Discrete Mathematics 342 (2019) 2204–2212

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# Exact Ramsey numbers in multipartite graphs arising from Hadamard matrices and strongly regular graphs

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ARTICLE INFO

*Article history:*
Received 13 January 2019
Received in revised form 24 April 2019
Accepted 27 April 2019
Available online 17 May 2019

*Keywords:*
Graph Ramsey number
Multipartite graph
Bipartite graph
Strongly regular graph
Hadamard matrix

ABSTRACT

In this paper we investigate bounds on set multipartite Ramsey numbers for the bipartite graph $K_{2,n}$, extending or improving well-known upper bounds by Chung and Graham, Irving, Lortz and Mengersen. Known constructions based on certain classes of combinatorial designs (projective plane, Hadamard matrix, strongly regular graph) yield near-optimal bounds. As the main goal, a new construction based on strongly regular graph and Hadamard matrix produces a sharp class, generalizing a classical bound by Exoo, Harborth, and Mengersen.



## 1. Introduction

### 1.1. Graph Ramsey numbers from designs

A great challenge in Combinatorics is that of determining the celebrated Ramsey numbers for graphs, defined as follows. Given simple graphs $G_1, \ldots, G_k$, the *Ramsey number* $r(G_1, \ldots, G_k)$ denotes the smallest positive integer $n$ such that any $k$-coloring of the edges of a complete graph $K_n$ on $n$ vertices contains a monochromatic copy of $G_i$ in color $i$ for some $i$, $1 \leq i \leq k$. We refer to the book [12] for an overview on Ramsey theory.

One remarkable feature of Ramsey theory is the use of tools from many fields of mathematics. Some of these connections received particular attention, as stated by T.D. Parsons in Mathematical Reviews MR664707 of [10].

> "Of all the results in Ramsey graph theory, the most intriguing are those which relate families of Ramsey numbers to other areas of mathematics, particularly algebra and combinatorial designs..."

Indeed, exact or near-optimal values of several Ramsey numbers depend on the existence of suitable combinatorial designs: projective plane [14,24], resolvable design [8], difference set [5], design admitting polarity [11,25], to cite a few references. Many of these connections are briefly described in [12,26].

Despite the importance, the topic is so difficult that advances on these links have been rarely discovered in the past 20 years. As far as we know, the last contribution seems to be [1] in 2006.

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1 The first author is partially supported by Capes, Brazil/MCT.
2 The second author is partially supported by CNPq/MCT, Brazil Grant: 311703/2016-0.

https://doi.org/10.1016/j.disc.2019.04.023

### 1.2. Ramsey numbers in multipartite graphs

Ramsey numbers can be described in a more general setting: when the host graph is multipartite. More formally, let $K_{c\times s}$ denote a complete multipartite graph having $c$ classes with $s$ vertices in each class. Given a positive integer $s$ and simple graphs $G_1, \ldots, G_k$, the *set multipartite Ramsey number* $M_s(G_1, \ldots, G_k)$ denotes the smallest positive integer $c$ such that any $k$-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of $G_i$ in color $i$ for some $i$, $1 \le i \le k$. As usual, the case where $G_i = G$ for $1 \le i \le k$ is abbreviated to $M_s(G; k)$. In particular, the classical Ramsey number $r(G_1, \ldots, G_k)$ can be regarded as $M_1(G_1, \ldots, G_k)$ since the graph $K_{c\times 1}$ is isomorphic to the complete graph $K_c$.

Set multipartite Ramsey numbers were introduced by Burger, Grobler, Stipp, and van Vuuren in [2,3], who studied $M_s(K_{m_1\times n_1}, K_{m_2\times n_2})$. The extension to many colors is established in [22] and closely related problems have been explored for special classes of graphs: paths [13,21], stripes versus small cycles [15], small cycles [21], stars [27].

### 1.3. The contribution

The main purpose of this paper is to discuss how certain classes of designs can be used for the computation of suitable set multipartite Ramsey numbers. We focus on the case where $G_i = K_{2,n_i}$ for any $i$. Since the numbers $r(K_{2,n_1}, \ldots, K_{2,n_k})$ have been investigated for $k = 2$ (see [9,19,20]) and for an arbitrary $k$ (see [5–7,14]), it seems to be natural analyzing the behavior of the corresponding extension $M_s(K_{2,n_1}, \ldots, K_{2,n_1})$.

