# The size multipartite Ramsey numbers for paths


Syafrizal Sy⋆, Edy Tri Baskoro and Saladin Uttunggadewa

Department of Mathematics
Institut Teknologi Bandung
Jl. Ganesa 10 Bandung 40132, Indonesia
syafrizal-sy301@students.itb.ac.id
{ebaskoro,s_uttunggadewa}@dns.math.itb.ac.id



**Abstract.** For graphs $G_1, G_2, \cdots, G_k$, the (generalized) *size multipartite Ramsey number* $m_j(G_1, G_2, \cdots, G_k)$ is the least natural number $m$ so that any colouring of the edges of $K_{j\times m}$ with $k$ colours will yield a copy of $G_i$ in the $i$th colour for some $i$. In this note, we determine the exact value of the size multipartite Ramsey number $m_j(P_s, P_t)$ for $s = 2, 3$ and all integers $t \geq 2$, where $P_t$ denotes a path on $t$ vertices.


## 1 Introduction

Recently, Burger and van Vuuren [3] studied one of generalisations of the classical Ramsey number as follows. Let $K_{n\times l}$ denote a complete, balanced, multipartite graph consisting of $n$ partite sets and $l$ vertices per partite set. Let $j, l, n, s$ and $t$ be natural numbers with $n, s \geq 2$. Then the *size multipartite Ramsey number* $m_j(K_{n\times l}, K_{s\times t})$ is the smallest natural number $\zeta$ such that an arbitrary colouring of the edges of $K_{j\times\zeta}$, using two colours red and blue, necessarily forces a red $K_{n\times l}$ or a blue $K_{s\times t}$ as subgraph.

In this paper, we generalize this concept by releasing completeness requirement in the forbidden graphs as follows. Let $j \geq 2$ be a natural number. For graphs $G_1, G_2, \cdots, G_k$, the (generalized) *size multipartite Ramsey number* $m_j(G_1, G_2, \cdots, G_k)$ is the smallest natural number $m$ so that any colouring of the edges of $K_{j\times m}$ with $k$ colours will yield a copy of $G_i$ in the $i$th colour for some $i$. The existence of all numbers $m_j(G_1, G_2, \cdots, G_k)$ for $j = 2$ follows from a result of Erdös and Rado [4]. For the case of $k = 2$, with $G_1, G_2$ are complete balanced multipartite graphs, the numbers can be derived from result Burger and van Vuuren [3]. The exact values of bipartite Ramsey numbers $b(P_s, P_t) = m_2(P_s, P_t)$ of two paths can be obtained from a special case of some results of Gyárfás and Lehel [6], and Faudree and Schelp [5]. Furthermore, Hattingh and Henning [7] determined the exact values of bipartite Ramsey numbers $b(P_m, K_{1,n})$. In this paper, we establish the exact values of the size multipartite Ramsey numbers $m_j(P_s, P_t)$ of two paths with $s = 2, 3$.

## 2 Main results

In this note, we prove the following theorem.

**Theorem 1.** *If $n \geq 3$ then $m_j(P_s, P_n) = \lceil \frac{n}{j} \rceil$ for $s = 2, 3$.*

⋆ Permanent address: Department Mathematics, Universitas Andalas, Padang, Indonesia

**Proof**. Let $k = \lceil \frac{n}{j} \rceil$. If all edges of $F = K_{j\times(k-1)}$ are colored by blue then $F$ contains neither red $P_2$ (and $P_3$) nor blue $P_n$ for $n \geq 3$. Therefore, $m_j(P_s, P_n) \geq k$ for $s = 2, 3$ and $n \geq 3$. It easy to see that $m_j(P_2, P_n) \leq k$, and so $m_j(P_2, P_n) = k$. Now, we prove that $m_j(P_3, P_n) \leq k$. Let all edges of $F = K_{j\times k}$ be colored by red or blue, so that $F$ contains no red $P_3$. To show that $F$ contains a blue path $P_n$ on $n$ vertices, consider the following three cases.

