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Partition dimension of trees - palm approach

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Abstract

The partition dimension of a graph is the minimum number of vertex partitions such that every vertex has different distances to the ordered partitions. Many resolving partition for trees have all vertices not in an end-path in the same partition. This reduces the problem of the partition dimension of trees into finding the partition dimension of palms, the end-paths from a branch. In this paper, we construct a resolving partition for trees using resolving partitions of their palms. We also study some bounds for the partition dimension of palms and also find the partition dimension of regular palm and olive tree.

Keywords: partition dimension, tree, palm.
Mathematics Subject Classification : 05C12, 05C15

1. Introduction

Let $G = (V, E)$ be a simple connected graph. A partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of the vertex-set V is called a *resolving partition* if for every two vertices, there exists a partition class $S_i \in \Pi$ with different distances to the two vertices, we say that this partition revolves the two vertices. To show Π is a resolving partition, it suffices to verify that vertices in the same partition is resolved. The *partition dimension* of G, denoted by $pd(G)$, is the minimum number of partitions in a resolving partition of G. The *representation* of a vertex v with respect to teh ordered partition Π is given by $r(v|\Pi) = (d(v, S_1), d(v, S_2), \cdots, d(v, S_k))$. Note that a resolving partition is equivalent to every vertices having different representations.

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Several methods have been developed to construct a resolving partition of a tree graph, see [5, 10, 11]. Many constructions focus on vertices in the *end-path*s, a path joining a leaf to its nearest branch, with vertices not in an end-path is in the same partition class. These constructions makes us belive that obtaining the partition dimension of palms, the union of all end-paths from a branch, will help obtain a better resolving partition for trees. We strengthen this observation by giving an upperbound for the partition dimension of trees using the partition dimension of their palms in the next section.

In our previous paper, we utilize the locating-coloring of palms to generate a locating-coloring of trees, see [9]. In this paper, we will do a similar approach for partition dimension of trees. The partition construction for trees and especially for palms we provide in this paper differ than the locating coloring used in [9].

A star S_n is the complete bipartite graph $K_{1,n}$. A *palm* $S_n(a_1, a_2, \cdots, a_n)$ for $n \geq 2$, is a graph obtained from a star S_n by subdividing the i^{th} edge $a_i - 1$ times. Since permuting a_i will result in an isomorphic graph, we always consider $\{a_i\}$ in a non decreasing sequence. Let us denote the vertex and edge set of a palm by

$V = \{a_0\} \cup \{a_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq a_i\}$, and
$E = \{a_0 a_{i,1} \mid 1 \leq i \leq n\} \cup \{a_{i,j} a_{i,j+1} \mid 1 \leq i \leq n, 1 \leq j \leq a_i - 1\}$.

The k^{th} *level* is the set of vertices of distance k to the *hub* vertex a_0, and the k^{th} *end-path* is the subgraph induced by the set $\{a_0\} \cup \{a_{k,j} : 1 \leq j \leq a_k\}$. A palm is called an end-palm if it is the union of all end-paths from a branch.

If $a_i = i$ for every i then this palm tree is called an *olive* tree and denoted by O_n, namely $O_n = S_n(1, 2, \cdots, n)$. Olive trees with partition dimension 3 and 4 were studied in [1] and [7]. Figure 1 is an example of an olive tree O_5. If $a_i = k$ for some k then the palm tree is called *regular*, and it is denoted by $S_n(k) := S_n(k, k, \cdots, k)$. Other graph theory related definitions can be found in [6].

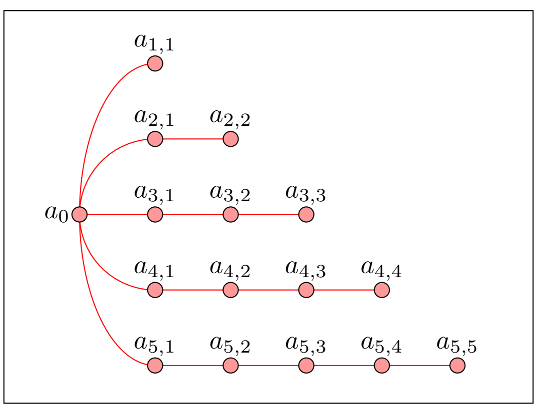

Figure 1. Graph $O_5 = S_5(1, 2, 3, 4, 5)$

2. Partition dimension of trees

In this section, we emphesize the importance of studying the partition dimension of palms by giving a construction of a resolving partition of trees using resolving partitions of their palms. This construction gives an upperbound for the partition dimension of trees using the partition dimensions of their palms.

