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Ramsey numbers in complete balanced multipartite graphs. Part II: Size numbers

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Abstract

The notion of a graph theoretic Ramsey number is generalised by assuming that both the original graph whose edges are arbitrarily bi-coloured and the sought after monochromatic subgraphs are complete, balanced, multipartite graphs, instead of complete graphs as in the classical definition. We previously confined our attention to diagonal multipartite Ramsey numbers. In this paper, the definition of a multipartite Ramsey number is broadened still further, by incorporating off-diagonal numbers, fixing the number of partite sets in the larger graph and then seeking the minimum cardinality of such partite sets that would ensure the occurrence of certain specified monochromatic multipartite subgraphs.

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1. Introduction

The classical graph theoretic Ramsey number $r(m,n)$ may be defined as the smallest natural number p with the property that, if the edges of the complete graph K_p are arbitrarily coloured using the colours red and blue, then a red K_m or a blue K_n will be forced as subgraph. In [3] we generalised this definition by taking both the original graph whose edges are to be bi-coloured and those which are sought as monochromatic subgraphs to be complete, balanced, multipartite graphs. However, we previously fixed the cardinality, j, of each partite set in the larger graph and sought the minimum number of partite sets, $\xi = M_j(K_{n\times l}, K_{s\times t})$,[1] of that cardinality that would ensure the occurrence of a red $K_{n\times l}$ or a blue $K_{s\times t}$ as subgraph in any (red, blue)-colouring of the edges of $K_{\xi\times j}$. We called the resulting number, $M_j(K_{n\times l}, K_{s\times t})$, the set multipartite Ramsey number. In this paper we rather fix the number of partite sets, and then seek the minimum cardinality of such partite sets that would ensure the occurrence of certain specified monochromatic multipartite subgraphs, and call this number the size multipartite Ramsey number.

Definition 1 (Size multipartite Ramsey numbers). Let j, l, n, s and t be natural numbers with $n,s \geqslant 2$. Then the size multipartite Ramsey number $m_j(K_{n\times l}, K_{s\times t})$ is the smallest natural number ζ such that an arbitrary colouring of the edges of $K_{j\times\zeta}$, using the two colours red and blue, necessarily forces a red $K_{n\times l}$ or a blue $K_{s\times t}$ as subgraph.

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[1] We denote a complete, balanced, multipartite graph consisting of n partite sets and l vertices per partite set by $K_{n\times l}$.

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The subgraphs in Definition 1 need not be vertex-induced subgraphs. This definition is a generalisation of that of the classical Ramsey numbers in the sense that if $r(\sigma,\lambda)=\tau$, then $m_\tau(K_{\sigma\times 1},K_{\lambda\times 1})=1$. The following symmetry property of off-diagonal multipartite Ramsey numbers holds.

Proposition 1 (Symmetry property). *If the multipartite Ramsey number $m_k(K_{n\times l},K_{s\times t})$ exists, then $m_k(K_{n\times l},K_{s\times t})=m_k(K_{s\times t},K_{n\times l})$.* □

Our goal in this paper is to determine new, small, off-diagonal size multipartite Ramsey numbers. After establishing a necessary and sufficient criterion for the existence of these numbers, as well as some basic properties of these numbers, in Section 2, we briefly review all known size multipartite Ramsey numbers and establish new small size numbers in Section 3. In Section 4, we turn to the problems of determining lower and upper bounds for larger size multipartite Ramsey numbers.

2. Existence and basic properties

The question of the existence of size multipartite Ramsey numbers is settled first.