As a goal of this work, density arguments allow us to produce upper bounds on $M_s(K_{2,n_1}, \ldots, K_{2,n_k})$ in Section 2.2, extending or improving previous bounds due to Chung and Graham [5], Irving [14], Lortz and Mengersen [19,20].

Moreover, we explore near-optimal bounds in Section 2.3 by using known constructions based on combinatorial designs or algebraic structures: projective planes (see Irving [14]) and finite fields (Lazebnik and Mubayi [16]). A celebrated connection is reported as follows.

**Theorem 1** (*Exoo, Harborth, and Mengersen [9]*)**.** *Let $n \ge 2$. Hence $M_1(K_{2,n}; 2) = 4n - 2$ if and only if there is a strongly regular graph with parameters $(4n - 3, 2n - 2, n - 2, n - 1)$.*

In particular, the existence of a strongly regular graph with parameters $(4n - 3, 2n - 2, n - 2, n - 1)$ produces $M_1(K_{2,n}; 2) \ge 4n - 2$. This is a special case of the main result of this work, stated below.

**Theorem 2.** *Suppose that there is a strongly regular graph with parameters $(4n - 3, 2n - 2, n - 2, n - 1)$ and a symmetric Hadamard matrix of order $m$. Then*

$$M_m(K_{2,m(n-1)+1}; 2) \ge 4n - 2. \tag{1}$$

According to Theorem 1, the existence of such graph produces the equality in (1) when $m = 1$. Moreover, we prove that the equality also holds for all sufficiently large $m$ relative to $n$, as established below.

**Theorem 3.** *Suppose that there is a strongly regular graph with parameters $(4n - 3, 2n - 2, n - 2, n - 1)$ and there is a symmetric Hadamard matrix of order $m$ with $m \ge 4n$. Then*

$$M_m(K_{2,m(n-1)+1}; 2) = 4n - 2.$$

Typically, the graph Ramsey numbers mentioned above are generated by just one class of designs. Perhaps surprisingly, Theorem 3 establishes a sharp class from two classes of designs. This new phenomenon seems to emphasize deep combinatorial questions that underlie the evaluation of Ramsey numbers, as pointed out by Ryser [28] and Parsons [25].

## 2. Bounds on $M_s(K_{2,n_1}, \ldots, K_{2,n_k})$

### 2.1. Lower bounds from a relationship

The link below extends a result due to Magnant and Yusko [21], which enables us to translate lower bounds on Ramsey numbers into lower bounds on multipartite Ramsey numbers.

**Theorem 4.** *Let $k$ and $s$ be positive integers, where $k \ge 2$. For simple graphs $G_1, \ldots, G_k$,*

$$\left\lfloor \frac{r(G_1, \ldots, G_k) - 1}{s} \right\rfloor + 1 \le M_s(G_1, \ldots, G_k).$$

**Proof.** Let $r = r(G_1, \ldots, G_k)$. By the choice of $r$, there is a $k$-coloring of the complete graph $K_{r-1}$ without a monochromatic copy of $G_i$ in color $i$, where $1 \le i \le k$. Define $c = \lfloor (r - 1)/s \rfloor$. Since $cs \le r - 1$, select a subgraph $H$ of $K_{r-1}$ isomorphic to $K_{c\times s}$ and consider the $k$-coloring restricted to $H$. Since $H$ contains no monochromatic copy of $G_i$ in color $i$, where $1 \le i \le k$, the result $M_s(G_1, \ldots, G_k) > \lfloor (r - 1)/s \rfloor = c$ follows. ■

We now discuss some classes of lower bounds on $M_s(K_{2,n}; k)$. To achieve this, we briefly review a few well-known lower bounds on Ramsey numbers. Irving [14] (independently, Chung and Graham [5]) proved that $r(K_{2,2}; k+1) \geq k^2+k+2$ for any prime power $k$, by using cyclic difference sets and projective planes. Lazebnik and Woldar [17] obtained the slightly improvement $r(K_{2,2}; k) \geq k^2+2$ if $k$ is an odd prime power.

Lazebnik and Mubayi [16] extended the previous result to all prime power (odd or even). Moreover, they also established that $r(K_{2,n}; k) \geq (n-1)k^2+1$, where $n-1$ and $k$ are powers of the same prime, by using properties of finite fields.

Combining all these results with Theorem 4, the following lower bounds are derived.