Theorem 2.1. *Let T be a tree with b end-palms, $P_1, P_2, \cdots, P_b$, then*

$$pd(T) \leq 1 - b + \sum_{i=1}^{b} pd(P_i).$$

Before proving Theorem 2.1, consider the following observation and lemma.

Observation 2.1. *Let G be a graph and $\Pi = \{S_1, \cdots, S_n\}$ a resolving partition of G, then*

1. *Permuting the order of S_i's (the indices) preserve the resolving property.*
2. *For every vertex v of G and index $i \in \{1, 2, \cdots, n\}$, there is a resolving partition where v is in the i^{th} partition.*

Lemma 2.1 ([2]). *Let G be a graph and xy a bridge of G. Let G_x and G_y be the component of $G - xy$ containing x and y respectively. Let $\Pi = \{S_1, \cdots, S_n\}$ be a partition of $V(G)$. If $S_i \subseteq V(G_x)$ and $S_j \subseteq V(G_y)$ for some S_i and S_j in Π, then the representation of any vertex $u \in V(G_x)$ and $v \in V(G_y)$ is different.*

Proof of Theorem 2.1. If $b = 1$, then T is a palm and the result follows. Let $b \geq 2$, $l_0 = 2$, and $l_i = 2 + \sum_{k=1}^{i}(pd(P_i) - 1)$ for other i. Consider the following partition construction.

1. For every palm P_i, let $\Pi_i = \{S_1, S_{l_{i-1}}, S_{l_{i-1}+1}, \cdots, S_{l_i - 1}\}$ be a resolving partition of P_i with the hub vertex in S_1. This is possible because of Observation 2.1.
2. put every other vertices (the one not in an end-path) in S_1.

Note that the partition construction above uses $l_b - 1 = 1 - b + \sum pd(P_i)$ partitions and every palm P_i contains a unique partition $S_{l_{i-1}}$. This partition is a resolving partition because two vertices in the same palm is resolved by the existing resolving partition in that palm, and two vertices not in the same palm have different representarion by Lemma 2.1. □

3. Partition dimension of palm

Now that we know the importance of the partition dimension of palms, we can focus this section to study the partition dimension of palms. The relation between the partition dimension of a palm and its maximum degree is given in the following theorem.

Theorem 3.1. *[4] If G is a graph with $pd(G) = k \geq 3$, then $\Delta(G) \leq 3^{k-1} - 1$.*

We will use the previous theorem to find the bounds for the partition dimension of palms.

Theorem 3.2. *Let $n \geq 2$ and $G = S_n(a_1, a_2, \cdots, a_n)$ be a palm, then*

$$\lfloor \log_3 n \rfloor + 2 \leq pd(G) \leq n.$$

Proof. The lower bound is a direct consequence of Theorem 3.1. The upper bound is achived by putting each end-path in a different partition, the hub vertex can be included in any partition. □

This bound is useful for characterizing infinite trees with finite dimensions, see [3] and [8].

It's not hard to see that the upper bounds in Theorem 3.2 is only satisfied by star graph. However, the lower ground is achieved by many palms, consider the following family of palms. A palm $S_n(a_1, a_2, \cdots, a_n)$ is said to have property $\mathbb{A}$ if $a_{3^k+1} \geq 2k+1$ and $a_{2\cdot 3^k+1} \geq 2k+2$ for all non negatif integers $k < \log_3 n$. For example if $\Delta(G) = 27$, then the requirements for property $\mathbb{A}$ is $a_3 \geq 2$, $a_4 \geq 3$, $a_7 \geq 4$, $a_{10} \geq 5$, and $a_{19} \geq 6$. Remember that $\{a_i\}$ is non decreasing so this means G has atleast different end-paths with 1 end-path of length at least 2, 3 end-paths of length at least 3, 3 end-paths of length at least 4, 9 end-paths of length at least 5, and 9 end-paths of length at least 6.

