



## Determination of all graphs whose eccentric graphs are clusters

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### Abstract

A disconnected graph  $G$  is called a cluster if  $G$  is not union of  $K_{2s}$  (1-factor) but union of complete graphs of order at least two. J. Akiyama, K. Ando and D. Avis showed in Lemma 2.1 of [2] that  $G$  is equi-eccentric if the eccentric graph  $G_e$  is a cluster or  $pK_2$ . And they also characterized all graphs whose eccentric graphs are complete graphs and  $pK_2$  in [2]. In this paper, we determined in Thoerm 2 all graphs whose eccentric graphs are clusters, which is an extension of Lemma 2.1 in [2].

*Keywords:* eccentricity, eccentric graph, cluster, distance, radius

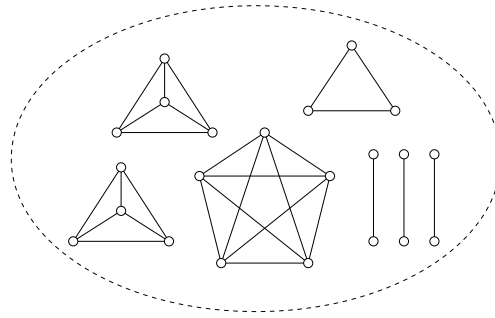
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### 1. Introduction

Let  $G = (V(G), E(G))$  be a simple undirected graph. A disconnected graph  $G$  is called a **cluster** if  $G$  is not union of  $K_{2s}$  (1-factor) but union of complete graphs  $\bigcup_{i=1}^n K_{p_i}$  ( $n \geq 2, p_i \geq 2$ ; Figure 1).

The **eccentricity**  $e(v)$  of a vertex  $v$  in  $V(G)$  is defined by  $e(v) = \max_{u \in V(G)} d(u, v)$ , where  $d(u, v)$  stands for the length of the shortest path in  $G$  between  $u$  and  $v$ . We denote by  $G_e = (V(G_e), E(G_e))$  the **eccentric graph** based on  $G$  (Figure 2), where the vertex set  $V(G_e)$  is identical to  $V(G)$  and  $uv \in E(G_e) \Leftrightarrow d(u, v) = \min(e(u), e(v))$ .

A similar graph, called the “furthest neighbor graph”, was introduced by Shamos [8]. Its vertex



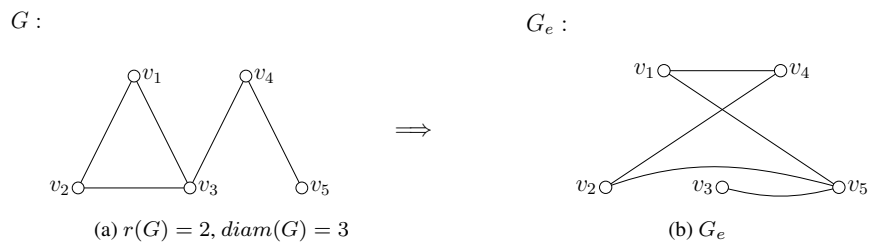
**Figure 1:** Example of a cluster  $G = 3K_2 \cup K_3 \cup 2K_4 \cup K_5$

set is a set of points in the plane, and the distance between two vertices is their Euclidean distance. Two vertices are joined if either one is the “furthest neighbor” of the other. Extremal properties of this graph are studied in [5].

A central vertex of a graph  $G$  is a vertex  $v$  with the property that the maximum distance between  $v$  and any other vertex is as small as possible, this distance being called the **radius**, denoted by  $r(G)$ . That is,  $r(G) = \min_{v \in V(G)} e(v)$ . The **diameter** of  $G$  denoted  $diam(G)$  is defined by  $diam(G) = \max_{v \in V(G)} e(v)$ .