**Corollary 5.** *Given positive integers $\alpha, \beta, s$ and a prime number $p$, let $k = p^\alpha$ and $n = p^\beta + 1$. The following lower bounds hold:*

1. $$M_s(K_{2,2}; k+1) \geq \left\lfloor \frac{k^2+k+1}{s} \right\rfloor + 1.$$

2. $$M_s(K_{2,2}; k) \geq \left\lfloor \frac{k^2+1}{s} \right\rfloor + 1.$$

3. $$M_s(K_{2,n}; k) \geq \left\lfloor \frac{(n-1)k^2}{s} \right\rfloor + 1.$$

### 2.2. Upper bounds from density argument

The research on Ramsey numbers for bipartite graphs $K_{2,n}$ is immensely difficult even for the specific case $k = 2$. Exoo et al. [9] established that

$$r(K_{2,n}, K_{2,n}) \leq 4n-2, \tag{2}$$

improving $r(K_{2,n}, K_{2,n}) \leq 4n$ in [5]. Many bounds on off-diagonal cases were deeply studied by Lortz and Mergensen [19, 20]. In particular, they proved

$$r(K_{2,n}, K_{2,n+1}) \leq 4n \quad \text{and} \quad r(K_{2,n}, K_{2,n+a}) \leq 4n+2a-3 \tag{3}$$

for $a \geq 2$ and were able to present classes where the bounds above are optimal for small $a$. The literature on multicolored Ramsey numbers is rarer. Chung and Graham [5] obtained the upper bound

$$r(K_{2,n}; k) \leq (n-1)k^2+k+2 \tag{4}$$

and improved slightly the case $n = 2$ (independently proved by Irving [14])

$$r(K_{2,2}; k) \leq k^2+k+1. \tag{5}$$

A closely related bound is established in [22], more precisely,

$$M_s(K_{2,2}; k) \leq \lceil (k^2+k+2s-1)/s \rceil. \tag{6}$$

Density arguments have been successfully explored to improve several upper bounds on Ramsey numbers for bipartite graphs, according to [14,18–20] (when $k = 2$) and [5–7,22] (for arbitrary $k$). An adaptation of this approach is displayed as follows.

**Lemma 6.** *Let $k \geq 2, s, n_1, \ldots, n_k$ be positive integers. Suppose that $c$ is a positive integer such that*

$$kcs\binom{\frac{(c-1)s}{k}}{2} > \sum_{i=1}^{k}(n_i-1)\binom{cs}{2}, \tag{7}$$

*then $M_s(K_{2,n_1}, \ldots, K_{2,n_k}) \leq c$.*

**Proof.** Let $c$ be a positive integer that satisfies (7). Given an arbitrary $k$-coloring of $K_{c\times s} = (V, E)$, let $H_i = (V, E_i)$ denote the spanning subgraph of $K_{c\times s}$ formed by all edges in color $i$, where $i = 1, \ldots, k$.

For each color $i$, let $\alpha_i$ denote the number of stars of type $K_{1,2}$ in $H_i$. A key part of the proof consists in estimating $\alpha_i$. To achieve this, a star $K_{1,2}$ in $H_i$ can be represented by a pair $(v, A)$ formed by a vertex $v$ (the center of the star) and a subset $A$ (the basis of the star) of $V$ with $|A| = 2$ such that the set of edges $\{va : a \in A\}$ is contained in $E_i$. Thus each vertex $v$ of the graph $K_{c\times s}$ is the center of $\binom{d_i(v)}{2}$ distinct stars of type $K_{1,2}$ in $H_i$, where $d_i(v)$ denotes the degree of $v$ in $H_i$.

Since the binomial $\binom{\cdot}{2}$ is a convex function, Jensen's inequality and the handshaking lemma $\sum_{v\in V} d_i(v) = 2|E_i|$ imply

$$\alpha_i = \sum_{v\in V}\binom{d_i(v)}{2} \geq |V|\binom{\frac{\sum_{v\in V} d_i(v)}{|V|}}{2} = |V|\binom{\frac{2|E_i|}{|V|}}{2} = cs\binom{\frac{2|E_i|}{cs}}{2}.$$

Since $\sum_{i=1}^{k} |E_i| = |E| = c(c-1)s^2/2$, an application of Jensen's inequality again yields

$$\sum_{i=1}^{k}\alpha_i \geq cs\left[\sum_{i=1}^{k}\binom{\frac{2|E_i|}{cs}}{2}\right] \geq cs\left[k\binom{\frac{\sum_{i=1}^{k}2|E_i|/cs}{k}}{2}\right] = kcs\binom{\frac{(c-1)s}{k}}{2}.$$

