirected power graph $\mathscr{G}(G)$ is given by the formula*

$$E(\mathscr{G}(G)) = \frac{1}{2} \sum_{g \in G} \{2o(g) - \phi(o(g)) - 1\}.$$

Corollary 1. ([23]) *The number of edges of the undirected power graph $\mathscr{G}(\mathbb{Z}_n)$ is given by $\frac{1}{2} \sum_{d|n} \{2d - \phi(d) - 1\} \phi(d)$.*

Several criteria for the nonsimplicity of G have been obtained in [71].

Theorem 10. ([72]) *If G is a finite simple group of order n, then $|E(\mathscr{G}(G))| \leq |E(\mathscr{G}(\mathbb{Z}_n))|$.*

Corollary 2. ([71]) *Let G be a finite group of order $n = p_1^{a_1} \cdots p_k^{a_k}$, where $k > 1$, $a_1, \ldots, a_k \geq 1$, and where $p_1, \ldots, p_k$ are pairwise distinct prime numbers. If $d(\mathscr{G}^*(G)) = 3$, then G is not simple.*

The *degree* of x in $\mathscr{G}(G)$ is denoted by $\deg(x)$.

Proposition 6. ([71]) *Let G be a finite group, and let p be the greatest prime divisor of $|G|$. If there exists a vertex x in $\mathscr{G}^*(G)$ with $\deg(x) \geq |G|/p$, then G is not simple.*

Proposition 7. ([71]) *Let G be a finite group, and let q be the smallest prime divisor of $|G|$. If there exists a vertex $x \neq e$ in $\mathscr{G}(G)$ such that $\deg(x) \geq |G|/q$, then $Z(G) \neq \{e\}$.*

9 Chromatic number

The *chromatic number* $\chi(G)$ of an undirected graph G is the smallest positive integer k such that the vertices of G can be coloured in k colours so that no two adjacent vertices share the same colour. A graph G is said to be *planar* if it can be drawn on a plane without any crossing of its edges.

Theorem 11. ([25]) *If $n = p^m$, for some prime p and for some $m \in \mathbb{N}$, then $\chi(\mathscr{G}(D_{2n})) = n$ and $\chi(\mathscr{G}(\mathbb{S}_n)) > n$. Moreover, for all such n with $n \geq 5$, the undirected power graph $\mathscr{G}(\mathbb{S}_n)$ is not planar.*

Theorem 12. ([25]) *The undirected power graph $\mathscr{G}(D_{2n})$ is not planar in each of the following two cases:*

(i) *$n = p^m$, for some prime p, $m \in \mathbb{N}$ and $n \geq 5$;*

(ii) *$n = 2p$, for some odd prime p.*

A *full exponent group* is a group G such that $\mathrm{Exp}(G) = o(x)$, for an element $x \in G$.

Theorem 13. ([69]) *Let G be a full exponent group such that $\mathrm{Exp}(G) = p_1^{b_1} \cdots p_k^{b_k}$, where $p_1, \ldots, p_k$ are prime numbers and $b_1, \ldots, b_k \in \mathbb{N}_0$. Then*

$$\omega(\mathscr{G}(G)) = \chi(\mathscr{G}(G)) = p_k^{b_k} + \sum_{j=0}^{k-2} \left(p_{k-j-1}^{b_{k-j-1}} - 1 \right) \left(\prod_{i=0}^{j} \phi(p_{k-i}^{b_{k-i}}) \right).$$

Conjecture 1. ([69]) *Theorem 13 remains valid for all finite groups.*

A complete list of the chromatic numbers of the undirected power graphs of non-abelian groups of order up to 14 are determined and listed in [25].

10 Clique number

Suppose $D = (V, E)$ is a graph. A subset X of the vertices of D is called a *clique* if it induces a complete graph of D. The maximum size of a clique in D is called the *clique number* of D and denoted by $\omega(D)$.

