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Note

Set and size multipartite Ramsey numbers for stars

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ABSTRACT

The set multipartite Ramsey number for stars $M_s(K_{1,n_1}, \dots, K_{1,n_k})$ is the smallest positive integer c such that any k-coloring of the edges of $K_{c\times s}$ contains a monochromatic star K_{1,n_i} in color i for some i, $1 \le i \le k$, where $K_{c\times s}$ denotes the complete multipartite graph having c classes with s vertices per each class. On the other hand, the size multipartite Ramsey number for stars, denoted by $m_c(K_{1,n_1}, \dots, K_{1,n_k})$, is the smallest positive integer s such that any k-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of K_{1,n_i} in color i for some i, $1 \le i \le k$. In this note we compute both $M_s(K_{1,n_1}, \dots, K_{1,n_k})$ and $m_c(K_{1,n_1}, \dots, K_{1,n_k})$, extending well-known results on the classical and the bipartite Ramsey numbers for stars, respectively.

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1. Introduction

As mentioned, $K_{c\times s}$ denotes the complete multipartite graph having c classes with s vertices per each class. Given graphs $G_1, \dots, G_k$, consider the extremal numbers:

- for a positive integer s, the *set multipartite Ramsey number* $M_s(G_1, \dots, G_k)$ denotes the smallest positive integer c such that any k-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of G_i in color i for some i, $1 \le i \le k$.
- for an integer $c \ge 2$, the *size multipartite Ramsey number* $m_c(G_1, \dots, G_k)$ denotes the smallest positive integer s (if it exists) such that any k-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of G_i in color i for some i, $1 \le i \le k$.

It is worth mentioning two particularly interesting cases:

- $M_1(G_1, \dots, G_k)$ can be regarded as the *classical Ramsey number* $r(G_1, \dots, G_k)$, since $K_{n\times 1}$ is isomorphic to the complete graph K_n on n vertices. The determination of these numbers has turned out to be a central problem in combinatorics.
- the number $m_2(G_1, \dots, G_k)$ produces the widely studied *bipartite Ramsey number* $b(G_1, \dots, G_k)$.

In 2004, Burger, Grobler, Stipp, and van Vuuren [2–4] investigated the numbers $M_s(G_1, G_2)$ and $m_c(G_1, G_2)$ where each G_i is a complete multipartite graph, which can be naturally extended to several colors, see [12]. Recently the numbers $m_c(G_1, G_2)$ have been investigated for special classes: stripes versus cycles, stars versus cycles, see [11] and its references.

In this work we focus on the case where each G_i is a star. Chvátal and Harary [6] and Harary [7] evaluated the numbers $r(K_{1,n}, K_{1,m})$, where $K_{1,n}$ denotes a star on $n+1$ vertices. As stated by Irving [10], "many of the more interesting problems emerge in cases $k > 2$". In particular, Irving pointed out that the chromatic index of certain complete graphs can be applied to compute $r(K_{1,2}, \dots, K_{1,2})$. See also the numbers $r(K_{1,3}, \dots, K_{1,3})$ in [10]. Burr and Roberts [5] extended these results as

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follows

$$r(K_{1,n_1},\ldots,K_{1,n_k})=\begin{cases}S-k+2 & \text{if } S-k \text{ is odd or } n_1,\ldots,n_k \text{ are odd;}\\ S-k+1 & \text{if } S-k \text{ is even and some } n_i \text{ is even,}\end{cases} \tag{1}$$

where $S=\sum_{i=1}^{k}n_i$.

As the first goal of this note we compute the following numbers, generalizing all the previous results.

Theorem 1. *Given integers $s\geq 1$ and $k,n_1,\ldots,n_k\geq 2$, let $S=\sum_{i=1}^{k}n_i$. Then*

$$M_s(K_{1,n_1},\ldots,K_{1,n_k})=\begin{cases}\dfrac{S-k}{s}+1 & \text{if } (S-k)/s \text{ is even, } s \text{ is odd and}\\ & \text{some } n_i \text{ is even;}\\ \left\lfloor\dfrac{S-k}{s}\right\rfloor+2 & \text{otherwise.}\end{cases}$$

Concerning the bipartite Ramsey numbers, $b(K_{1,n},K_{1,n})=2n-1$ is reported by Beineke and Schwenk [1]. Its extension to arbitrary number of colors was proved by Hattingh and Henning [8], more precisely:

$$b(K_{1,n_1},\ldots,K_{1,n_k})=S-k+1 \tag{2}$$

where $S=\sum_{i=1}^{k}n_i$.