Combining with the hypothesis in (7),

$$\sum_{i=1}^{k}\alpha_i > \sum_{i=1}^{k}(n_i-1)\binom{cs}{2},$$

hence $\alpha_j > (n_j - 1)\binom{cs}{2}$ holds for some color $j$. By the pigeonhole principle, there is a subset $A$ with $|A| = 2$ that is basis of more than $n_j - 1$ stars of type $K_{1,2}$ in $H_j$, that is, $H_j$ contains a copy of $K_{2,n_j}$. ■

We are ready to present upper bounds on $M_s(K_{2,n_1}, \ldots, K_{2,n_k})$.

**Theorem 7.** *Given positive integers $k \geq 2, n_1, \ldots, n_k$, denote $S = \sum_{i=1}^{k} n_i$.*

1. $M_1(K_{2,n_1}, \ldots, K_{2,n_k}) = r(K_{2,n_1}, \ldots, K_{2,n_k}) \leq k(S-k+1)+2$.
2. *For every $s \geq 2$,* $M_s(K_{2,n_1}, \ldots, K_{2,n_k}) \leq \left\lceil \frac{k(S-k+1)+2s-1}{s} \right\rceil$.
3. *In particular,* $M_s(K_{2,n}; k) \leq \left\lceil \frac{(n-1)k^2+k+2s-1}{s} \right\rceil$ *for any $s \geq 2$.*

**Proof.** The proof is based on an application of Lemma 6. Elementary calculations reveal that the inequality (7) is equivalent to

$$s^2c^2 - s(2s+k+kS-k^2)c + (s^2+sk+kS-k^2) > 0. \tag{8}$$

For the case where $s = 1$, note that the number $c = 2+k+kS-k^2$ satisfies the inequality above. Thus Lemma 6 implies the first part.

In order to establish the second part, assume $s \geq 2$. Since $k$ and $S$ are also fixed, let $f(c)$ denote the quadratic function induced by the left side of the inequality in (8). Write $t = k(S-k+1)+2s$ and $c_0 = (t-1)/s$. The result follows if we show that $f(\lceil c_0\rceil) > 0$. An analysis on its derivative shows that the real function $f$ is increasing in the interval $[t/(2s), \infty)$. Combining with the fact that $c_0 \geq t/(2s)$, we obtain

$$f(\lceil c_0\rceil) \geq f(c_0) = (s-1)^2 + k(s-1) > 0.$$

Lemma 6 completes the argument.

Part 3 is an immediate consequence of part 2. ■

Theorem 7 yields $r(K_{2,n_1}, K_{2,n_2}) \leq 2n_1 + 2n_2$, fairly close to those in (2) and (3). Indeed, our bound differs from (2) by exactly 2, and differs from those bounds in (3) by exactly 2 and 3, respectively. For the case where $n_1 = \cdots = n_k = n$, part 1 of Theorem 7 presents an alternative proof of the bound in (4). Moreover, the case where $n = 2$ of part 3 derives the bound in (6) for $s \geq 2$.

*2.3. The gap between lower bound and upper bound*

Suppose now that both $n-1$ and $k$ are powers of the same prime, denote $L = \lceil (n-1)k^2/s\rceil$ and $N = \lceil (k-1)/s\rceil + 2$. Part 3 of Corollary 5 and part 3 of Theorem 7 produce

$$L \leq M_s(K_{2,n}; k) \leq \left\lceil \frac{(n-1)k^2+k+2s-1}{s} \right\rceil \leq L+N \tag{9}$$

for any $s \geq 2$. Thus the gap between the lower and upper bounds is at most $N$, which does not depend on $n$. Table 1 illustrates numerical applications of (9) for a few instances, revealing that the gap can be very small.

Let $s$ be a positive integer and let $n-1$ be a power of 2. Part 3 of Corollary 5 yields

$$\left\lfloor \frac{4n-4}{s} \right\rfloor + 1 \leq M_s(K_{2,n}; 2).$$

**Table 1**
Bounds on $M_s(K_{2,n}; k)$.

| $n$ | $k$ | $s$ | Lower bound $L$ | Upper bound $L+N$ |
|---|---|---|---|---|
| 3 | 8 | 7 | 19 | 22 |
| 3 | 16 | 5 | 103 | 108 |
| 9 | 8 | 7 | 74 | 77 |
| 9 | 16 | 5 | 410 | 415 |
| $n$ | $k$ | $s \geq k-1$ | $L$ | $\leq L+3$ |
| $n$ | $k$ | $s \geq (k-1)/2$ | $L$ | $\leq L+4$ |
| $n$ | $k$ | $s \geq (k-1)/3$ | $L$ | $\leq L+5$ |

A slight improvement and extension are obtained in the next results, whose construction depends on suitable strongly regular graphs.