Theorem 3.3. *Let $G = S_n(a_1, a_2, \cdots, a_n)$ be a palm with property $\mathbb{A}$, then $pd(G) = \lfloor \log_3 n \rfloor + 2$.*

Proof. If $n = 2$, G is a path and the result follows, so let $n \geq 3$ be an integer. Note that that $pd(G) = \lfloor \log_3 n \rfloor + 2$ is equivalent to

$$pd(G) = k \iff 3^{k-2} \leq n \leq 3^{k-1} - 1. \tag{1}$$

By Theorem 3.2, $pd(G) \geq \lfloor \log_3 n \rfloor + 2$. Now we give an algorithm to make a partition $\Pi = \{S_1, \cdots, S_k\}$ of with $k = \lfloor \log_3 n \rfloor + 2$.

1. For $i = 1, \cdots, n$; write $i = 1 + (i-1)_3$ where $(i-1)_3$ is written as a $(k-1)$-digit numbers in base 3, allowing the first digit to be zero, this is always possible by (1). For example if $n = 26$, then $k = 4$ and write $20 = 1 + (201)_3$ and $9 = 1 + (022)_3$.
2. Define n distinct integer-sequences $A_l = \{a_1^l, a_2^l, a_3^l, \cdots\}, 0 \leq l \leq n-1$, with the following algoritm.
   - Initially, define each A_l as the sequence of all 1s, i.e., $A_l = \{1, 1, 1, 1, 1, 1, 1, 1, 1, \cdots\}$.
   - Write $l = 1 + (\overline{l_k l_{k-1} \cdots l_3 l_2})_3$ as in step 1.
   - For $t = 2, \cdots, k$, if $l_t \neq 0$ then change the value of a_s^l with t for every $s \geq 2t - 4 + l_t$.

   For example if $n = 26$, then $k = 4$ and the sequences A_l for $l = 1, 15, 20$ and 25 are as follows:

   $$A_1 = A_{1+(000)_3} = \{1, 1, 1, 1, 1, 1, 1, \cdots\} \qquad A_{15} = A_{1+(112)_3} = \{1, 2, 3, 3, 4, 4, 4, \cdots\}$$
   $$A_{20} = A_{1+(201)_3} = \{2, 2, 2, 2, 2, 4, 4, \cdots\} \qquad A_{25} = A_{1+(220)_3} = \{1, 1, 1, 3, 3, 4, 4, \cdots\}$$

3. Assign $a_{i,j} \in S_{a_j^i}$. Also assign $a_0 \in S_k$ if $n = 3^{k-2}$ and $a_0 \in S_1$ otherwise.

Now we prove that the representations of all vertices are different. Note that the sequence A_l has the following properties: (a) A_l is an increasing sequence, (b) A_l is different for every l, (c) $a_1^l \in \{1, 2\}$ for every l, (d) If A_l contains a term with value t ($l_t \neq 0$), then the first term with value t is a_{2t-3}^l (if $l_t = 1$) or a_{2t-2}^l (if $l_t = 2$), and (e) If $n > 3^{k-2}$, then for every $t \in \{2, \cdots, k\}$ there exists an l such that $a_{2t-3}^l = t$.

(A) First, consider if $3^{k-2} + 1 \leq n \leq 3^{k-1} - 1$. The representation of the hub is $r(a_{0,0}|\Pi) = (0, 1, 3, 5, \cdots, 2k-3)$ by the above properties (d) and (e). Suppose there is another vertex with the same representation, say $r(a_{i,j}) = r(a_{0,0})$ with $(i, j) \neq (0, 0)$. We will prove that A_i contains every value t, $1 \leq t \leq k$. Suppose otherwise, A_i does not contains any term of value t for some t, then the shortest path from $a_{i,j}$ to a vertex in S_t contains the hub vertex, which means $d(a_{i,j}, S_t) > d(a_{0,0}, S_t)$.

Now, for every t, $2 \leq t \leq k$, let $a_{s_t}^i$ be the first term in A_i which is equal to t. From the property (d), we have $s_t \leq 2t-2$ and since $s_t - j = d(a_{i,j}, a_{i,s_t}) = d(a_{i,j}, S_t) = d(v, S_t) = 2t-2$, then $j = 0$ and $s_t = 2t-2$ which means that $i = 1 + (222 \cdots 2)_3$. That implies $i = 3^{k-1} > n$, a contradiction.