Theorem 1 (Partial existence of size numbers). *The size multipartite Ramsey number $m_j(K_{n\times l},K_{s\times t})$ exists for any $n,s\geqslant 2$ and $l,t\geqslant 1$ if and only if $j\geqslant r(n,s)$.*

Proof. We first show that $m_j(K_{n\times l},K_{s\times t})$ exists for any $n,s\geqslant 2$ and $l,t\geqslant 1$ if $j\geqslant r(n,s)$. It is known that the diagonal bipartite Ramsey number $m_2(K_{2\times l},K_{2\times l})$ exists, and in fact that $m_2(K_{2\times l},K_{2\times l})\leqslant\binom{2l}{l}-1$ for all $l\geqslant 1$. This result is due to Hattingh and Henning [7]. Therefore, it is always possible to find an arbitrarily large monochromatic bipartite graph in any edge bi-colouring of a bipartite graph F, if the partite sets of F are "large enough". Now consider a complete, balanced multipartite graph G consisting of $r(n,s)$ "large enough" partite sets (the meaning of the phrase "large enough" will be made precise later in the proof). Colour the edges of G according to the following algorithm:

(1) Index each partite set of G as H_0.
(2) Select any two partite sets of G for which the connecting edges have not yet been coloured. If no edges incident to neither of these partite sets have yet been coloured, then colour all edges between these partite sets arbitrarily, using the colours red and blue. Note that a complete, balanced, monochromatic, bipartite graph will be forced by this sub-colouring and index both of the partite sets of this monochromatic subgraph as H_1. Else, if edges incident to one or both of the selected partite sets of G have been coloured previously, then select those subsets of vertices from each partite set of G with highest indices, say H_k and H_m, respectively, where $m\leqslant k$. An arbitrary bi-colouring (using the colours red and blue) of the edges between the vertices within the one partite set indexed as H_k and $|H_k|$ of the vertices amongst those within the other partite set indexed as H_m will force a complete, balanced, monochromatic, bipartite graph; index both partite sets of this monochromatic subgraph as H_{k+1}.
(3) Repeat step 2 until there are coloured edges between subsets of all pairs of partite sets.

This results in an expansive colouring[2] of $K_{r(n,s)\times|H_\alpha|}$ as subgraph of G, induced by some edge bi-colouring of $K_{r(n,s)}$, where H_α is the maximal index utilised in the above algorithm. Hence we will have a red $K_{n\times|H_\alpha|}$ or a blue $K_{s\times|H_\alpha|}$ as subgraph of G. Colour the remaining edges of G arbitrarily, using the colours red and blue. By choosing the original partite sets of G so large that $|H_\alpha|\geqslant\max\{l,t\}$, we will therefore have forced a red $K_{n\times l}$ or a blue $K_{s\times t}$ as subgraph of G via the above edge bi-colouring.

Finally, we show that $m_j(K_{n\times l},K_{s\times t})$ does not exist for any $n,s\geqslant 2$ and $l,t\geqslant 1$ if $j<r(n,s)$. Suppose $1\leqslant j<r(n,s)$ for some $n,s\geqslant 2$, then there exists a (red, blue)-colouring of the edges of K_j that contains neither a red K_n nor a blue K_s as subgraph. But, since $K_n\subseteq K_{n\times l}$ and $K_s\subseteq K_{s\times t}$ for any $l,t\geqslant 1$, the expansive colouring of $K_{j\times k}$ induced by this specific colouring of $E(K_j)$ contains neither a red $K_{n\times l}$ nor a blue $K_{s\times t}$ as subgraph, no matter how large we choose $k\geqslant 1$. □

It is possible to establish bounds on the size multipartite numbers in terms of bounds on the set multipartite numbers, and vice versa, as is done in the following theorem.

[2] See the definition of an expansive colouring in [3].

Theorem 2 (Set numbers versus size numbers).

(1) $m_k(K_{n\times l}, K_{s\times t}) > j$ *if and only if* $M_j(K_{n\times l}, K_{s\times t}) > k$, $\forall\ l,t \geqslant 1$ *and* $n,s \geqslant 2$.
(2) $m_k(K_{n\times l}, K_{s\times t}) \leqslant j$ *if and only if* $M_j(K_{n\times l}, K_{s\times t}) \leqslant k$, $\forall\ l,t \geqslant 1$ *and* $n,s \geqslant 2$.