A graph is a **self-center** [4] or **r-equi-eccentric** (or briefly **r-equi**) [1] if  $e(v) = r(G) = diam(G)$  for all vertices  $v \in V(G)$  (see Figure 3). If  $S \subset V(G)$ , then we say that  $e(S) = i$  if  $e(v) = i$  for all  $v \in S$ , and we denote by  $\langle S \rangle$  the subgraph induced by  $S$ . We denote by  $N(v)$  the **neighborhood** of a vertex  $v$  of  $G$  consisting of the vertices in  $G$  adjacent to  $v$ . The **closed neighborhood**  $N[v]$  of  $v$  is defined by  $N[v] = N(v) \cup \{v\}$ .

All other definitions and notations used in this paper might be found in [4] or [6].



**Figure 2:** Example of a graph and its eccentric graph



**Figure 3:** Examples of a 3-equi and a 4-equi-eccentric graph

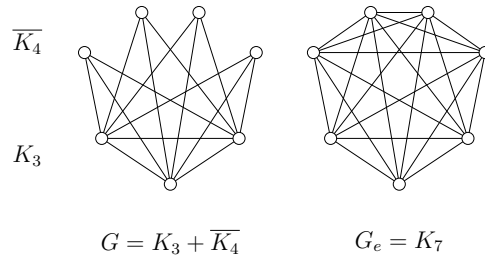
**Lemma A** ([2, Lemma 2.1]). *If  $G_e = \bigcup_{i=1}^n K_{p_i}$ , where  $\sum_{i=1}^n p_i = p, p_i \geq 2 (i = 1, 2, \dots, n)$  and  $n \geq 2$ , then  $G$  is equi-eccentric.*

**Theorem A** ([2, Theorem 2.2]). *If  $G$  is a graph of order  $2p$  then  $G_e = pK_2$  if and only if  $G$  is radius critical, that is,  $r(G - v) = r(G) - 1$  for all  $v \in V(G)$ .*

**Theorem B** ([2, Theorem 2.1]). *For any connected graph  $G$  of order  $p$ ,  $G_e = K_p$  if and only if for all  $v \in V(G)$  either  $e(v) = 1$  or  $e(N(v)) = 1$ .*

Theorem B can be restated as follows and we give a different proof of it from one in [2]:

**Theorem B'.** *A graph  $G$  of order  $p$  whose eccentric graph is a complete graph  $K_p$  if and only if  $G$  is a join of a complete graph  $K_m$  and  $n$  isolated vertices  $\overline{K_n}$  i.e.,  $G = K_m + \overline{K_n}, m+n = p, m \neq 0$ .*



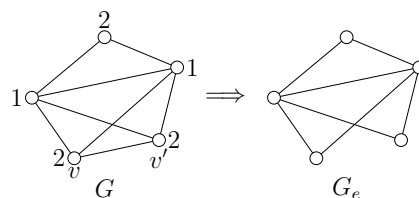
**Figure 4:** Example of Theorem B'

*Proof.* We divide our proof into two cases depending on  $r(G) = 1$  or not.

Case 1 Suppose that  $r(G)=1$ .

Let  $S(i)$  be a set of all vertices  $v$  of  $G$  with  $e(v) = i$ . Since  $S(i) = \emptyset$  for every  $i(\geq 3)$ , we have that  $V(G) = S(1) \cup S(2), S(1) \cap S(2) = \emptyset$ . Let  $|S(1)|, |S(2)|$  be  $m, n$ , respectively. The induced subgraph  $\langle S(1) \rangle$  of  $G$  is a complete subgraph  $K_m$  of  $G$ .

If there exists at least one edge  $vv' \in E(G)$  where both  $v, v' \in S(2)$ , then we have that  $vv' \notin E(G_e)$ , which implies that  $G_e$  is not a complete graph (Figure 5). Therefore,  $\langle S(2) \rangle$  must be a totally disconnected graph  $\overline{K_n}$ . That is,  $G$  must be a  $K_m + \overline{K_n}$ . Conversely, if  $G = K_m + \overline{K_n}$ , then  $G_e$  is a complete graph  $K_{m+n}$ .



**Figure 5:** Example of Case 1 with the value of eccentricity of vertices in  $G$

Case 2 Suppose that  $r(G) \geq 2$ .