The second goal of this note generalizes these results, as stated below.

Theorem 2. *Given integers $c,k,n_1,\ldots,n_k\geq 2$, let $S=\sum_{i=1}^{k}n_i$. Then*

$$m_c(K_{1,n_1},\ldots,K_{1,n_k})=\begin{cases}\dfrac{S-k}{c-1} & \text{if } (S-k)/(c-1) \text{ and } c \text{ are odd and}\\ & \text{some } n_i \text{ is even;}\\ \left\lceil\dfrac{S-k+1}{c-1}\right\rceil & \text{otherwise.}\end{cases}$$

Recall that a matching in $G=(V(G),E(G))$ is a set of edges, no two of which are adjacent. The *chromatic index* (also called *edge chromatic number*) $\chi'(G)$ of a graph G denotes the minimum r such that there exists a partition of $E(G)$ into r matchings.

Only two well-known results are required for our purpose: (i) the celebrated Vizing's Theorem on the chromatic index of a graph and (ii) the chromatic index of a complete multipartite graph due to Hoffman and Rodger [9]. We show Theorem 1 in Section 2. Since the proof of Theorem 2 is very similar, we present only a sketch of its proof in Section 3.

2. Proof of Theorem 1

In order to facilitate the understanding, let us split the proof of Theorem 1 into parts. We begin with a simple but very useful general upper bound, which is sharp for several classes.

Proposition 3. *Given integers $k,n_1,\ldots,n_k\geq 2$, let $S=\sum_{i=1}^{k}n_i$. For any $s\geq 1$,*

$$M_s(K_{1,n_1},\ldots,K_{1,n_k})\leq\lfloor(S-k)/s\rfloor+2.$$

Proof. Denote $c=\lfloor(S-k)/s\rfloor+2$. Given an arbitrary k-coloring of $K_{c\times s}$, let H_i be the subgraph of $K_{c\times s}$ formed by all edges in color i, where $1\leq i\leq k$. Select a vertex v of $K_{c\times s}$. Note that $\sum_{i=1}^{k}d_i(v)=(c-1)s$, where $d_i(v)$ denotes the degree of v in H_i. By the choice of c, we have

$$\sum_{i=1}^{k}d_i(v)=(c-1)s>(S-k)=\sum_{i=1}^{k}(n_i-1), \tag{3}$$

thus $d_j(v)>n_j-1$ holds for some j, $1\leq j\leq k$. Hence there is a monochromatic copy of K_{1,n_j} in color j. ■

As usual, $\Delta(G)$ denotes the maximum degree of G. A cornerstone of graph theory states that the parameter $\chi'(G)$ is very close to the trivial lower bound $\Delta(G)$, more specifically:

Theorem 4 (*Vizing's Theorem [13]*)**.** *For a simple graph G,*

$$\Delta(G)\leq\chi'(G)\leq\Delta(G)+1.$$

The key ingredient of Irving's result about $r(K_{1,2},\ldots,K_{1,2})$ is based on the statement: since a matching does not contain a copy of $K_{1,2}$, the chromatic index may induce almost optimal coloring with no copy of $K_{1,2}$. With the same spirit, Vizing's Theorem plays a central role to extend such idea. More formally:

Proposition 5. *Given integers $k, n_1, \dots, n_k \geq 2$, let $S = \sum_{i=1}^{k} n_i$. For any $s \geq 1$,*

$$\lfloor (S-k-1)/s \rfloor + 2 \leq M_s(K_{1,n_1}, \dots, K_{1,n_k}).$$

Proof. Let $c = \lfloor (S-k-1)/s \rfloor + 1$. Vizing's Theorem states $\chi'(K_{c\times s}) \leq s(c-1)+1$, that is, there is a $\big(s(c-1)+1\big)$-coloration of $K_{c\times s}$ with no monochromatic copy of $K_{1,2}$. This coloring is denoted by $\phi : E(K_{c\times s}) \to \mathbb{Z}_{s(c-1)+1}$.

By the choice of c, $s(c-1)+1 \leq S-k = \sum_{i=1}^{k}(n_i - 1)$, thus we can select a function $\psi : \mathbb{Z}_{s(c-1)+1} \to \{1, \dots, k\}$ such that $|\psi^{-1}(i)| = |\{x \in \mathbb{Z}_{s(c-1)+1} : \psi(x) = i\}| \leq n_i - 1$ for any $i = 1, \dots, k$. The composition $\psi \circ \phi : E(K_{c\times s}) \to \{1, \dots, k\}$ is a k-coloration of $K_{c\times s}$ with no monochromatic copy of K_{1,n_i} in color i for any $i = 1, \dots, k$. Thus the statement follows. ■

An immediate consequence is established below.