A subset Y of D is called an *independent set* if it induces a null subgraph in D. The maximum size of an independent set is called the *independence number* of D and denoted by $\alpha(D)$.

Theorem 14. ([69]) *Let $n = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1, \ldots, p_k$ are prime numbers and $a_1, \ldots, a_k \in \mathbb{N}_0$. Then the clique number and chromatic number of the undirected power graph $\mathscr{G}(\mathbb{Z}_n)$ are determined by the following formula*

$$\omega(\mathscr{G}(\mathbb{Z}_n)) = \chi(\mathscr{G}(\mathbb{Z}_n)) = p_k^{a_k} + \sum_{j=0}^{k-2} \left(p_{k-j-1}^{a_{k-j-1}} - 1 \right) \left(\prod_{i=0}^{j} \phi(p_{k-i}^{a_{k-i}}) \right).$$

The following theorem describes the directed power graphs $\overrightarrow{\mathscr{G}}(G)$ of all finite abelian groups G.

Theorem 15. ([39]) *Let G be a finite abelian group, and let a, b be any elements of G. Suppose that the prime factorization of the order of a is $|a| = \prod_{i=1}^{n} p_i{}^{t_i}$, where $1 \leq t_i \leq w_{i,m_i}$. Then*

(a) *a belongs to a clique of order*

$$\prod_{i=1}^{n} (p_i{}^{t_i} - p_i{}^{t_i-1}),$$

where we replace $(p_i{}^{t_i} - p_i{}^{t_i-1})$ by 1 if $t_i = 0$;

(b) *(a, b) is an edge of the directed power graph of G if and only if, for every $i = 1, \ldots, n$,*

$$p_i{}^{w_{i,j}} \,\big|\, v_{i,j,k} u_{i,j,k}^{\phi(p_i{}^{w_{i,j}})-1} p_i{}^{r_{i,j,k}-s_{i,j,k}} - v_{i,j',k'} u_{i,j',k'}^{\phi(p_i{}^{w_{i,j'}})-1} p_i{}^{r_{i,j',k'}-s_{i,j',k'}},$$

for all $1 \leq j \leq j' \leq m_i$, and $1 \leq k \leq k' \leq q_{i,j'}$.

(c) *If w_{i,h_i} is the smallest exponent in G_{p_i} such that $t_i \leq w_{i,h_i}$ then $\overrightarrow{\mathscr{G}}(G)$ contains*

$$\prod_{i=1}^{n} \frac{(p_i{}^{w_{i,1}})^{q_{i,1}} (p_i{}^{w_{i,2}})^{q_{i,2}} \cdots (p_i{}^{w_{i,h_i-1}})^{q_{i,h_i-1}} \left((p_i{}^{t_i})^{q_{i,h_i}+\cdots+q_{i,m_i}} - (p_i{}^{t_i-1})^{q_{i,h_i}+\cdots+q_{i,m_i}}\right)}{(p_i{}^{t_i} - p_i{}^{t_i-1})}$$

cliques of order $\prod_{i=1}^{w} (p_i{}^{t_i} - p_i{}^{t_i-1})$, for each t_i. If $t_i = 0$ for any i then we replace $(p_i{}^{t_i} - p_i{}^{t_i-1})$ by 1.

11   Planarity

Theorem 16. ([69]) *Let G be a finite group. Then the undirected power graph $\mathscr{G}(G)$ is planar if and only if $\pi_e(G) \subseteq \{1, 2, 3, 4\}$.*

Corollary 3. ([69]) *The undirected power graph of a cyclic group of order n is planar if and only if $n \in \{2, 3, 4\}$.*

Corollary 4. ([23]) *The undirected power graph of U_n is planar if and only if n divides 240.*

Corollary 5. ([69]) *$\chi(\mathscr{G}(D_{2n})) = \omega(\mathscr{G}(D_{2n})) = \chi(\mathscr{G}(\mathbb{Z}_n))$.*

