Discrete Mathematics 339 (2016) 2775–2784

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Multicolored set multipartite Ramsey numbers[☆]

Emerson L. Monte Carmelo[*], Juliana Sanches

Departamento de Matemática, Estadual de Maringá, Brazil

ARTICLE INFO

Article history:
Received 3 February 2015
Received in revised form 27 January 2016
Accepted 9 May 2016
Available online 13 June 2016

Keywords:
Ramsey number
Multipartite graph
Coloring
Density argument

ABSTRACT

Set multipartite Ramsey numbers were introduced by Burger, Grobler, Stipp and van Vuuren in 2004, generalizing the celebrated Ramsey numbers. In this work we extend set multipartite Ramsey numbers to an arbitrary number of colors. Growth properties, connections with classical Ramsey numbers, general lower and upper bounds are obtained, including some improvements of known bounds. We then focus on the case where a monochromatic bipartite graph is required by exploring density arguments and connections with well-known results.

1. Introduction

Let K_r denote the complete graph on r vertices. Given positive integers $n_1 \geq 2$ and $n_2 \geq 2$, the celebrated Ramsey number $r(n_1, n_2)$ denotes the smallest natural number r such that every red–blue coloring of the edges of K_r contains a red copy of K_{n_1} or a blue copy of K_{n_2}. Determining Ramsey numbers has been a great challenge in combinatorics since 1930. Indeed, the only known exact values for the diagonal case are $r(2, 2) = 2$, $r(3, 3) = 6$ and $r(4, 4) = 18$, but $r(5, 5)$ still remains an open problem. Up-to-date tables on bounds are available in [19]. We refer to the book [15] for an overview on Ramsey theory.

A large number of concepts, variants, and extensions have been widely investigated in many directions. In particular, Burger et al. [6,7] generalized the Ramsey numbers by assuming that both the original graph and the sought after monochromatic graph are complete, balanced and multipartite graphs. More precisely, let $K_{c\times s}$ denote the multipartite graph having c classes with s vertices per each class. In particular, note that $K_{1\times s}$ denotes the complement of K_s (s isolated vertices), and $K_{2\times s}$ corresponds to the bipartite graph $K_{s,s}$.

Given positive integers s, n_1, m_1, n_2, m_2, with $n_1, n_2 \geq 2$, the *set multipartite Ramsey number* $M_s(K_{n_1\times m_1}, K_{n_2\times m_2})$ is the smallest natural number c such that every red–blue coloring of the edges of $K_{c\times s}$ contains either a red $K_{n_1\times m_1}$ or a blue $K_{n_2\times m_2}$. It is worth mentioning that these numbers can be regarded as an extension of the classical Ramsey numbers. Indeed, $M_1(K_{n_1\times 1}, K_{n_2\times 1}) = r(K_{n_1}, K_{n_2}) = r(n_1, n_2)$ because $K_{n\times 1}$ is isomorphic to K_n.

Many results on $M_s(K_{n_1\times m_1}, K_{n_2\times m_2})$ are presented in [6,7]. In addition to proving the existence of these numbers, the authors obtain growth properties, relationships with classical Ramsey numbers as well as several results concerning general lower and upper bounds.

In this article we extend the set multipartite Ramsey numbers to an arbitrary number of colors, described in Section 2. As the first goal, most results in [7] are extended in Sections 3 and 4, including general lower and upper bounds. As the

[☆] A preliminary version of this work appeared as an extended abstract in the proceedings of the 5th Latin-American Workshop on Cliques in Graphs, held in Buenos Aires, Argentina, on November, 2012 *Multicolored Ramsey Numbers in Multipartite Graphs* Matemática Contemporânea, 42 (2014) 115–122, http://mc.sbm.org.br/images/pdf/42/a12.pdf.

[*] Corresponding author.
E-mail addresses: elmcarmelo@uem.br (E.L. Monte Carmelo), juu.sanches@gmail.com (J. Sanches).

http://dx.doi.org/10.1016/j.disc.2016.05.005
0012-365X/

second goal, we focus on the case where a monochromatic bipartite graph is required. For this purpose, we explore density arguments similar to Turán numbers as well as an extension of a link by Chvátal and Harary [10] in Section 5. Then sharper upper bounds for certain classes of parameters are obtained, improving previous bounds under certain parameters.

In Section 6, a new link with classical Ramsey numbers for multipartite graphs is established too, which allows us to derive lower bounds from several known results of literature: an exact value due to Bialostocki and J. Schönheim [3], a bound by Lazebnik and Woldar [18], for instance. Moreover, a sharp asymptotic result for the four cycle $K_{2,2}$ completes this article, generalizing a well-known result by Irving [16] and, independently, by Chung and Graham [9].

2. Existence and relationships with classical Ramsey numbers

2.1. Existence

We begin with the definition of set multipartite Ramsey number for several colors.

Definition 1. Given positive integers $s \geq 1$, $n_i \geq 2$, $m_i \geq 1$ for $1 \leq i \leq k$, *the set multipartite Ramsey number* $M_s(K_{n_1\times m_1}, \ldots, K_{n_k\times m_k})$ denotes the smallest c such that for each k-coloring of the edges of $K_{c\times s}$, there is a monochromatic copy of $K_{n_i\times m_i}$ with color i for some $1 \leq i \leq k$.