Given a vertex $v$ of a graph $G=(V,E)$, we denote by $N(v)$ the set of neighbors of $v$. Recall that a graph $G$ is *strongly regular with parameters* $(p,k,\lambda,\mu)$ when

- $G$ has $p$ vertices;
- $G$ is $k$-regular;
- if $vw$ is an edge of $G$, then $|N(v)\cap N(w)|=\lambda$;
- if $vw$ is not an edge of $G$, then $|N(v)\cap N(w)|=\mu$.

It is worth mentioning that strongly regular graphs are closely related to quasi-symmetric designs and designs with polarity [4].

**Corollary 8.** *Suppose that there is a strongly regular graph with parameters* $(4n-3, 2n-2, n-2, n-1)$. *For any* $s \geq 2$,

$$\left\lfloor \frac{4n-3}{s} \right\rfloor + 1 \leq M_s(K_{2,n}; 2) \leq \left\lceil \frac{4n-3}{s} \right\rceil + 2.$$

**Proof.** The upper bound is an immediate consequence of Theorem 7 part 3. A combination of Theorems 1 and 4 produces the lower bound. ■

**Example 9.** A classical construction states that a strongly regular graph with parameters $(4n-3, 2n-2, n-2, n-1)$ exists whenever $4n-3$ is a prime power, see [9]. Given a prime $p \equiv 1 (mod\ 4)$, let $s = p^{\alpha}$ and $n = (p^{\beta}+3)/4$ with $\beta > \alpha$. An application of the previous result produces

$$p^{\beta-\alpha} + 1 \leq M_s(K_{2,n}; 2) \leq p^{\beta-\alpha} + 2.$$

Since this gap does not exceed 1, any improvement would be very desirable.

Let us mention an intriguing behavior: constructions from combinatorial designs sometimes determine Ramsey numbers up to an error of 1, see [10,11,24,25] for instance.

## 3. A new lower bound from strongly regular graphs and Hadamard matrices

This section deals with the proof of Theorem 2. To achieve this purpose, let us build a lower bound on $M_s(K_{2,n}; 2)$. Unlike the constructions in [1,5,8,11,14,24,25], our method needs not only one but two classes of combinatorial designs. Two preliminary results are required.

**Lemma 10.** *Let* $G=(V(G),E(G))$ *be a strongly regular graph with parameters* $(4n-3, 2n-2, n-2, n-1)$. *For distinct vertices a and b, denote*

$$G_1 = \{c \in V(G) - \{a,b\} : ac \in E(G) \text{ and } bc \in E(G)\},$$

$$G_2 = \{c \in V(G) - \{a,b\} : ac \notin E(G) \text{ and } bc \notin E(G)\},$$

$$G_3 = \{c \in V(G) - \{a,b\} : ac \in E(G) \text{ and } bc \notin E(G)\},$$

$$G_4 = \{c \in V(G) - \{a,b\} : ac \notin E(G) \text{ and } bc \in E(G)\}.$$

*The inequality* $|G_i| \leq n-1$ *holds for any i, where* $1 \leq i \leq 4$.

**Proof.** We firstly analyze the case where $ab \in E(G)$. The choice of $G$ yields $|G_1| = n-2$. Since the complement $\overline{G}$ of $G$ is also a strongly regular graph with parameters $(4n-3, 2n-2, n-2, n-1)$ (see [4]), the equality $|G_2| = n-1$ holds. It is easy to see that $|G_1|+|G_3|+|\{b\}| = 2n-2$ and $|G_1|+|G_4|+|\{a\}| = 2n-2$. A combination of the previous equalities yields $|G_3| = |G_4| = n-1$.

The proof for the case where $ab \notin E(G)$ follows analogously. ■

Recall that a *Hadamard matrix* of order $m$ is an $m \times m$ matrix $H$ in which every entry is $\pm 1$ such that $HH^t = mI_m$, where $I_n$ denotes the identity matrix of order $m$.