Corollary 6. ([69]) *The undirected power graph $\mathscr{G}(SD_{2^n})$ is a union of a complete graph of order 2^n and 2^n copies of K_2 that share the identity vertex. For $n > 3$, this graph is neither Eulerian, nor Hamiltonian, nor nonplanar. Moreover, $\chi(\mathscr{G}(SD_{2^n})) = \omega(\mathscr{G}(SD_{2^n})) = \alpha(\mathscr{G}(SD_{2^n})) = 2^n$.*

Theorem 17. ([69]) *Suppose that $n = p_1^{b_1} \cdots p_k^{b_k}$ is the prime power decomposition of n and $m = b_1 + \cdots + b_k$. Then $\alpha(\mathscr{G}(\mathbb{Z}_n))$ is the coefficient of the middle term or the two middle terms of $\prod_{j=1}^{m}(1 + x + \cdots + x^{b_j})$.*

It is easily seen that the undirected power graph of a p-group G is a union of complete graphs of order p sharing the identity vertex if and only if G has exponent p. The following theorem describes all finite groups with this property.

Theorem 18. ([69]) *The undirected power graph $\mathscr{G}(G)$ is a union of complete graphs which share the identity element of G if and only if G is an EPPO-group and for any two distinct maximal cyclic subgroups A and B their intersection $A \cap B$ is equal to $\{e\}$.*

Corollary 7. ([69]) *If G is an EPO-group then the undirected power graph $\mathscr{G}(G)$ is a union of complete graphs sharing the identity element of G.*

Proposition 8. ([69]) *A finite group G is an EPPO-group if and only if the vertices of every maximal clique of the undirected power graph $\mathscr{G}(G)$ form a maximal cyclic subgroup of G.*

11.1 Isomorphism

The following theorem shows that undirected power graphs can characterize all finite simple groups.

Theorem 19. ([69]) *Let G_1 be a finite group that is a simple group, or a cyclic group, or a symmetric group, or a dihedral group, or a generalised quaternion group. If G_2 is a finite group such that $\mathscr{G}(G_1) \cong \mathscr{G}(G_2)$, then $G_1 \cong G_2$.*

Theorem 20. ([22])*Let G_1 and G_2 be finite abelian groups such that $\mathscr{G}(G_1) \cong \mathscr{G}(G_2)$. Then $G_1 \cong G_2$.*

It is explained in [22] that Theorem 20 does not generalize to infinite abelian groups. Indeed, let $C_{p^{infty}}$ be the group $\mathbb{Q}_p/\mathbb{Z}$, where p is a prime and Q_p is the group of rational numbers with p-power denominators. Then $\mathscr{G}(C_{p^\infty})$ is a countably infinite complete graph, independent of the chosen prime.

It is also impossible to drop the condition that G be abelian from Theorem 20 either. Indeed, let G be a finite group of exponent 3, that is, satisfying $x^3 = e$ for all $x \in G$. Then the undirected power graph $\mathscr{G}(G)$ consists of $(|G| - 1)/2$ triangles sharing the identity vertex. The elementary abelian group (the direct product of cyclic groups of order 3) has exponent 3. However, there are nonabelian groups of exponent 3 as well. The smallest one is the group of order 27 with presentation

$$G = \langle x, y \mid x^3 = y^3 = [x, y]^3 = 1 \rangle,$$

where $[x, y]$ stands for the commutator $x^{-1}y^{-1}xy$.

It is noted in [22] that a short calculation with GAP [80] and its package GRAPE [78] revealed that there are two pairs of groups of order 16 with isomorphic undirected power graphs, and two pairs of groups of order 27. As the order of the group increases, isomorphic power graphs become more common. For example, for order 32 there are one quadruple, two triples, and eight pairs. It is also possible to use Magma [18] for experimental verification of properties of this sort.