For any vertex  $v$  of  $G$ , there is a vertex  $v'$  which is adjacent to  $v$ , implying that  $d(v, v') = 1$ . Since  $e(v) \neq 1 \neq e(v')$ , we have that  $vv'$  is not an edge of  $G_e$ . Therefore,  $G_e$  cannot be a complete graph. □

## 2. Main results

**Theorem 1.** If  $r(G) \geq 2$ , then  $(G + \overline{K_n})_e = \overline{(G + \overline{K_n})} = \overline{G} \cup K_n, n \geq 2$ .

*Proof.* Since  $r(G) \geq 2$ , there is no vertex  $v$  such that  $e(v) = 1$  in  $G$ . Therefore,  $G + \overline{K_n}$  is 2-equi (Figure 6). Hence  $uv \in E(G + \overline{K_n})$  if and only if  $uv \notin E((G + \overline{K_n})_e)$ , which implies that  $(G + \overline{K_n})_e$  is isomorphic to the complement of  $(G + \overline{K_n})$ ; i.e.,  $\overline{G + \overline{K_n}}$ . Moreover, the complement  $\overline{G + \overline{K_n}}$  is  $\overline{G} \cup K_n$ . □

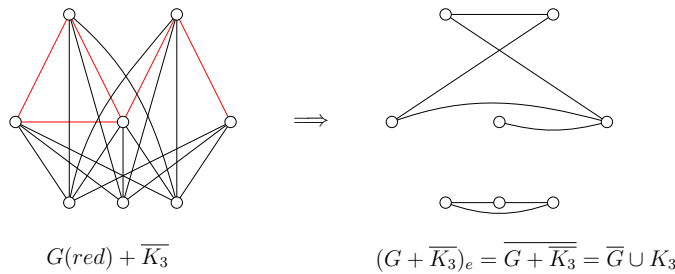


Figure 6: Example of Theorem 1

Putting  $G = K(m_1, m_2, \dots, m_n)$ , we obtain the Corollary 1.

**Corollary 1.** For  $m_i \geq 2$  for  $i(1 \leq i \leq n)$  and  $\ell \geq 2$ ,  
 $(K(m_1, m_2, \dots, m_n) + \overline{K_\ell})_e = K_{m_1} \cup K_{m_2} \cup \dots \cup K_{m_n} \cup K_\ell$ .

**Proposition 1.**  $(C_{2p})_e = pK_2$  (see Figure 7a).  $(C_{2p+1})_e = C_{2p+1}$  (see Figure 7b).

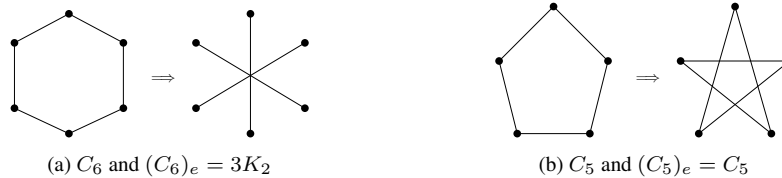


Figure 7:  $C_6$  and  $C_5$ , and its eccentric graphs

**Theorem 2.** A graph whose eccentric graph is a cluster, i.e.,  $G_e = \cup_{i=1}^n K_{p_i} (n \geq 2, p_i \geq 2$  for any  $i(1 \leq i \leq p)$ , and at least one  $p_i$  is not 2,  $\sum_{i=1}^n p_i = p)$ , if and only if  $G$  is a complete  $n$ -partite graph  $K(p_1, p_2, \dots, p_n)$ .

*Proof.* Since  $n \geq 2$ ,  $G_e = \cup_{i=1}^n K_{p_i}$  is not connected. Then, there exists no vertex  $v \in V(G)$  such that  $e(v) = 1$  (if not,  $G_e$  would be connected). Let  $k = \text{diam}(G)$  and let  $x$  and  $y$  be any pair of vertices in  $G$  such that  $d(x, y) = k$ . Note that  $xy$  is an edge of  $G_e$ .