Hadamard matrices are particularly interesting as they generate special classes of symmetric designs, the so-called Hadamard 2-designs and Hadamard 3-designs. We refer to [29] for a survey on this topic.

**Lemma 11.** *Let $M = (m_{i,j})$ be a Hadamard matrix of order $m \geq 2$. For distinct integers $1 \leq i, j \leq m$, denote*

$$I_1 = \{k \in \{1, \dots, m\} : \ m_{i,k} = 1 \text{ and } m_{j,k} = 1\},$$

$$I_2 = \{k \in \{1, \dots, m\} : \ m_{i,k} = -1 \text{ and } m_{j,k} = -1\},$$

$$I_3 = \{k \in \{1, \dots, m\} : \ m_{i,k} = 1 \text{ and } m_{j,k} = -1\},$$

$$I_4 = \{k \in \{1, \dots, m\} : \ m_{i,k} = -1 \text{ and } m_{j,k} = 1\}.$$

*Then $|I_1| + |I_2| + |I_3| + |I_4| = m$ and $|I_1| + |I_2| = m/2$.*

**Proof.** Given distinct lines $i$ and $j$ of a Hadamard matrix $M$, note that $\sum_{k=1}^{m} m_{i,k} \cdot m_{j,k} = 0$, thus $|I_1| + |I_2| - |I_3| - |I_4| = 0$. Since $|I_1| + |I_2| + |I_3| + |I_4| = m$, we immediately conclude $|I_1| + |I_2| = m/2$. ■

The proof above is a simple adaptation of that for the well-known necessary condition: if there is a Hadamard matrix of order $m \geq 3$, then $m \equiv 0 \ (mod\ 4)$. Note that Lemma 11 also holds when $m = 2$.

We are ready to prove Theorem 2.

**Proof of Theorem 2.** Let $G = (V(G), E(G))$ be a strongly regular graph with parameters $(4n-3, 2n-2, n-2, n-1)$, where $V(G) = \{1, \dots, 4n-3\}$. Let $M = [m_{i,j}]_{m \times m}$ be a symmetric Hadamard matrix of order $m$. The vertex class of $K_{(4n-3)\times m}$ can be partitioned into the classes $L_1, \dots, L_{4n-3}$, where $L_a = \{(a, 1), \dots, (a, m)\}$ for each $a = 1, \dots, 4n-3$. Choose the coloring $\psi : E(K_{(4n-3)\times m}) \longrightarrow \{-1, 1\}$ defined by the rule

$$\psi\big((a, i)(b, j)\big) = \begin{cases} \ \ m_{i,j} \text{ if } ab \in E(G) \\ -m_{i,j} \text{ if } ab \notin E(G). \end{cases}$$

The symmetric matrix $M$ assures that $\psi$ is a well-defined coloring of the edges of $K_{(4n-3)\times m}$. It is enough to prove the following statement.

**Claim 1.** *Any two distinct vertices of $K_{(4n-3)\times m}$ are simultaneously connected to at most $m(n-1)$ vertices in each color.*

In order to prove Claim 1, we introduce the following notation. Given distinct vertices $v_1 = (a, i)$ and $v_2 = (b, j)$ and a color $w \in \{-1, 1\}$, let $\delta(v_1, v_2, w)$ denote the number of vertices of $K_{(4n-3)\times m}$ that are simultaneously connected to both $v_1$ and $v_2$ with color $w$. Thus Claim 1 corresponds to the inequality $\delta(v_1, v_2, w) \leq m(n-1)$. For this purpose, the analysis is divided into three cases:

*Case 1:* $a \neq b$ and $i = j$. We estimate how many vertices $(c, k)$ are simultaneously connected to $v_1 = (a, i)$ and $v_2 = (b, i)$ by color $w$. Consider an arbitrary $k \in \{1, \dots, m\}$. If $w = m_{i,k}$, then the construction of $\psi$ implies that $ac$ and $bc$ are both edges in $E(G)$. If $w = -m_{i,k}$, then $ac \notin E(G)$ and $bc \notin E(G)$. As mentioned before, $G$ and its complement $\overline{G}$ are strongly regular graphs with parameters $(4n-3, 2n-2, n-2, n-1)$. In both situations, there are at most $n-1$ possibilities in selecting $c$ for each $k$. Therefore, there are at most $m(n-1)$ choices for $(c, k)$. The argument ensures that $\delta(v_1, v_2, w) \leq m(n-1)$.