**Case 1.** $j = 2$.
Let $V_1$=$\{a_1, a_2, ..., a_k\}$ and $V_2$=$\{b_1, b_2, ..., b_k\}$ be the partite sets of $F$. If all edges of $F$ are blue then the proof is complete. Now, suppose $F$ contains $r$ red edges, $r \leq k$. Since there is no red $P_3$, these red edges are independent. Without loss of generality, we may assume that the $r$ red edges are: $a_1b_1, a_2b_2, \cdots, a_rb_r$. If $r$ is odd then $a_1b_2a_3b_4 \cdots a_{r-2}b_{r-1}a_rb_1a_2b_3a_4 \cdots b_{r-2}a_{r-1}b_ra_{r+1}b_{r+1}a_{r+2}b_{r+2} \cdots a_kb_k$ is a blue path with at least $n$ vertices in $F$. Now, if $r$ is even then we have a blue path $a_1b_2a_3b_4 \cdots a_{r-3}b_{r-2}a_{r-1}b_ra_{r-2}b_{r-3}a_{r-4} \cdots b_3a_2b_1a_r$ $b_{r-1}a_{r+1}b_{r+1} \cdots a_kb_k$ with at least $n$ vertices in $F$.

**Case 2.** $j = 3$.
If all edges of $F$ are blue then it is finished. Let $V_1$, $V_2$ and $V_3$ be the partite sets of $F$. Now, assume, without loss of generality, there exist $r$, $s$ and $t$ red edges connecting $V_1$ to $V_2$, $V_1$ to $V_3$, and $V_2$ to $V_3$, respectively. By considering these red edges, partition $V_1, V_2$ and $V_3$ as follow: $V_1 = R_1 \cup X \cup S_1$, $V_2 = R_2 \cup Y \cup T_2$ and $V_3 = S_3 \cup Z \cup T_3$, where $|R_1| = |R_2| = r$, $|S_1| = |S_3| = s$ and $|T_2| = |T_3| = t$ so that all edges connecting $R_1$ to $R_2$, $S_1$ to $S_3$ and $T_2$ to $T_3$ are red. Next, without loss of generality, assume $r \leq s \leq t$. This implies that $|Z| \leq |Y| \leq |X|$. Observe that there exist three independent blue paths: (i) path ${}_aP_b$ of $2r$ vertices connecting all vertices of $R_1$ and some of $S_3$ with the initial vertex $a \in R_1$ and the terminal vertex $b \in S_3$, (ii) path ${}_cP_d$ of $2r$ vertices connecting all vertices of $R_2$ and some of $T_3$ with the initial vertex $c \in R_2$ and the terminal vertex $d \in T_3$, (iii) path ${}_eP_f$ of $2s$ vertices connecting all vertices of $S_1$ and some of $T_2$ with the initial vertex $e \in T_2$ and the terminal vertex $f \in S_1$, see Fig.1.(i). We can the join all these paths into one larger blue path ${}_aP_f$:=${}_aP_{bc}P_{de}P_f$. This path has $4r + 2s$ vertices, see Fig.1.(ii).

Let denote by $A, B$ and $C$ the subsets of $T_2$, $S_3$ and $T_3$, respectively, which contain all vertices not in the above three blue paths. Then, we have $|Y| + |A| = |X|$ and $|B| + |Z| + |C| = |X| + |B| = |X| + (s - r)$, and $(s - r) = |Y| - |Z| \leq |X|$. We will show that there exists a blue path connecting $X$, $Y \cup A$ and $B \cup Z \cup C$ with at least $3|X| + (s - r)$ vertices.