Proof. (1) The inequality $M_j(K_{n\times l}, K_{s\times t}) > k$ holds if and only if there exists an arbitrary bi-colouring of the edges of $K_{k\times j}$ that contains neither a red $K_{n\times l}$ nor a blue $K_{s\times t}$ as subgraph, which may equivalently be restated as $m_k(K_{n\times l}, K_{s\times t}) > j$.
(2) This result follows from the previous result by a double contra-positive argument. □

It is also possible to establish growth properties for size multipartite Ramsey numbers. The proof of the following result is similar to that of Proposition 2 in [3], and will not be repeated here.

Proposition 2 (Growth properties). *Let* $n,s,\alpha,\gamma \geqslant 2$ *and* j, k, l, t, β *and* δ *be natural numbers. Then*

(1) $m_j(K_{n\times l}, K_{s\times t}) \leqslant m_j(K_{\alpha\times\beta}, K_{\gamma\times\delta})$ *if* $n \leqslant \alpha$, $l \leqslant \beta$, $s \leqslant \gamma$ *and* $t \leqslant \delta$ (*when both size multipartite Ramsey numbers exist*).
(2) $m_j(K_{n\times l}, K_{s\times t}) \leqslant m_k(K_{n\times l}, K_{s\times t})$ *if* $k \leqslant j$ (*when both size multipartite Ramsey numbers exist*). □

There are similar results to those of Propositions 2(1) and 2(2) for the classical and set multipartite Ramsey numbers (see [3, Proposition 2]), but note that the strictness of inequality property in the latter case does not necessarily hold for the size multipartite numbers $m_j(\cdot,\cdot)$, no matter which of the inequalities $n \leqslant \alpha$, $l \leqslant \beta$, $s \leqslant \gamma$ or $t \leqslant \delta$ are strict. Exactly when strict inequality occurs (as well as minimal bounds on the gaps in such strict inequalities) is characterised by the next result, whose proof is similar to that of Theorem 2 in [3], and will not be repeated here.

Theorem 3 (Gaps between size numbers). *For all integers* $n \geqslant 3$, $s \geqslant 2$ *and* $j,l,t \geqslant 1$, $m_j(K_{n\times l}, K_{s\times t}) \geqslant m_j(K_{(n-1)\times l}, K_{s\times t}) + \lceil t/\lfloor j/s \rfloor \rceil - 1$. □

We establish the following asymptotic limit for size multipartite Ramsey numbers.

Theorem 4 (Size number asymptotic limit). $m_j(K_{n\times l}, K_{s\times t}) \to 1$ *as* $j \to \infty$ *for any* $n,s \geqslant 2$ *and* $l,t \geqslant 1$.

Proof. We know, by Proposition 2(2), that the sequence $m_j(K_{n\times l}, K_{s\times t})$ is non-increasing for increasing j and any fixed values of $n,s \geqslant 2$ and $l,t \geqslant 1$. Therefore we only need to show that there exists a natural number k such that $m_k(K_{n\times l}, K_{s\times t}) = 1$. It is clear that $k = r(nl, st)$ is such a number, since every (red, blue)-colouring of the edges of $K_k \equiv K_{k\times 1}$ contains a red K_{nl} (in which case it also contains a red $K_{n\times l}$) or a blue K_{st} (in which case it also contains a blue $K_{s\times t}$). □

Note that the value of k in the proof of Theorem 4 is expected to be very conservative in the sense that the asymptotic unary limit of $m_j(K_{n\times l}, K_{s\times t})$ may possibly be attained long before $j = r(nl, st)$. Finally, the next result follows as corollary of Theorem 1.

Corollary 1 (Set number asymptotic limit). $M_j(K_{n\times l}, K_{s\times t}) \to r(n,s)$ *as* $j \to \infty$ *for any* $n,s \geqslant 2$ *and* $l,t \geqslant 1$.