Corollary 6. *Given integers $k, n_1, \dots, n_k \geq 2$, let $S = \sum_{i=1}^{k} n_i$. If s does not divide $S-k$, then*

$$M_s(K_{1,n_1}, \dots, K_{1,n_k}) = \lfloor (S-k)/s \rfloor + 2.$$

Proof. If s does not divide $S-k$, note that $\lfloor (S-k-1)/s \rfloor = \lfloor (S-k)/s \rfloor$. An application of Propositions 3 and 5 ensures the result. ■

In view of the result above, we may assume that s divides $S-k$ from now on. In this case, the upper bound $\chi'(K_{c\times s}) \leq s(c-1)+1$ due to Vizing's theorem is not so powerful to evaluate the exact value of $M_s(K_{1,n_1}, \dots, K_{1,n_k})$. Fortunately, a slight improvement can be derived from the next statement.

Theorem 7 (*Hoffman and Rodger [9]*). *If c or s is even, then $\chi'(K_{c\times s}) = \Delta(K_{c\times s}) = s(c-1)$.*

As an application, we can determine another part of Theorem 1, more specifically:

Proposition 8. *Given integers $k, n_1, \dots, n_k \geq 2$. Let $S = \sum_{i=1}^{k} n_i$ and suppose that s divides $S-k$. If $(S-k)/s$ is odd or s is even, then*

$$M_s(K_{1,n_1}, \dots, K_{1,n_k}) = (S-k)/s + 2.$$

Proof. Proposition 3 states the upper bound. For the lower bound, let $c = (S-k)/s + 1$. By hypothesis, c or s is even. Hence Theorem 7 ensures that $\chi'(K_{c\times s}) \leq s(c-1)$. By a similar argument used in the proof of Proposition 5, we obtain a k-coloration of $K_{c\times s}$ with no monochromatic copy of K_{1,n_i} in color i for any $i = 1, \dots, n$, that is, $M_s(K_{1,n_1}, \dots, K_{1,n_k}) > c$. ■

We analyze now the unique case where the upper bound in Proposition 3 is not optimal.

Proposition 9. *Given positive integers $k, n_1, \dots, n_k \geq 2$ with n_i even for some $i = 1, \dots, k$, let $S = \sum_{i=1}^{k} n_i$. If $(S-k)/s$ is an even integer and s is odd, then*

$$M_s(K_{1,n_1}, \dots, K_{1,n_k}) = \frac{S-k}{s} + 1.$$

Proof. Let $c = (S-k)/s + 1$. Proposition 5 produces $M_s(K_{1,n_1}, \dots, K_{1,n_k}) \geq c$. Suppose for a contradiction that $M_s(K_{1,n_1}, \dots, K_{1,n_k}) > c$. By assumption, there exists a k-coloring of $K_{c\times s} = (V, E)$ that contains no copy of K_{1,n_i} in color i for any $i = 1, \dots, k$. Let $H_i = (V, E_i)$ denote the subgraph of $K_{c\times s}$ induced by the color i and let $d_i(v)$ be the degree of a vertex $v \in V$ in H_i. Since K_{1,n_i} is not subgraph of H_i, $d_i(v) \leq n_i - 1$ holds for any vertex $v \in V$. Note that

$$(c-1)s = \sum_{i=1}^{k} d_i(v) \leq \sum_{i=1}^{k} (n_i - 1) = S - k = (c-1)s.$$

The inequality above ensures that $d_i(v) = n_i - 1$ for every vertex $v \in V$ and for every $i = 1, \dots, k$. By hypothesis, there is a color j such that n_j is even. By Euler's identity,

$$2|E_j| = \sum_{v\in V} d_j(v) = \sum_{v\in V} (n_j - 1) = cs(n_j - 1).$$

Thus $cs(n_j - 1)$ is even, contradicting the assumption that c, s and $(n_j - 1)$ are all odd numbers. ■

Corollary 6, Proposition 8, and Proposition 9 ensure Theorem 1 except when $(S-k)/s$ is an even integer, s is odd, and n_i is odd for every $i = 1, \dots, k$. For this remaining case, we need a more involved construction.