Partition the sets $C = C_1 \cup C_2$ such that $C_2$ consists of all end-vertices of red edges connecting $A$ and $C$, and so $|C_2| = |A| = (t - s)$ and $|C_1| = |B| = (s - r)$. Partition the sets $X = X_1 \cup X_2$ such that $|X_2| = |C_2|$; Clearly $|X_1| = |Y|$. Suppose $D = B \cup Z \cup C_1$. Note that $|X_2| = |A| = |C_2|$. Let $C_2 = \{a_1, a_2, \cdots, a_m\}$, $X_2 = \{b_1, b_2, \cdots, b_m\}$, and $A = \{c_1, c_2, \cdots, c_m\}$, where $m = t - s$. Then we obtain a blue path $a_1b_1c_1a_2b_2c_2 \cdots a_mb_mc_m$. This path has $3(t - s)$ vertices, and is denoted by ${}_{a_1}P_{c_m}$. Since $fa_1$ is a blue edge then by joining the two paths ${}_aP_f$ and ${}_{a_1}P_{c_m}$ we have a blue path with $4r + 3t - s$ vertices. This resulting path, denote by ${}_aP_{c_m}$, starts from $a$ and ends at $c_m$. Next, we consider the following three subcases.

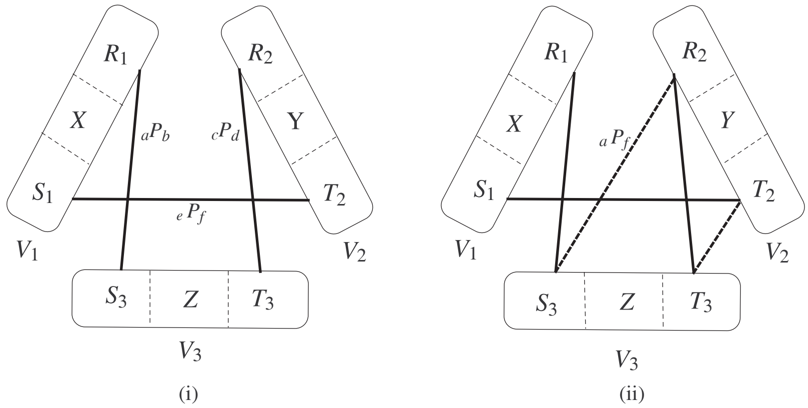


**Fig. 1.** The three blue paths form a larger blue path starting from vertex $a \in R_1$ and ending at $f \in S_1$.

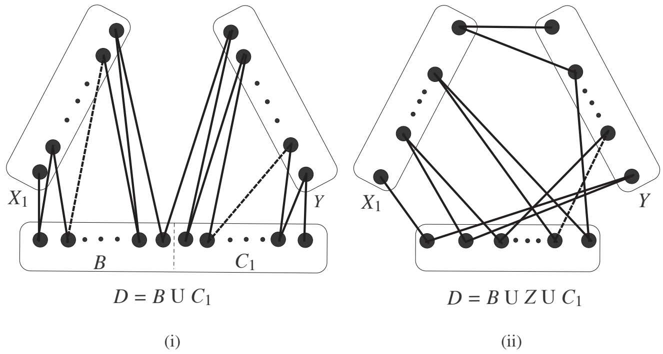


**Fig. 2.** (i) A blue path $_gP_h$ (ii) A blue path $_uP_v$

**Subcase 2.1.** $|Z| = 0$.
Since $|Z| = 0$ then $|D| = |B \cup C_1| = 2|Y| = 2|X_1| = 2(s - r)$. Then, we obtain a blue path by connecting all vertices in $X_1$ with a half of $D$ alternatingly, and continuing connecting the other half of $D$ with all the vertices in $Y$ alternatingly. This path starts at some vertex $g \in X_1$ and ends at $h \in C_1$, and is denoted by ${}_gP_h$ (see Fig.2.(i)). Note that this path has $4(s - r)$ vertices. Since $c_mg$ is a blue edge then by joining the two paths ${}_aP_{c_m}$ and ${}_gP_h$ we have a blue path with $3(s + t)$ vertices. This resulting path uses all vertices of $F$, and so $F$ contains a blue path with at least $n$ vertices.