Let  $z_k$  be a vertex in  $N(x)$  such that  $d(z_k, y) = k - 1$  (Figure 8a). For any vertex  $z$  in  $N(x)$ , if every path between  $z$  and  $y$  passes through  $x$ , then  $d(z, y) > k$ , which is contradiction. Therefore, there exists at least one path between  $z$  and  $y$  not passing through  $x$  for any  $z \in N(x)$ .

If  $d(y, z) < k - 1$  for some  $z \in N(x)$ ,  $d(x, y) = d(y, z) + d(z, x) < k$ , which is a contradiction.

If  $d(y, z) > k - 1$  for some  $z \in N(x)$ , that is  $d(y, z) = k (= \text{diam}(G))$ ,  $e(z) = k$  (Figure 8b). And then,  $xy \in G_e$ ,  $yz \in G_e$ , but  $xz \notin G_e$ , since  $d(x, z) = 1 \neq e(x)$  or  $e(z)$ . It means that  $G_e$  is not  $\cup K_{p_i}$  (i. e., vertices  $x, y$  and  $z$  do not form a part of a complete graph), which is a contradiction. Then,  $d(y, z) = k - 1$  for any vertex  $z$  in  $N(x)$ , if  $G_e$  is a cluster.

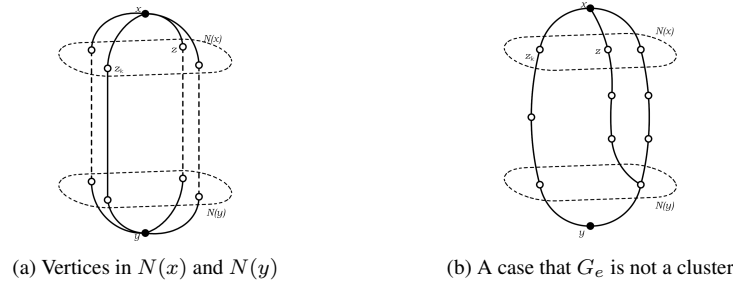


Figure 8

Due to the above results,  $G$  includes a  $k$ -equi-eccentric graph  $G'$  which is composed of all paths  $xz_i \cup z_iy$  where  $z_i \in N(x)$ .

Note that the number of  $N(y)$  is more than 1. Otherwise, for a unique point  $w \in N(y)$ ,  $e(w) = d(x, w)$  and  $d(w, y) = 1$ . Then,  $xy \in G_e$ ,  $xw \in G_e$  but  $yw \notin G_e$ , which means that vertices  $x, y$  and  $w$  do not form a part of complete graph, that is,  $G_e$  cannot be a cluster.

Any  $G'$  can be constructed by applying one of or combination of following operations to  $G_L$ , which is a union of  $n$  disjoint paths  $P_i = xz_i \cup z_iy$  ( $i = 1, 2, \dots, n$ ) where  $n = |N(x)|, |V(G_L)| = |V(G')|$  and the length of  $P_i$  is  $k$  (Figure 9):

Operation 1 Add an edge between points  $p_i$  and  $p_j$  where  $d(p_i, x) + 1 = d(p_j, x)$ .

Operation 2 Identify a point  $p_i$  and some points  $p_j$ s where  $1 < d(p_i, x) = d(p_j, x) = \ell < k - 1$ , but at least two of them for each  $\ell$  must be distinct. And add  $m$  points  $p_k$ s and edges so that  $m$  is just the reduced number above and  $d(p_k, x) \neq \ell$ .

For any  $z \in N(x)$ , there must exist at least one point  $w \in N(y)$  such that  $e(z) = d(z, w) = k$ , if  $G_e$  is a cluster. We can classify the situation into the following two cases by the maximum number of  $w$  for all  $z \in N(x)$ .

Case 1 The maximum number of  $w$  is more than 1 for all  $z \in N(x)$ .