Proposition 10. *Given positive integers $k, n_1, \dots, n_k \geq 2$ with n_i odd for all $i = 1, \dots, k$. Let $S = \sum_{i=1}^{k} n_i$. If $(S-k)/s$ is an even integer and s is odd, then*

$$M_s(K_{1,n_1}, \dots, K_{1,n_k}) = \frac{S-k}{s} + 2.$$

Proof. The upper bound is already known according to Proposition 3. Let $q = (S-k)/s$. Denote the $q+1$ classes of vertices of the multipartite graph $K_{(q+1)\times s}$ by $Ł_1, \ldots, Ł_q, Ł_\infty$, where $Ł_a = \{(a,1), \ldots, (a,s)\}$ for any $a = 1, \ldots, q, \infty$. We need to build a k-coloration of $K_{(q+1)\times s}$ with no monochromatic copy of K_{1,n_i} in color i for any $i = 1, \ldots, k$. This coloring depends on three auxiliary functions, which are described in the steps below.

Step 1: Initially, consider the $(S-k)$-coloration $\psi : E(K_{(q+1)\times s}) \longrightarrow \mathbb{Z}_{S-k}$ defined by

$$\psi\big(\{(a,b),(c,d)\}\big) = \begin{cases} \overline{(a+c)s+b+d} & \text{if } a, c \le q; \\ \overline{2as+b+d} & \text{if } c = \infty; \\ \overline{2cs+b+d} & \text{if } a = \infty; \end{cases}$$

where for any integer x, $\bar{x}$ denotes the equivalence class of x modulo $S-k$.

The coloring of the neighbor of a vertex (a,b) has distinct behaviors when $a \neq \infty$ or $a = \infty$, as described in the two next claims.

Claim 1: For each $a = 1, \ldots, q$ and for each $b = 1, \ldots, s$, the $S-k(=qs)$ incident edges at the vertex (a,b) of $K_{(q+1)\times s}$ have $S-k$ different colors. To prove this claim, suppose for a contradiction that there are distinct vertices, say, (c_1,d_1) and (c_2,d_2) with $c_1 \neq a \neq c_2$ such that

$$\psi\big(\{(a,b),(c_1,d_1)\}\big) = \psi\big(\{(a,b),(c_2,d_2)\}\big). \tag{4}$$

We divide the analysis into three cases:

1. If $c_1 = \infty$ and $c_2 = \infty$. In this case, the equality (4) implies $\overline{d_1} = \overline{d_2}$, that is, $d_1 = d_2$. Since $c_1 = c_2$, we obtain the contradiction $(c_1,d_1) = (c_2,d_2)$.
2. If $c_1 \le q$ and $c_2 \le q$. Since $a, c_1, c_2 \le q$, (4) implies $\overline{(c_1-c_2)s} = \overline{d_2-d_1}$. But $|d_2-d_1| < s$, thus $d_2-d_1 = 0$ holds. Consequently, $\overline{(c_1-c_2)s} = \bar{0}$. The condition $|(c_1-c_2)s| \le (q-1)s < S-k$ implies $(c_1-c_2)s = 0$, that is, $c_1 = c_2$, which contradicts the assumption $(c_1,d_1) \neq (c_2,d_2)$.
3. If $c_1 = \infty$ and $c_2 \le q$. In this case, (4) implies $\overline{(a-c_2)s} = \overline{d_2-d_1}$. Proceeding as in the case 2, we conclude the absurd $a = c_2$.

Claim 2: Let $b \in \{1, \ldots, s\}$. In the coloration ψ, the $S-k$ incident edges at the vertex (∞,b) can be partitioned into $(S-k)/2$ colors of type $A_b = \{\bar{b} + \overline{2js+l} : 1 \le j \le q/2, 1 \le l \le s\}$. Moreover, there are exactly two edges of each one of these $(S-k)/2$ colors.

In order to proceed the proof of Claim 2, suppose that

$$\psi\big(\{(\infty,b),(c_1,d_1)\}\big) = \psi\big(\{(\infty,b),(c_2,d_2)\}\big).$$