Theorem 21. ([69]) *Let G be a finite group. The undirected power graph $\mathscr{G}(G)$ is bipartite if and only if G is an elementary abelian group of even order.*

Theorem 22. ([1]) *Among all finite groups of a given order, the cyclic group of that order has the largest number of edges in its directed power graph.*

Theorem 23. ([69]) *The undirected power graph $\mathscr{G}(\mathbb{Z}_{p^n})$ has the maximum number of edges among all undirected power graphs of p-groups of order p^n.*

A conjecture was recorded in [69, Conjecture 2] that Theorem 23 remains valid for finite groups of any order. This conjecture was proved in [70].

Theorem 24. ([70]) *The undirected power graph $\mathscr{G}(\mathbb{Z}_n)$ has the maximum number of edges among all undirected power graphs of groups of order n.*

Theorem 25. ([22]) *Let G_1 and G_2 be finite groups with $\overrightarrow{\mathscr{G}}(G_1) \cong \overrightarrow{\mathscr{G}}(G_2)$. Then G_1 and G_2 have the same numbers of elements of each order.*

Conjecture 2. ([22]) *Let G_1 and G_2 be finite groups with $\mathscr{G}(G_1) \cong \mathscr{G}(G_2)$. Then G_1 and G_2 have the same numbers of elements of each order.*

It is noted in [22] that this conjecture has been verified computationally using GAP [80] for all groups of order up to 511. The converse to the conjecture fails for groups of order 16.

Theorem 26. ([22]) *The only finite group G for which $\mathrm{Aut}(G) = \mathrm{Aut}(\mathscr{G}(G))$ is the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Theorem 27. ([21]) *Two finite groups which have isomorphic undirected power graphs must have isomorphic directed power graphs.*

Corollary 8. ([21]) *Two finite groups whose undirected power graphs are isomorphic have the same numbers of elements of each order.*

Theorem 28. ([21]) *Let G be a finite group, and let S be the set of vertices of the undirected power graph $\mathscr{G}(G)$ which are joined to all other vertices. Suppose that $|S| > 1$. Then one of the following occurs:*

(a) *G is cyclic of prime power order and $S = G$;*

(b) *G is cyclic of order n that is not a power of a prime, and S consists of the identity and the generators of G, so that $|S| = 1 + \phi(n)$;*

(c) *G is generalised quaternion, and S contains the identity and the unique involution in G, so that $|S| = 2$.*

11

12 Connected graphs

Let us define a binary relation ϱ on S by $a\varrho b \Leftrightarrow a^m = b^m$, for some $m \in \mathbb{N}$.

Theorem 29. ([23]) *Let S be a finite semigroup, and let $a, b \in S$, $a \neq b$. Then a and b are connected by a path in the graph $\mathscr{G}(S)$ if and only if $a\varrho b$.*

Corollary 9. ([23]) *Let S be a finite semigroup. Then the connected components of the undirected power graph $\mathscr{G}(S)$ are precisely the sets C_e defined by*

$$C_e = \{x \in S \mid x\varrho e\} = \{x \in S \mid \exists m \in \mathbb{N} : x^m = e\},$$

for all idempotents e in S.

A description of all idempotents of $\mathbb{Z}_m$ is given in [23, §3]. In particular, it is shown that the graph $\mathscr{G}(\mathbb{Z}_m)$ is disconnected for all $m > 1$ and that it is Eulerian if and only if $n = 2$.

Corollary 10. ([23]) *Let S be a finite semigroup, then $\mathscr{G}(S)$ is connected if and only if S contains a single idempotent.*

Proposition 9. ([23]) *Let S be a semigroup such that $\mathscr{G}(S)$ is connected. Then S contains at most one idempotent.*

Corollary 11. ([23]) *Let S be a regular semigroup. If $\mathscr{G}(S)$ is connected, then S is a group.*

Theorem 30. ([23]) *Let G be a group. Then the undirected power graph $\mathscr{G}(G)$ is connected if and only if G is a torsion group.*

The minimum number $\kappa(D)$ of vertices of D which need to be removed to make the remaining graph disconnected is called the *connectivity* of D.