*Case 2:* $a = b$ and $i \neq j$. Since $i \neq j$, $m \geq 2$ holds trivially. Note that there are $2n-2$ neighbors of the vertex $a$ in $G$. By Lemma 11, the number of vertices simultaneously connected to $v_1 = (a, i)$ and $v_2 = (a, j)$ by color $w$ is exactly $|I_1|(2n-2) + |I_2|(2n-2)$. Lemma 11 again implies that

$$\delta(v_1, v_2, w) = (|I_1| + |I_2|)(2n-2) = (m/2)(2n-2) = m(n-1).$$

*Case 3:* $a \neq b$ and $i \neq j$. We analyze the contribution of each color separately.

*Case 3.1:* Firstly consider the subcase where $w = 1$. A vertex $(c, k)$ is simultaneously connected to $v_1 = (a, i)$ and $v_2 = (b, j)$ by color 1 if and only if one of the following situations occurs:

– $ac \in E(G)$, $bc \in E(G)$, $m_{i,k} = 1$, and $m_{i,j} = 1$;
– $ac \notin E(G)$, $bc \notin E(G)$, $m_{i,k} = -1$, and $m_{i,j} = -1$;
– $ac \in E(G)$, $bc \notin E(G)$, $m_{i,k} = 1$, and $m_{i,j} = -1$;
– $ac \notin E(G)$, $bc \in E(G)$, $m_{i,k} = -1$, and $m_{i,j} = 1$.

Adding up the contributions of these four situations,

$$\delta(v_1, v_2, 1) = |G_1||I_1| + |G_2||I_2| + |G_3||I_3| + |G_4||I_4|.$$

Lemma 10 states that $|G_i| \leq n-1$ for any $1 \leq i \leq 4$ and Lemma 11 ensures that $|I_1|+|I_2|+|I_3|+|I_4| = m$. Combining these facts,

$$\delta(v_1, v_2, 1) \leq (n-1)(|I_1|+|I_2|+|I_3|+|I_4|) = m(n-1).$$

*Case 3.2:* It remains the subcase where $w = -1$. A vertex $(c, k)$ is simultaneously connected to $v_1 = (a, i)$ and $v_2 = (b, j)$ by color $-1$ if and only if one of the situations occurs:

– $ac \in E(G)$, $bc \in E(G)$, $m_{i,k} = -1$, and $m_{i,j} = -1$;
– $ac \notin E(G)$, $bc \notin E(G)$, $m_{i,k} = 1$, and $m_{i,j} = 1$;
– $ac \in E(G)$, $bc \notin E(G)$, $m_{i,k} = -1$, and $m_{i,j} = 1$;
– $ac \notin E(G)$, $bc \in E(G)$, $m_{i,k} = 1$, and $m_{i,j} = -1$.

By a similar argument used when $w = 1$, we conclude

$$\begin{aligned}\delta(v_1, v_2, -1) &= |G_1||I_2| + |G_2||I_1| + |G_3||I_4| + |G_4||I_3| \\ &\leq (n-1)(|I_1|+|I_2|+|I_3|+|I_4|) \\ &\leq m(n-1)\end{aligned}$$

Since $\delta(v_1, v_2, w) \leq m(n-1)$ for all possible cases, there is not a monochromatic copy of $K_{2,m(n-1)+1}$ in the coloring induced by $\psi$. This argument concludes the proof. ■

It is worth mentioning that the construction in Theorem 1 corresponds to the particular case where $m = 1$. A closer look reveals that the influence of the Hadamard matrix $M = [1]$ is imperceptible on that construction, since $-1$ does not appear.

## 4. A sharp class

Suppose that there is a strongly regular graph with parameters $(4n-3, 2n-2, n-2, n-1)$. Exoo et al. [9] showed that the lower bound in (1) is optimal when $m = 1$ (Theorem 1). However, the upper bound in Corollary 8 combined with Theorem 2 does not produce the exact value of $M_m(K_{2,m(n-1)+1}; 2)$ for any $m$. Fortunately, this upper bound can be improved for sufficient large $m$, more precisely.