In this case,  $|N(x)| \geq 3$  and  $|N(y)| \geq 3$ . For some  $z_i$  in  $N(x)$ , there exist at least two

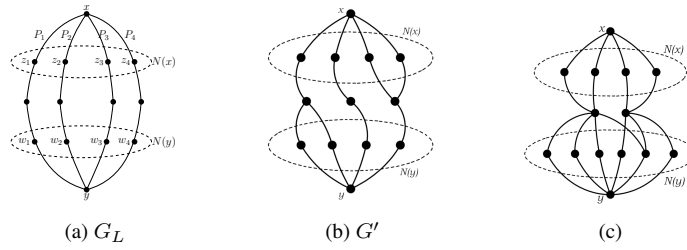


Figure 9: Examples of  $G_L$  and  $G'$  in the case of  $k = 4$

vertices  $w_j$  and  $w_\ell$  in  $N(y)$  such that  $d(z_i, w_j) = k, d(z_i, w_\ell) = k$ . That is, any paths  $z_i y$  with  $d(z_i, y) = k - 1$  never pass through  $w_j$  or  $w_\ell$  (Figure 10a). Then both  $z_i w_j$  and  $z_i w_\ell$  are edges of  $G_e$ , thus  $w_j w_\ell$  must also be an edge of  $G_e$ . Since  $d(w_j, w_\ell) = 2$  and  $e(w_j) = e(w_\ell) = k, k$  must be 2, which implies that  $N(x) = N(y)$  (i.e.,  $z_i$  coincides with  $w_i$  for  $i = 1, 2, \dots, m$ ) and  $G$  is 2-equi-eccentric (Figure 10b).

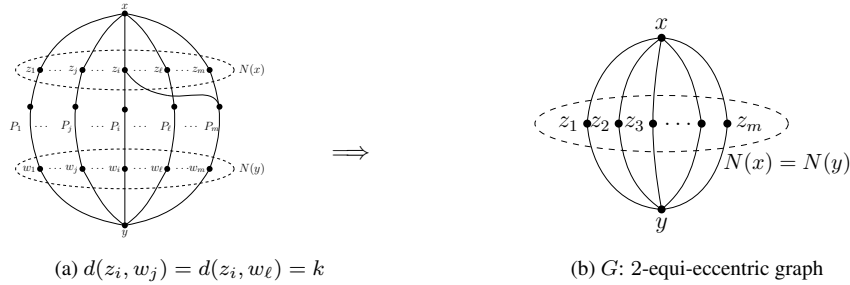


Figure 10

Case 1-1  $G'$  includes all vertices of  $G$ .

There are two cases (i) and (ii) depending on whether some pairs of  $N(x)$  are joined by edges of  $G$  or not.

(i) If no pair of  $N(x)$  is joined by an edge of  $G, G$  must be a complete bipartite graph  $K(2, m)$  (Figure 11a) and  $G_e$  is  $K_2 \cup K_m$  (Figure 11b).

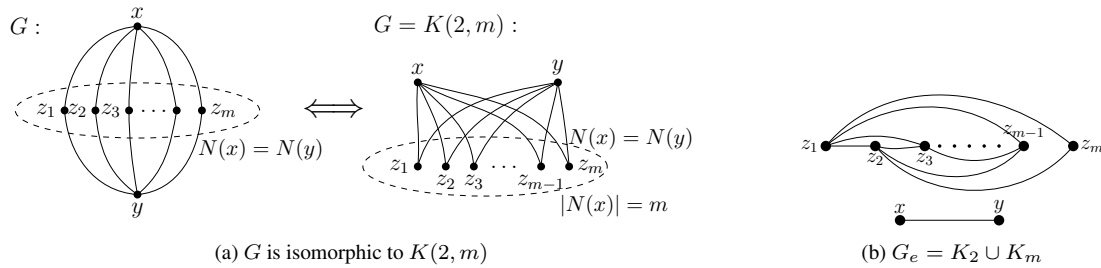


Figure 11: Example of  $G$  and  $G_e$  of Case 1-1 (i)

- (ii) If some pairs of  $N(x)$  are joined by edges of  $G$ ,  $G_e$  is a cluster if and only if  $N(x)$  induces a complete  $n$ -partite graph  $K(m_1, m_2, \dots, m_n)$ ,  $m_i \geq 2$  for  $1 \leq i \leq n$  by Corollary 1.