If G is finite group, then we define

$$M(G) = \{x \in G \mid x <_{\max} G\}.$$

Theorem 31. ([69]) *Let G be a finite group that is not a cyclic group. Define $M(G) = \{g \in G \mid g <_{\max} G\}$. For any element x of G, put $r(x) = \cup_{y \in M(G) - \langle x \rangle}(\langle x \rangle \cap \langle y \rangle)$. Then*

$$\kappa(\mathscr{G}(G)) \leq \min\{|r(x)| \mid \langle x \rangle <_{\max} G\}.$$

13 2-connected graphs

A cut vertex in an undirected graph is a vertex whose removal increases the number of connected components of the graph. A graph is said to be 2-*connected* if it does not have a cut vertex.

Theorem 32. ([72]) *Suppose that G is a p-group. The undirected power graph $\mathscr{G}(G)$ is 2-connected if and only if G is a cyclic group or generalized quaternion 2-group.*

Theorem 33. ([72]) *Let G be a nilpotent group. If G is not a p-group then the undirected power graph $\mathscr{G}(G)$ is 2-connected.*

It is noted in [72] that the converse to Theorem 33 is not true. Indeed, the group $G = \mathbb{Z}_5 \times \mathbb{S}_3$ is not nilpotent, but its undirected power graph is 2-connected. Besides, the dihedral group D_{2p}, where p is a prime, is a solvable group such that the undirected power graph $\mathscr{G}(D_{2p})$ is not 2-connected. Moreover, the following proposition holds.

Proposition 10. ([72]) *If A and B are groups of coprime orders such that A is cyclic of prime order, then the undirected power graph $\mathscr{G}(A \times B)$ is 2-connected.*

Question 1. ([72]) *Does there exist a nonabelian simple group with a 2-connected power graph?*

14 Diameter

The *distance $d(x, y)$* between vertices x and y of a connected graph $D = (V, E)$ is the length of a shortest path connecting them. The maximum possible distance in D is called the *diameter* of D and is denoted by $d(D)$. For example, in [9] a voltage assignment approach was used to construct large digraphs with small diameter.

The deletion of a single edge or vertex from a graph can change its diameter. Since the identity element of a group is adjacent to all other vertices of the undirected power graph of the group, one can delete any vertex, which is not the identity of the group, from the graph without changing its diameter. It would be interesting to determine when the same property holds for the identity vertex too. The following theorem gives a complete description of the structure of finite groups satisfying this property.

Theorem 34. ([71]) *Let G be a finite group. Then $d(\mathscr{G}(G)) = d(\mathscr{G}^*(G))$ if and only if G is nilpotent and its Sylow subgroups are cyclic groups or generalized quaternion 2-groups.*

Theorem 35. ([71]) *Let G be a finite group such that $d(\mathscr{G}^*(G)) = 3$. Then the Sylow subgroups of G are generalized quaternion 2-groups or cyclic and G is not nilpotent.*

15 Girth

The girth of an undirected graph G is denoted by $g(G)$ and is defined as the length of a shortest cycle in G.

Proposition 11. ([72]) *The girth of undirected power graphs $\mathscr{G}(G)$ is equal to 3 if and only if G is not an elementary abelian 2-group. Moreover, if $\mathscr{G}(G)$ is 2-connected then $g(\mathscr{G}(G) - e) = 3$.*

Proposition 12. ([72]) *The undirected power graph of a finite group G is a tree if and only if G is an elementary abelian 2-group.*

16 Combinatorial properties expressed as unavoidable regularities

Combinatorial properties of groups and semigroups defined in terms of all infinite subsets enjoying certain unavoidable regularities have been investigated by many authors, and a survey of this direction of research has been given in [30], (cf. [28, 29, 37, 49]).

The following combinatorial property of this sort has been defined in terms of directed power graphs in [39]. Let $D = (V, E)$ be a directed graph. A semigroup S is said to be *power D-saturated* if and only if, for every infinite subset T of S, the directed power graph of S has a subgraph isomorphic to D with all vertices in T.