Then $\overline{2(c_1-c_2)s} = \overline{d_2-d_1}$. Since $|d_2-d_1| < s$, we have $d_2-d_1 = 0$ and consequently $\overline{2(c_1-c_2)s} = \bar{0}$. Thus there is an integer z such that $2(c_1-c_2)s = z(S-k) = zqs$. In particular, $2(c_1-c_2) = zq$. The conditions $c_1, c_2 \le q$ reveal that two situations can hold: $c_1 - c_2 = 0$ or $|c_1-c_2| = q/2$. If (c_1,d_1) and (c_2,d_2) are distinct and $c_2 \ge c_1$, then $(c_2,d_2) = (c_1+q/2, d_1)$. On the other hand, if $(c_2,d_2) = (c_1+q/2,d_1)$, we have $\psi\big(\{(\infty,b),(c_1,d_1)\}\big) = \psi\big(\{(\infty,b),(c_2,d_2)\}\big)$. The facts above ensure that the $S-k$ incident edges at the vertex (∞,b) are of $(S-k)/2$ colors, being two of each color. It remains to prove that this set of colors is A_b, which is a consequence of the facts that $\psi\big(\{(\infty,b),(c_1,d_1)\}\big) = \bar{b} + \overline{2c_1s+d_1}$ and $\psi\big(\{(\infty,b),(c_1,d_1)\}\big) = \psi\big(\{(\infty,b),(c_1-q/2,d_1)\}\big)$ if $c_1 > q/2$.

Step 2: Consider the function $\omega : \mathbb{Z}_{S-k} \longrightarrow \mathbb{Z}_{S-k}$ defined as follows: given $\bar{x} \in \mathbb{Z}_{S-k}$ with $1 \le x \le S-k(=qs)$, there are unique integers m and r with $0 \le m \le q-1$ and $1 \le r \le s$ such that $x = ms + r$. Define $\omega(\bar{x}) = \bar{x} = \overline{ms+r}$ if m is even and $\omega(\bar{x}) = \bar{x} - \bar{s} = \overline{(m-1)s+r}$ if m is odd. Note that $Im(\omega) = \{\overline{2ms+r} : 0 \le m \le q/2-1, 1 \le r \le s\}$ and $|Im(\omega)| = (q/2)s = (S-k)/2$.

Claim 3: In the $(S-k)/2$-coloration $\omega \circ \psi : E(K_{(q+1)\times s}) \longrightarrow Im(\omega)$, a vertex (a,b) is connected to exactly two vertices by a same color, say $\bar{x} \in Im(\omega)$. Indeed, we have $\omega^{-1}(\bar{x}) = \{\bar{x}, \bar{x}+\bar{s}\}$, thus, if $a \le q$, it follows from Claim 1 that in the $(S-k)/2$-coloration $\omega \circ \psi$, the vertex (a,b) is connected to two vertices by color $\bar{x}$. On the other hand, if $a = \infty$, observe that just one element of $\omega^{-1}(\bar{x}) = \{\bar{x}, \bar{x}+\bar{s}\}$ is on set $A_b = \{\bar{b} + \overline{2ms+l} : 1 \le m \le q/2, 1 \le r \le s\}$. This way, it follows from Claim 2 that in $(S-k)/2$-coloration $\omega \circ \psi$, the vertex (∞,b) is connected to two vertices by color $\bar{x}$.

Step 3: Note that $(n_1-1)/2, \ldots, (n_k-1)/2$ are integers satisfying

$$\sum_{i=1}^{k}(n_i-1)/2 = (S-k)/2 = |Im(\omega)|.$$

Thus, we can select a function $\phi : Im(\omega) \longrightarrow \{1, \ldots, k\}$ in a way that $|\phi^{-1}(i)| = (n_i-1)/2$ for every $i = 1, \ldots, k$.

We are ready to present the desired coloring. Consider the k-coloration $\phi \circ (\omega \circ \psi) : E(K_{(q+1)\times s}) \longrightarrow \{1, \ldots, k\}$. As in $(S-k)/2$-coloration $\omega \circ \psi$, any vertex is connected to exactly two others vertices by a same color $\bar{x} \in Im(\omega)$, it follows that in k-coloration $\phi \circ (\omega \circ \psi)$, any vertex of $K_{(q+1)\times s}$ is connected to exactly others $2|\phi^{-1}(i)| = n_i - 1$ vertices by color i, for

$i = 1, \ldots, k$. Thus, the k-coloration $\phi \circ (\omega \circ \psi)$ contains no monochromatic copy of K_{1,n_i} in color i, for any $i = 1, \ldots, k$. This ensures that $M_s(K_{1,n_1}, \ldots, K_{1,n_k}) \geq q + 2 = (S - k)/s + 2$ and concludes the proof. ■

The result above concludes the proof of Theorem 1.

3. Proof of Theorem 2

The proof of Theorem 2 can be summarized into five statements, whose proofs are analogous to Propositions 3 and 5 and 8–10, respectively.

Claim A: $m_c(K_{1,n_1}, \ldots, K_{1,n_k}) \leq \left\lceil \frac{S-k+1}{c-1} \right\rceil$.