That is,  $G$  is  $K(m_1, m_2, \dots, m_n) + \overline{K_2} = K(2, m_1, m_2, \dots, m_n)$  and  $G_e$  is  $K_{m_1} \cup K_{m_2} \cup \dots \cup K_{m_n} \cup K_2$ . Figure 12 illustrates:  $N(x)$  induces a complete bipartite graph  $K(2, 3)$  and  $G = K(2, 3) + \overline{K_2} = K(2, 2, 3)$ , respectively.

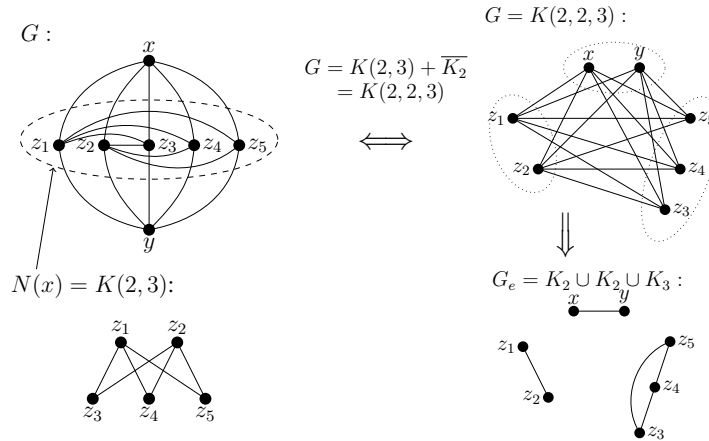


Figure 12:  $N(x)$  induces a complete bipartite graph  $K(2, 3)$

Case 1-2  $G'$  does not include all vertices of  $G$ .

Let  $v_1, v_2, \dots, v_n$  be vertices which are included in none of  $G'$ . Since (1)  $G$  is connected, (2)  $diam(G) = 2$ , and (3)  $v_i$  belongs to neither  $N(x)$  or  $N(y)$ ,  $v_i z_j$  must be an edge of  $G$  for at least one vertex of  $N(x)$  (say  $z_j$ ) (Figure 13a).

- (i) If no pair of  $N(x)$  has an edge of  $G$ ,  $v_i (i = 1, 2, \dots, \ell)$  is joined by an edge with every vertex in  $N(x)$  (Figure 13b), since otherwise  $e(v_i) = 3 > diam(G)(= 2)$ , which is contradiction (Figure 13a). Therefore,  $G$  must be  $K(m, \ell + 2)$  (Figure 13b), and  $G_e$  is  $K_m \cup K_{\ell+2}$ .
- (ii) If some pairs of  $N(x)$  are joined by edges of  $G$ ,  $N(x)$  must form  $K(m_1, m_2, \dots, m_n)$  to make  $G_e$  be a cluster by Corollary 1 (Figure 14). Especially,  $K(2, 2, \dots, 2)$  is the power  $(C_{2n})^{n-1}$  of  $C_{2n}$ . If  $N(x)$  is  $(C_{2n})^{n-1}$ ,  $G$  is also a power  $(C_{2(n+1)})^n$  of  $C_{2(n+1)}$  and  $G_e$  is  $(n + 1)K_2$ , which is not a cluster (Figure 15 is the case of  $p = 4$ ).

Let  $v_1, v_2, \dots, v_\ell$  be vertices not in  $N(x) \cup \{x, y\}$ . For  $v_i (i = 1, 2, \dots, \ell)$ , since (1)  $v_i$  does not belong to  $N(x)$ , (2)  $diam(G) = 2$ , and (3) if there exists  $z_i \in N(x)$  such that  $v_i z_i \notin G, v_i z_i \in G_e, v_i x \in G_e$  but  $x z_i \notin G_e, v_i z$  must be an edge of  $G$  for all  $z$  in  $N(x)$  (Figure 16a). Therefore,  $G$  is  $K(m_1, m_2, \dots, m_n) + \overline{K_{2+\ell}}$ , i.e.,  $K(m_1, m_2, \dots, m_n, 2 + \ell)$  (Figure 16b), and  $G_e$  is  $K_{m_1} \cup K_{m_2} \cup \dots \cup K_{m_n} \cup K_{2+\ell}$  (Figure 16c).