A directed graph is said to be *acyclic* if it has no directed cycles.

Denote by T_∞ the transitive tournament with the vertex set $\mathbb{N}$ and the edge set $E(T_\infty) = \{(m, n) \mid m > n\}$.

If S is finite or if D is a null graph, then it is obvious that S is power D-saturated. Therefore the following theorems deal only with infinite groups or semigroups and with directed graphs which have edges.

Theorem 36. ([39]) *Let $D = (V, E)$ be a directed graph with $E \neq \emptyset$, and let G be an infinite group. Then G is power D-saturated if and only if G is a center-by-finite torsion group, the center $Z(G)$ has a finite number of primary components, each primary component of $Z(G)$ is finite or quasicyclic, the order of $G/Z(G)$ is not divisible by p for each quasicyclic p-subgroup of G, and D is isomorphic to a subgraph of T_∞.*

For a skew field K, the set of all $n \times n$ matrices with entries in K is denoted by $M_n(K)$. A matrix is said to be *monomial* if every row and column contains at most one nonzero entry. If G is a group, then the set of all $n \times n$ monomial matrices over $G^0 = G \cup \{0\}$ forms a semigroup denoted by $M_n(G)$. Thus a *matrix semigroup* is a subsemigroup of $M_n(K)$ or $M_n(G)$, for some n, K and G.

Theorem 37. ([43]) *Let $D = (V, E)$ be a directed graph with $E \neq \emptyset$, K a skew field, G a group, and let S be an infinite matrix semigroup in $M_n(K)$ or in $M_n(G)$. Then S is power D-saturated if and only if D is acyclic and all but a finite number of elements of S are contained in the union of a finite number of center-by-finite torsion groups H_i, where $i = 1, \ldots, k$, such that the center $Z(H_i)$ of each H_i has a finite number of primary components, each primary component of $Z(H_i)$ is finite or quasicyclic, and the order of $H_i/Z(H_i)$ is not divisible by p for each quasicyclic p-subgroup of H_i.*

Theorem 38. ([40]) *Let $D = (V, E)$ be a finite graph with $E \neq \emptyset$, and let S be an infinite commutative semigroup. Then S is power D-saturated if and only if the following conditions hold:*

(i) D *is acyclic,*

(ii) S *is periodic,*

(iii) S *has a finite number of idempotents,*

(iv) *all but a finite number of elements of S belong to the union of all subgroups of S,*

(v) *every subgroup of S has a finite number of primary components,*

(vi) *for every prime p, each p-subgroup of S is either finite or quasicyclic.*

Theorem 39. *([42]) Let D be a finite directed graph that is not null, and let S be an infinite semigroup. Then the following conditions are equivalent:*

(i) *the directed power graph of S is D-saturated;*

(ii) *D is acyclic and S^0 has a finite ideal series*

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = S^0 \tag{1}$$

where every infinite Rees quotient I_j/I_{j-1} is a Rees matrix semigroup that has a finite sandwich matrix with entries in a center-by-finite torsion group H_j such that each primary component of the center of H_j is finite or quasicyclic, the center of H_j has only a finite number of primary components, and the index of the center is not divisible by p for each quasicyclic p-subgroup of H_j.

17   Open questions

A *Moore graph* or digraph is a graph or digraph that meets the Moore bound or directed Moore bound, respectively. Let us refer to the survey [68] and articles [10–16, 76, 77] for more information and examples of previous results pertaining to the Moore graphs and Moore bound.

Problem 2. *Does there exist any undirected or directed power graph of a group or a semigroup, which is a Moore graph or digraph? Determine for which positive integers k there exist directed power graphs that are at distance k from achieving the optimality in terms of the Moore bound. Furthermore, for each positive integer k, describe all (a) groups and (b) semigroups whose directed power graphs are at distance k from achieving the Moore bound.*

Let us refer to the survey [20] for previous results on graceful labellings.