3. Known and new small size numbers

There are only a few size multipartite Ramsey numbers known to the authors. These are $m_2(K_{2\times 2}, K_{2\times 3}) = 9$ and $m_2(K_{2\times 2}, K_{2\times 4}) = 14$ due to Hattingh and Henning [6], $m_2(K_{2\times 3}, K_{2\times 3}) = 17$ due to Beineke and Schwenk [1], $m_2(K_{2\times 4}, K_{2\times 4}) = 48$ due to Irving [8] and the complete class of $(K_{2\times 2}, K_{2\times 2})$ multipartite Ramsey numbers, as listed in Table 1.

Bounds for small, diagonal as yet undetermined size multipartite Ramsey numbers may be found in [9]. The following proposition provides values for simple general classes of multipartite Ramsey numbers.

Proposition 3 (Basic size multipartite numbers).

(1) $m_j(K_{2\times 1}, K_{s\times t}) = \lceil t/\lfloor j/s \rfloor \rceil$ *for all* $t \geqslant 1$ *and* $j \geqslant s \geqslant 2$.
(2) $m_j(K_{n\times 1}, K_{s\times 1}) = 1$ *for all* $n,s \geqslant 2$ *and* $j \geqslant r(n,s)$.

Table 1
The class of $(K_{2\times 2}, K_{2\times 2})$ size multipartite Ramsey numbers

j	$m_j(K_{2\times 2}, K_{2\times 2})$
1	∞[a]
2	5[b]
3	3[b]
4	2[b]
5	2[b]
$\geqslant 6$	1[b]

[a]By Theorem 1.
[b]Due to Day, et al. [4].

Proof. (1) Suppose $t \geqslant 1$ and $j \geqslant s \geqslant 2$, and define c as the smallest natural number such that $K_{s\times t} \subseteq K_{j\times c}$. It is clear that any two vertices in the same partite set of $K_{j\times c}$ may not be in separate partite sets of $K_{s\times t}$. This means that, to find a $K_{s\times t}$ as subgraph in $K_{j\times c}$, we must group together *full* partite sets of $K_{j\times c}$ to form s new partite sets. This gives $\lfloor j/s \rfloor$ partite sets of $K_{j\times c}$ in each grouping. To ensure that there are t vertices in each grouping, there must be at least $c = \lceil t/\lfloor j/s \rfloor \rceil$ vertices per partite set in $K_{j\times c}$.

(2) From classical Ramsey theory we know that $r(n,s) = w$ (say) partite sets is sufficient to force a red $K_{n\times 1}$ or a blue $K_{s\times 1}$ as subgraph of any (red, blue)-colouring of the edges of $K_{w\times 1}$. Therefore, $m_j(K_{n\times 1}, K_{s\times 1}) \leqslant 1$ for all $j \geqslant w$. But then it follows from Definition 1 that $m_j(K_{n\times 1}, K_{s\times 1}) = 1$ for all $j \geqslant w$. □

Finally, we conclude this section by fully establishing the new class of $(K_{2\times 2}, K_{3\times 1})$ size multipartite Ramsey numbers.

Theorem 5 (The class of $(K_{2\times 2}, K_{3\times 1})$ size multipartite numbers).

(1) $m_1(K_{2\times 2}, K_{3\times 1}) = m_2(K_{2\times 2}, K_{3\times 1}) = \infty$.
(2) $m_3(K_{2\times 2}, K_{3\times 1}) = 3$ *and* $m_4(K_{2\times 2}, K_{3\times 1}) = 2$.
(3) $m_5(K_{2\times 2}, K_{3\times 1}) = m_6(K_{2\times 2}, K_{3\times 1}) = 2$.
(4) $m_j(K_{2\times 2}, K_{3\times 1}) = 1$ *for all* $j \geqslant 7$.

Proof. (1) By Theorem 1, since $r(2,3) = 3$.