**Subcase 2.2.** $0 < |Z| < |Y|$.
Since $|Z| < |Y|$ then $|D| = |B \cup Z \cup C_1| < 2|Y|$. Then, we obtain a blue path ${}_uP_v$ connecting all vertices in $X_1$ with all vertices in $Y$ through all vertices in $D$ one by one each time, until all the vertices in $D$ have been totally used. If there are still some vertices in $X_1$ (and so in $Y$) left then connect directly these remaining vertices alternatingly, see Fig.2.(ii). Since $c_mu$ is a blue edge then by joining the two paths ${}_aP_{c_m}$ and ${}_uP_v$, we have a blue path with $3(|Y| + r + t)$ vertices. This resulting path contains all the vertices of $F$, and so $F$ has contains a blue path with at least $n$ vertices.

**Subcase 2.3.** $|Z| = |Y| \neq 0$.
Since $|Z| = |Y|$, then $s - r = 0$ and so $|D| = |Z|$. Then we obtain a blue path ${}_wP_z$ connecting all vertices in $D$, $X_1$, and $Y$ alternatingly, where $w \in D$ and $z \in Y$. Since $c_mw$ is a blue edge then by joining the two paths ${}_aP_{c_m}$ and ${}_wP_z$, we have a blue path with $3(|Y|+r+t)$ vertices. This resulting path will contains all the vertices of $F$.

**Case 3.** $j \geq 4$.
Let $V_1, V_2, \cdots, V_j$ be the partite sets of $F$. Trivially, if all edges of $F$ are blue then it is finished. If $j$ even by Case 1 we have $\frac{j}{2}$ blue paths connecting all vertices $V_1$ to $V_2$, $V_3$ to $V_4$, $\cdots$, $V_{j-1}$ to $V_j$. Each path has $2k$ vertices. Since $F$ has no a red $P_3$ then we can concatenate these $\frac{j}{2}$ paths into one blue path of $kj$ vertices. This final path will have at least $n$ vertices. If$j$ is odd then by Case 1 we obtain $\frac{j-3}{2}$ blue paths connecting all vertices $V_1$ to $V_2$, $V_3$ to $V_4$, $\cdots$, $V_{j-4}$ to $V_{j-3}$ independently. Each path has $2k$ vertices. By using the method in Case 2 we get another blue path connecting all vertices in $V_{j-2}$, $V_{j-1}$ and $V_j$. Again, since $F$ has no red $P_3$, we can join all these paths into one with at least $n$ vertices.$\square$

**Corollary 1.** *If $n \geq 3$ then $m_j(P_s, C_n) = \lceil \frac{n}{j} \rceil$ for $s = 2, 3$.*

**Proof**. Let ${}_xP_y$ be the final blue path obtained in the proof of Theorem 1. This path consists of at least $n$ vertices. Since $xy$ is a blue edge then by joining the two vertices $x$ and $y$, we have a blue cycle $C_n$ with at least $n$ vertices.$\square$

## References


1. A.P. Burger, P.J.P. Grobler, E.H. Stipp and J.H. van Vuuren, Diagonal Ramsey numbers for multipartite graph, *Utilitas Math.*, 66 (2004), 137-163.
2. A.P. Burger and J.H. van Vuuren, Ramsey numbers in Complete Balanced Multipartite Graphs. Part I: Set Numbers, *Discrete Math.*, 283 (2004) 37-43.

3. A.P. Burger and J.H. van Vuuren, Ramsey numbers in Complete Balanced Multipartite Graphs. Part II: Size Numbers, *Discrete Math.*, February 20, 2004.
4. P. Erdös and R. Rado, A partiton calculus in set theory, *Bull. Amer. Math. Soc.* 62 (1956) 229-489.
5. R.J. Faudree and R.H. Schelp, Path-path Ramsey type numbers for complete bipartite graph, *J. Combin. Theory Ser.* B 19 (1975), 161-173.
6. A. Gyárfás and J. Lehel, A Ramsey-type problem in directed and bipartite graphs, *Periodica Math. Hungar.* 3 (1973), 299-304.
7. J.H. Hattingh and M.A. Henning, Star-path bipartite Ramsey numbers, *Discrete Math.* 185 (1998), 255-258.

1