**Proposition 12.** *For integers $s, n \geq 2$ such that $s \geq 2\sqrt{n}+1$,*

$$M_s(K_{2,n}; 2) \leq \left\lceil \frac{4n-4}{s} \right\rceil + 2.$$

**Proof.** Denote $c_0 = \lceil (4n-4)/s \rceil + 2$. By Lemma 6, it is enough to show that $c = c_0$ satisfies the inequality

$$cs\binom{\frac{(c-1)s}{2}}{2} > (n-1)\binom{cs}{2},$$

or equivalently,

$$s^2c^2 - (2s+4n-2)sc + s^2 + 2s + 4n - 4 > 0.$$

Let $\beta = 2s+4n-2$ and $\gamma = s^2+2s+4n-4$. Consider the real function $f(c) = s^2c^2 - \beta sc + \gamma$ for all $c \in \mathbb{R}$. We need to show that $f(c_0) > 0$. Since $f$ is a convex quadratic function and the highest root of $f$ is $\left(\beta + \sqrt{\beta^2 - 4\gamma}\right)/(2s)$, it is sufficient to prove that

$$c_0 > \left(\beta + \sqrt{\beta^2 - 4\gamma}\right)/(2s).$$

The hypothesis $s \geq 2\sqrt{n}+1$ implies that $(s-1)^2 \geq 4n$. Consequently, the inequality $s^2 - 2s - 4n + 15/4 \geq 0$ holds, which is equivalent to $(\beta-4)^2 \geq \beta^2 - 4\gamma + 1$. Combining the arguments above,

$$c_0 = \left\lceil \frac{\beta + \sqrt{(\beta-4)^2}}{2s} \right\rceil \geq \left\lceil \frac{\beta + \sqrt{\beta^2 - 4\gamma + 1}}{2s} \right\rceil > \frac{\beta + \sqrt{\beta^2 - 4\gamma}}{2s}.$$

The proof is complete. ■

Although the improvement is very subtle, it is enough to show Theorem 3, as follows.

**Proof of Theorem 3.** Theorem 2 ensures that $M_m(K_{2,m(n-1)+1}; 2) \geq 4n-2$. On the other hand, the hypothesis $m \geq 4n$ implies $(m-1)^2 \geq 4m(n-1)+4$, and consequently $m \geq 2\sqrt{m(n-1)+1}+1$. An application of Proposition 12 concludes that $M_m(K_{2,m(n-1)+1}; 2) \leq 4n-2$. ■

As alluded before, there is a strongly regular graph with parameters $(4n-3, 2n-2, n-2, n-1)$ whenever $4n-3$ is a prime power.

**Theorem 13.** *Let $p$ be a prime number and $\alpha$ be a positive integer with $p^\alpha \equiv 1$ (mod 4). If there is a symmetric Hadamard matrix of order $4r$ with $4r > p^\alpha$, then*

$$M_{4r}(K_{2,r(p^\alpha-1)+1}; 2) = p^\alpha + 1.$$

**Proof.** Let $n = (p^\alpha + 3)/4$ and $m = 4r$. Since $n$ is an integer and $4n - 3$ is a prime power, there is a strongly regular graph with parameters $(4n - 3, 2n - 2, n - 2, n - 1)$. Furthermore, the conditions $4r > p^\alpha$ and $p^\alpha \equiv 1(mod 4)$ ensure $m = 4r \geq p^\alpha + 3 = 4n$. Thus the result follows from Theorem 3. ■

A famous conjecture states the existence of a symmetric Hadamard matrix of order $4r$ for all $r$. If this conjecture were valid, the magnitude of Theorems 2, 3 and 13 increases. Classes of such matrices are known for many parameters. For instance, the existence of a symmetric Hadamard matrix of order $4m^4$ for all odd $m$ is proved in [23]. Thus an application of Theorem 13 yields the exact class below.

**Corollary 14.** *Consider positive integers $\alpha, m, p$ such that $p$ is prime, $m$ is odd, $p^\alpha \equiv 1$ (mod 4) and $4m^4 > p^\alpha$. We have*

$$M_{4m^4}(K_{2,m^4(p^\alpha-1)+1}; 2) = p^\alpha + 1. \tag{10}$$

**Example 15.** In particular, for fixed $p$ and $\alpha$ under the conditions above, the equality (10) is attained for all sufficiently large odd $m$. For example,

$$M_{4m^4}(K_{2,8m^4+1}; 2) = 10$$

for all odd $m \geq 3$.

### Acknowledgments

The first author is partially supported by Capes. The second author is partially supported by CNPq/MCT grant: 311703/2016-0. The authors are grateful to A. G. Castoldi for several remarks. The authors would like to thank the anonymous referee for suggestions that improved this paper.

### Conflict of interest statement

None.

### Declaration of competing interest

The authors declare that they had no conflicts of interest with respect to their authorship or the publication of this article.

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