(2) By Theorem 3(2) in [3], $M_2(K_{2\times 2}, K_{3\times 1}) = 4 > 3$, so that $m_3(K_{2\times 2}, K_{3\times 1}) > 2$ by Theorem 2(1). By Theorem 3(3) in [3], $M_3(K_{2\times 2}, K_{3\times 1}) \leqslant 3$, so that $2 < m_3(K_{2\times 2}, K_{3\times 1}) \leqslant 3$ by Theorem 2(2). Now $1 < m_4(K_{2\times 2}, K_{3\times 1}) \leqslant 2$ follows in a similar fashion, because $M_1(K_{2\times 2}, K_{3\times 1}) = 7 > 4$ and $M_2(K_{2\times 2}, K_{3\times 1}) \leqslant 4$.

(3) By Theorem 3(1) in [3], $M_1(K_{2\times 2}, K_{3\times 1}) = 7 > 6$, so that $1 < m_6(K_{2\times 2}, K_{3\times 1}) \leqslant m_5(K_{2\times 2}, K_{3\times 1}) \leqslant m_4(K_{2\times 2}, K_{3\times 1}) = 2$ by Theorem 2(1) and Proposition 2(2).

(4) By Theorem 3(1) in [3], $M_1(K_{2\times 2}, K_{3\times 1}) \leqslant j$ for all $j \geqslant 7$. Hence it follows by Theorem 2(2) that $1 \leqslant m_j(K_{2\times 2}, K_{3\times 1}) \leqslant 1$ for all $j \geqslant 7$. □

4. Bounds on size numbers

It is possible to provide a simple lower bound for size multipartite Ramsey numbers.

Proposition 4 (Direct lower bound). *For all integers* $j, l, t \geqslant 1$ *and* $n, s \geqslant 2$, $m_j(K_{n\times l}, K_{s\times t}) \geqslant \min\{\lceil nl/j \rceil, \lceil st/j \rceil\}$.

Proof. The graphs $K_{n\times l}$ and $K_{s\times t}$ have nl and st vertices respectively. Hence there must be at least $\min\{\lceil nl/j \rceil, \lceil st/j \rceil\}$ vertices per partite set in a complete, balanced, multipartite graph comprising j partite sets in order to possibly contain $K_{n\times l}$ or $K_{s\times t}$ as subgraph. □

Using the probabilistic method described by Erdös and Spencer [5], it is possible to establish the following general size multipartite Ramsey lower bound.

Theorem 6 (Probabilistic lower bound).

$$m_j(K_{n\times l},K_{s\times t}) > \min\left\{ \sqrt[nl]{n!(l!)^n 2^{l^2\binom{n}{2}-1}},\ \sqrt[st]{s!(t!)^s 2^{t^2\binom{s}{2}-1}} \right\} \Big/ j.$$

for all $n,s \geqslant 2$ *and* $j,l,t \geqslant 1$.

The result follows immediately from Theorem 4 in [3], via Theorem 2(1) in this paper.

Hattingh and Henning [7] established the bipartite set size upper bound

$$m_2(K_{2\times l},K_{2\times l}) \leqslant \binom{2l}{l} - 1 \quad \text{for all} \quad l \geqslant 1. \tag{4.1}$$

The following result, for which a proof may be found in Burger, et al. [2] and in Stipp [9], is a more general, yet weaker, result than the upper bound in (4.1).

Theorem 7 (Diagonal bipartite upper bound).

$$m_j(K_{2\times l},K_{2\times l}) \leqslant \max\left\{ 2l-1, \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{j-1} \right\rceil \right\}$$

for all $j \geqslant 2$ *and* $l \geqslant 1$. □

5. Conclusion

In this paper the notion of a graph theoretic Ramsey number was generalised by replacing the requirement of a complete graph in the classical definition by that of a complete, balanced, multipartite graph following the general approach by Burger, et al. [2] in the diagonal special case. The notion of a size multipartite Ramsey number involved fixing the number of partite sets in the larger graph and then seeking the minimum cardinality of such partite sets that would ensure the occurrence of certain specified monochromatic multipartite subgraphs. The existence of these generalised Ramsey numbers was established and some new, small size numbers were found, as well as lower and upper bounds for larger size numbers.

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