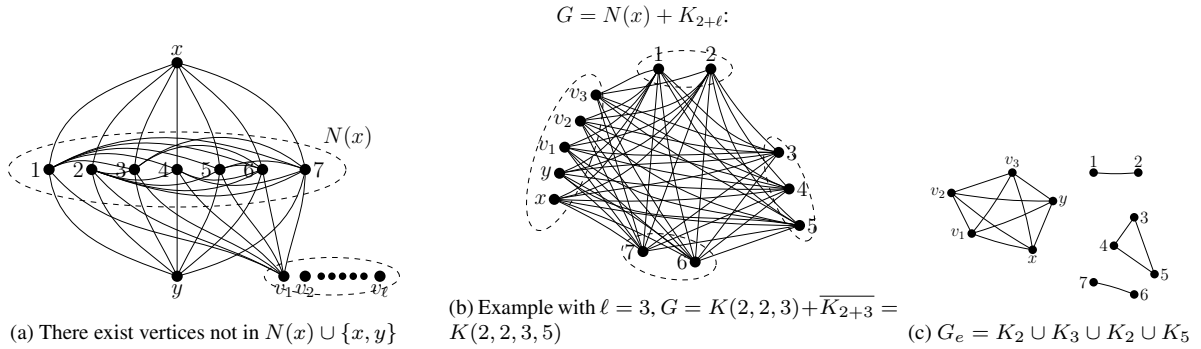


Figure 16:  $z_i$  is simply stated by  $i$ .

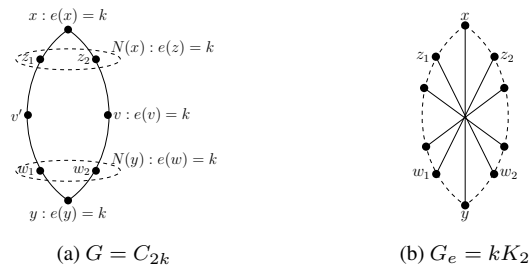


Figure 17: Example of  $G$  and  $G_e$  of Case 2-1

$d(v_i, v_c)$  for any vertex  $v_c$  on  $G'$  (Figure 18a). Let  $y'$  be a vertex on  $G'$  such that  $d(x', y') = k$ . For  $x'$  and  $y'$ , we repeat the analogous argument as one for  $x$  and  $y$ . Then we see that there are at least three paths of length  $k$  from  $x'$  to  $y'$ . If  $k \neq 2$ ,  $G_e$  cannot be a cluster as seen in Case 1. If  $k = 2$  and there is only one vertex  $v_i$  other than  $x$  and  $y$  in  $N(x')$  (Figure 18b where  $p = 5$ ),  $G$  is  $K(3, 2)$  which is a 2-equi-eccentric graph, and thus  $G_e$  is  $K_3 \cup K_2$  (Figure 18c).