Before we prove the representations of all vertices are different, note that if the i^{th} end-path contains a vertex in S_t then the nearest vertex in S_t from a vertex $a_{i,j}$ is always a vertex in this end-path. If the i^{th} end-path does not contain a vertex in S_t then $d(a_{i,j}, S_t) = 2t + j - 3$ by the properties (d) and (e).

Now we prove that $r(a_{i,j}|\Pi) \neq r(a_{l,m}|\Pi)$ for $(i, j) \neq (l, m)$.

Case I: $j = m$**.** Since $i \neq l$ then $A_i \neq A_l$ and $a_s^i \neq a_s^l$ for some s. If $s < j = m$, S_z with $z = \min\{a_s^i, a_s^l\}$ is going to distinguish $r(a_{i,j}|\Pi)$ and $r(a_{l,m}|\Pi)$ because A_i and A_l are monotone. If $s > j = m$, S_z with z is the largest between the value of a_s^i and a_s^l is going to distinguish $r(a_{i,j}|\Pi)$ and $r(a_{l,m}|\Pi)$ because A_i and A_l is monotone.

Case II: $j \neq m$**.** For a contradiction, suppose $r(a_{i,j}|\Pi) = r(a_{l,m}|\Pi)$. Without loss of generality, assume $j < m$. First note that $i \neq 0$, because if $i = 0$ then $l \neq 1$ which means that there exists a term in A_l which is not 1 (say $a_s^l = t > 1$). This means that $d(a_{0,1}, S_t) = 2t-2 > 2t-3 \geq d(a_{l,m}, S_t)$ by the properties (d) and (e). Next we prove $\underline{m = j + 1}$.

If $a_{i,j} \in S_1$ then $a_{l,m} \in S_1$. We know that $1 \leq j < m$, so $m \geq 2$. This means that the second term of A_l is 1 and A_l does not contain a term with value 2, so $d(a_{l,m}, S_2) = m + 1$. In any case whether A_i contains a term 2 or not, we obtain $d(a_{i,j}, S_2) \leq j + 1$, which means that $d(a_{l,m}, S_2) > d(a_{i,j}, S_2)$. Therefore $a_{i,j} \notin S_1$.

Let $a_s^i = t$ be the first term in A_i which is not 1. Note that $d(a_{i,j}, S_1) = d(a_{i,j}, a_{i,s-1}) = j - s + 1$, then $d(a_{l,m}, S_1) = j - s + 1$ which means that $a_{m-j+s-1}^l = 1$. Now $m - j + s - 1 \geq s$, therefore $a_s^l = a_{m-j+s-1}^l = 1$. Since $r(a_{i,j}|\Pi) = r(a_{l,m}|\Pi)$ and $j > m$, then A_l also contains a term which equals to t. Since $a_s^i = t$ is the first term in A_i which is equal to t, by the property (d), the only possible way is for the first term of A_l which equal to t must be a_{s+1}^l and $m - j + s - 1 = s$ which means that $\underline{m = j + 1}$.

Now we prove that A_i and A_l both contain all the numbers t, $2 \leq t \leq k$. Suppose otherwise, there exists some $t \in \{2, 3, \cdots, k\}$ not in either A_i or A_l or both. If t is not contained in both of them, then $d(a_{l,m}, S_t) = 2t + m - 3 > 2t + j - 3 = d(a_{i,j}, S_t)$ because $m > j$. If t is not contained in A_i but contained in A_j then $d(a_{l,m}, S_t) = |m - (2t-3)| = \max\{m, 2t-3\} - \min\{m, 2t-3\} \leq m + 2t - 3 - 1 = j + 2t - 3 = d(a_{i,j}, S_t)$ because $m = j + 1$ and $t \geq 2$. If t is not contained in

A_l but contained in A_i then $d(a_{l,m}, S_t) = m + 2t - 3 > j \geq d(a_{i,j}, S_t)$ because $m > j$ and $t \geq 2$.