Problem 3. *Describe all groups and semigroups whose undirected power graphs have gaceful labellings.*

Edge-magic total labellings of graphs were considered, for example, in [81].

Problem 4. *For each undirected power graph of a group or a semigroup, determine the number of edge-magic total labellings of the graph.*

We refer to the survey [5] and papers [6, 7] for information on the edge antimagic labellings of graphs.

Problem 5. *For each undirected power graph determine the number of edge antimagic labellings of the graph.*

Let us refer to [8, 79] for preliminaries on super edge-antimagic labellings of graphs.

Problem 6. *For each undirected power graph of a group or a semigroup, determine the number of super edge-antimagic labellings of the graph.*

The readers are referred to [17] for background information on the Ramsey numbers of graphs.

Problem 7. *Determine the Ramsey numbers of the undirected power graphs of all groups and semigroups.*

Let us refer to [2–4] for more information concerning the face antimagic labellings of planar graphs.

Problem 8. *For each planar undirected power graph of a group or a semigroup, determine the number of face antimagic labellings of the graph.*

It is also essential to investigate the relations of power graphs and other important classes of graphs associated to groups and semigroups. One of the largest classes of graphs of this sort is that of Cayley graphs.

Let G be a semigroup, and let S be a nonempty subset of G. As in [38], we say that the *Cayley graph* $\mathrm{Cay}(G, S)$ of G relative to S is defined as the directed graph with vertex set G and edge set $E(S)$ consisting of those ordered pairs (x, y) such that $sx = y$ for some $s \in S$. The set S is called the *connection set* of the Cayley graph $\mathrm{Cay}(G, S)$. The Cayley graph $\mathrm{Cay}(G, S)$ is also called the *Cayley digraph*. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *undirected* if, $(u, v) \in E(S)$ implies $(v, u) \in E(S)$. On the other hand, the underlying undirected graph $\overleftrightarrow{\mathrm{Cay}}(G, S)$ of $\mathrm{Cay}(G, S)$ can be called the *undirected Cayley graph* of G relative to S.

Cayley graphs of semigroups have broad applications in versatile areas, as evidenced by the extensive bibliography given in the survey [44]. Besides, Cayley graphs of semigroups are closely related to automata theory, as explained in the monograph [34] and papers [32, 36, 41]. This is a very large topic, and so without trying to be complete we indicate only a few pertinent papers [19, 31, 35, 55–67, 75, 76, 83–85] just as an illustration.

Problem 9. *Describe all directed and undirected power graphs of groups and semigroups that can be represented as Cayley graphs and, respectively, undirected Cayley graphs of groups or semigroups.*

Let $D = (V, E)$ be a directed graph. The *graph algebra* $\mathrm{Alg}(D)$ of D is the set $V \cup \{0\}$ equipped with multiplication defined by the rule

$$xy = \begin{cases} x & \text{if } x, y \in V \text{ and } (x, y) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let us refer to the monograph [34] and papers [36, 46–48] on graph algebras.

Problem 10. *Describe all directed graphs whose graph algebras are power-associative.*

The following problem makes sense for directed graphs and undirected graphs with appropriate adjustments.

Problem 11. *For each power-associative graph algebra $\mathrm{Alg}(D)$, determine the girth, diameter, connectivity, and the chromatic number of its directed power graph $\overrightarrow{\mathscr{G}}(\mathrm{Alg}(D))$ and undirected power graph $\mathscr{G}(\mathrm{Alg}(D))$. Find conditions necessary and sufficient for the directed power graph $\overrightarrow{\mathscr{G}}(\mathrm{Alg}(D))$ and undirected power graph $\mathscr{G}(\mathrm{Alg}(D))$ to be planar.*

Problem 12. *Describe all directed graphs D such that the graph algebra $\mathrm{Alg}(D)$ is power-associative and its undirected power graph has a graceful labelling.*

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