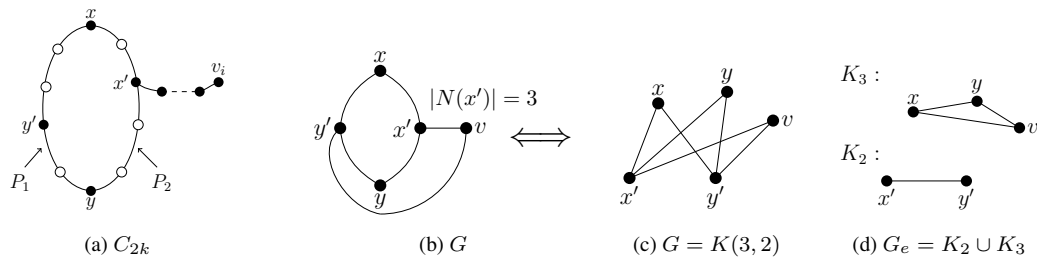


Figure 18: Example of Case 2-2 where  $|N(x')| = 3$

If  $k = 2$  and there are  $m' (\geq 2)$  vertices  $v_1, v_2, \dots, v_{m'}$  other than  $x', y', x$  and  $y$  in  $N(x')$  (where  $m' = p - 4$ ),  $G$  is  $K(m' + 2, 2)$  (Figure 19a) and  $G$  is a 2-equi-eccentric graph. Thus  $G_e$  is  $K_2 \cup K_{m'+2}$  (Figure 19b).

Case 2-3  $|N(x)| = 2$  and  $|N(y)| > 2$ .

If the number of  $w$  for each  $z_i \in N(x)$  is 1, there must exist just one  $w_{z_i} \in N(y)$  such

that no path  $z_i y$  passes through  $w_{z_i}$  and every path  $z_i y$  passes through all  $w_j$  other than  $w_{z_i}$  (Figure 20). But such a graph  $G'$  cannot be a  $k$ -equi-eccentric, so  $G_e$  is not a cluster. We do not need to consider this case.

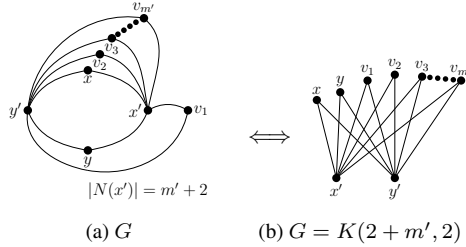


Figure 19: Example of Case 2-2 where  $|N(x')| > 3$

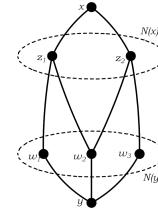


Figure 20: Example of Case 2-3

Case 2-4  $|N(x)| > 2$ .

If the number of  $w$  for each  $z_i \in N(x)$  is 1, there must exist just one  $w_{z_i} \in N(y)$  such that no path  $z_i y$  passes through  $w_{z_i}$  and every path  $z_i y$  passes through all  $w_j$  other than  $w_{z_i}$ . If  $G$  satisfies both this condition and “ $G_e$  is a cluster”,  $G$  is radius critical (Figure 21). Therefore,  $G_e$  is  $\frac{n}{2}K_2$  by Theorem A, which is not a cluster.

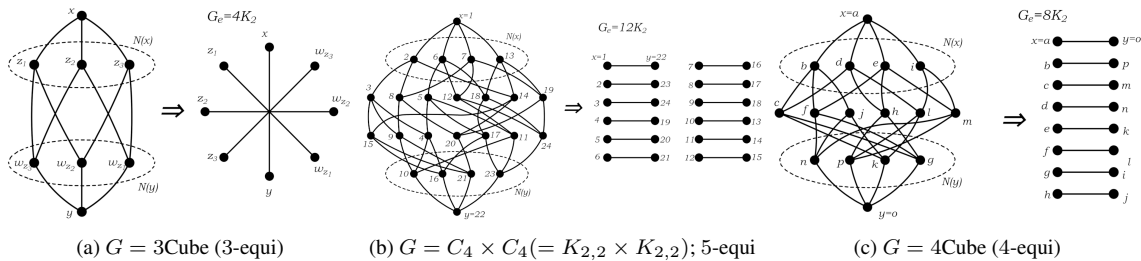


Figure 21: Examples of Case 2-4

□

### 3. Further research

In this article, we determined that all graphs whose eccentric graphs are clusters if and only if the graph is a complete  $n$ -partite graph  $K(p_1, p_2, \dots, p_n)$ ,  $p \geq 2$ . For further research, one can try to determine all graphs that have the same eccentric graph. One can also try to find more graph operations which preserve eccentricity like Mycielski’s operation, or to find practical applications of eccentric graphs.

### Acknowledgement

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