Let $s = \left\lceil \frac{S-k+1}{c-1} \right\rceil$ and take an arbitrary k-coloration of $K_{c\times s}$. Using the same notation and performing as in Proposition 3, the inequality in (3) is true. Thus there is a monochromatic copy of $K_{1,j}$ for some color j, $1 \leq j \leq k$.

Claim B: $\left\lceil \frac{S-k}{c-1} \right\rceil \leq m_c(K_{1,n_1}, \ldots, K_{1,n_k})$.

Let $s = \left\lfloor \frac{S-k-1}{c-1} \right\rfloor$. The arguments are similar to those used in Proposition 5. An application of Vizing's Theorem ensures that there is a k-coloration of $K_{c\times s}$ with no copy of K_{1,n_i} in color i for any $i = 1, \ldots, n$. Thus

$$m_c(K_{1,n_1}, \ldots, K_{1,n_k}) \geq s + 1 = \left\lfloor \frac{S-k-1}{c-1} \right\rfloor + 1 = \left\lceil \frac{S-k}{c-1} \right\rceil.$$

As an immediate combination of both claims above, $m_c(K_{1,n_1}, \ldots, K_{1,n_k}) = \left\lceil \frac{S-k+1}{c-1} \right\rceil$ holds when $c - 1$ does not divide $S - k$. From now on, we assume that $c - 1$ divides $S - k$, say, $S - k = (c - 1)z$.

Claim C: If c or z is even, then $m_c(K_{1,n_1}, \ldots, K_{1,n_k}) = \left\lceil \frac{S-k+1}{c-1} \right\rceil$.

Indeed, the upper bound is derived by Claim 1. It remains to show the lower bound $m_c(K_{1,n_1}, \ldots, K_{1,n_k}) > z$. Since c or z is even, Theorem 7 ensures $\chi'(K_{c\times z}) = z(c-1)$. Thus there is a $\big(z(c-1)\big)$-coloration of $K_{c\times z}$ with no monochromatic copy of $K_{1,2}$, denoted by $\phi : E(K_{c\times z}) \longrightarrow \mathbb{Z}_{z(c-1)}$. Since $z(c-1) = S-k = \sum_{i=1}^{n}(n_i-1)$, we can select a function $\varphi : \mathbb{Z}_{z(c-1)} \longrightarrow \{1, \ldots, k\}$ such that $|\varphi^{-1}(i)| = n_i - 1$ for any $i = 1, \ldots, k$. The composition $\varphi \circ \phi : E(K_{c\times z}) \longrightarrow \{1, \ldots, k\}$ is a k-coloration of $K_{c\times z}$ with no copy of K_{1,n_i} in color i for any $i = 1, \ldots, k$.

Claim D: Let c and z be odd. If n_j is even for some $j \in \{1, \ldots, k\}$, then $m_c(K_{1,n_1}, \ldots, K_{1,n_k}) = z = \frac{S-k}{c-1}$.

Indeed, Claim B states that $m_c(K_{1,n_1}, \ldots, K_{1,n_k}) \geq z$. Suppose for a contradiction that $m_c(K_{1,n_1}, \ldots, K_{1,n_k}) > z$. Thus there is a k-coloration of $K_{c\times z}$ with no copy of K_{1,n_i} in color i for any $i = 1, \ldots, k$. Proceeding as the proof in Proposition 9, we obtain $2|E_j| = cz(n_j - 1)$, where $|E_j|$ denotes the number of edges colored by color j. The last equality contradicts the assumption that c, z, and $(n_j - 1)$ are all odd numbers.

Claim E: Let both c and z be odd. If n_i is odd for any $i \in \{1, \ldots, k\}$, then $m_c(K_{1,n_1}, \ldots, K_{1,n_k}) = z + 1 = \left\lceil \frac{S-k+1}{c-1} \right\rceil$.

The proof is essentially based on Proposition 10. With the change of variables $c = q+1$ and $z = s$, the following statements hold: $S - k = qs$, q is even, s is odd, and n_i is odd for any $i = 1, \ldots, k$. As an application of Proposition 10, we can build a k-coloration of $K_{c\times z} (= K_{(q+1)\times s})$ with no copy of K_{1,n_i} in color i for any $i = 1, \ldots, n$. Thus $m_c(K_{1,n_1}, \ldots, K_{1,n_k}) \geq z+1 = \left\lceil \frac{S-k+1}{c-1} \right\rceil$. The upper bound follows from Claim A.

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