Since $m = j + 1$, $r(a_{i,j}|\Pi) = r(a_{l,m}|\Pi)$, and both A_i and A_l contain all t, $1 \leq t \leq k$, and also the properties (d) and (e), the first term in A_i which equal to $t \geq 2$ must be a^i_{2t-3} and first term in A_l which equal to $t \geq 2$ must be a^l_{2t-2}. This implies that $i = 1 + (111 \cdots 1)_3$ and $l = 1 + (222 \cdots 2)_3$ and $l = 3^{k-1} - 1 \geq n > l$, a contradiction. Therefore, the representations of all vertices are different.

(B) Now consider if $n = 3^{k-2}$. For every integer $i \in [0, 3^{k-2})$, the representation of i as a $(k-1)$-digit number in base 3 will always have the first digit to be zero. This means that k is not contained in any sequence A_l and $S_k = \{v\}$. If $r(a_{i,j}|\Pi) = r(a_{l,m}|\Pi)$ then their distances to S_k must be the same, which means they are at the same level. A similar argument as in Case 2.1 can be used to show that $r(a_{i,j}|\Pi) \neq r(a_{l,m}|\Pi)$.

To conclude, we have constructed a resolving partition of G with $k = \lfloor \log_3 n \rfloor + 2$ colors, and so $pd(G) = \lfloor \log_3 n \rfloor + 2$. □

Theorem 3.2 can be expressed in the following way:

Corollary 3.1. *Let G be an $\mathbb{A}$ palm. For $k \geq 2$, $pd(G) = k$ if and only if $3^{k-2} \leq \Delta(G) \leq 3^{k-1} - 1$.*

Since the lowerbound and upperbound in Theorem 3.2 is achieved, by adapting the method in the proof of Theorem 4.1(2) in [9], for every k between $\lfloor \log_3 n \rfloor + 2$ and n, there is an olive G with $\Delta(G) = n$ and $pd(G) = k$.

One particular palm with property $\mathbb{A}$ is the olive tree.

Corollary 3.2. *For $n \geq 2$, $pd(O_n) = \lfloor \log_3 n \rfloor + 2$.*

Note that for $k = 3$ and $k = 4$ Corollary 3.2 corrects the results stated in Theorem 7 and 8 in [1]. The corrected results are (1) $pd(O_n) = 3$ if and only if $3 \leq n \leq 8$ and (2) $pd(O_n) = 4$ if and only if $9 \leq n \leq 26$. Figure 3 shows a resolving partition of olive O_8.

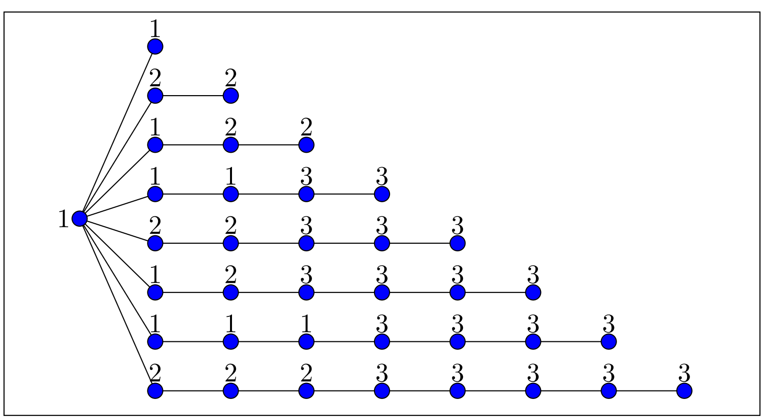

Figure 2. A resolving partition of O_8.

The complexity of the locating chromatic number for regular palms are given in [9].

Theorem 3.4. *[9] for every positive integer k, $\chi_L(S_n(k)) = \Theta\left(n^{\frac{1}{k}}\right)$.*

The coloring algorithm in the proof of this theorem can be adapted for partition dimension by setting the color classes as partitions. This means we have the following theorem.

Theorem 3.5. *For every fixed positive integer k, $pd(S_n(k)) = \Theta\left(n^{\frac{1}{k}}\right)$.*

We end this paper by giving a conjecture for a stronger value for the partition dimension of regular palms.

Conjecture 1. *For $k \geq 2$, $pd(S_n(k)) = (1 + o(1))\left(\frac{k-1}{2}\right)\sqrt[k]{4n}$.*

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