In particular, note that the classical Ramsey number $r(n_1, \ldots, n_k)$ for k colors can be regarded as $M_1(K_{n_1\times 1}, \ldots, K_{n_k\times 1})$. The case where $n_i = n$ and $m_i = m$ for every $1 \leq i \leq k$ is simplified by $M_s(K_{n\times m}; k)$.

The existence for the general case is guaranteed as follows.

Theorem 2. *Given positive integers $s \geq 1$, $n_i \geq 2$ and $m_i \geq 1$ for $1 \leq i \leq k$, the number $M_s(K_{n_1\times m_1}, \ldots, K_{n_k\times m_k})$ is well-defined and*

$$M_s(K_{n_1\times m_1}, \ldots, K_{n_k\times m_k}) \leq r(n_1m_1, \ldots, n_km_k).$$

Proof. Let $c = r(n_1m_1, \ldots, n_km_k)$. Given an arbitrary k-coloring of $K_{c\times s}$, choose one vertex of each class and let V be the set formed by these c chosen vertices. Because the graph induced by V is isomorphic to K_c, the coloring of $K_{c\times s}$ above induces a k-coloring of K_c. By definition of c, this induced coloring contains a monochromatic copy of $K_{n_im_i}$ for some color i. Since $K_{c\times s}$ contains K_c and $K_{n_i\times m_i}$ is a subgraph of $K_{n_im_i}$, thus a monochromatic copy of $K_{n_i\times m_i}$ occurs. ■

2.2. A lower bound from classical Ramsey numbers

Exploring relationships between multipartite Ramsey numbers and classical Ramsey numbers seems to be a natural source of research, like Theorem 2. Another link is based on the concept of expansive coloring, investigated in [7,12]. A coloring of edges in $K_{c\times s}$ is called an *expansive coloring* if it satisfies the property: all edges induced by each pair of classes in $K_{c\times s}$ have the same color.

Theorem 3. *Given positive integers $s \geq 1$, $n_i \geq 2$, $m_i \geq 1$ for $1 \leq i \leq k$, the following inequality holds*

$$\max\{r(n_1, \ldots, n_k), \min\{\lceil m_i/s\rceil n_i : 1 \leq i \leq k\}\} \leq M_s(K_{n_1\times m_1}, \ldots, K_{n_k\times m_k}).$$

Proof. The proof is divided into two parts:

Part 1: We first prove that $r(n_1, \ldots, n_k) \leq M_s(K_{n_1\times m_1}, \ldots, K_{n_k\times m_k})$. Let $c = r(n_1, \ldots, n_k)$. By the choice of c, there is a k-coloring G of K_{c-1} that does not contain any monochromatic copy of K_{n_i} in color i, where $1 \leq i \leq k$. Take the expansive coloring H of $K_{(c-1)\times s}$ induced by G, namely, the edges between the classes C_u and C_v in $K_{(c-1)\times s}$ are colored with the color of the edge uv in G. Suppose for a contradiction that H contains a monochromatic copy of K_{n_i} for some color i. These n_i vertices in K_{n_i} have to belong to distinct classes of H, which induce a monochromatic copy of K_{n_i} in G because H is expansive. This is a contradiction with the construction of G. Since K_{n_i} is isomorphic to $K_{n_i\times 1}$ and $K_{n_i\times 1}$ is a subgraph of $K_{n_i\times m_i}$, the graph $K_{(c-1)\times s}$ does not contain a monochromatic copy of $K_{n_i\times m_i}$ with color i.

Part 2: It remains to prove the second inequality, i.e,

$$\min\{\lceil m_i/s\rceil n_i : 1 \leq i \leq k\} \leq M_s(K_{n_1\times m_1}, \ldots, K_{n_k\times m_k}).$$

Let q be a positive integer such that $q < \min\{\lceil \frac{m_i}{s}\rceil n_i : 1 \leq i \leq k\}$. We claim that any arbitrary k-coloring of $K_{q\times s}$ does not contain a copy of $K_{n_i\times m_i}$, where $1 \leq i \leq k$. Two cases are analyzed in order to prove this statement:

Case 1: if $m_i \leq s$. In this case, $q < \lceil m_i/s\rceil\, n_i \leq n_i$; and consequently K_{n_i} is not a subgraph of $K_{q\times s}$. Combining with the fact that $K_{n_i} \subseteq K_{n_i\times m_i}$, we have that $K_{n_i\times m_i}$ is not a subgraph of $K_{q\times s}$.

Case 2: if $m_i > s$. Suppose for a contradiction that $K_{n_i\times m_i}$ is a subgraph of $K_{q\times s}$ with the color i. For this case, note that two vertices in the same class of $K_{q\times s}$ are forbidden to belong to distinct classes in $K_{n_i\times m_i}$. We need at least $\lceil m_i/s\rceil$ classes in $K_{q\times s}$ in order to obtain each class of $K_{n_i\times m_i}$. Then we need at least $\lceil m_i/s\rceil\, n_i$ classes in $K_{q\times s}$ in order to obtain a copy of $K_{n_i\times m_i}$, and consequently the contradiction $q \geq \lceil m_i/s\rceil\, n_i$ holds. ■

A simple argument shows us that $M_s(K_{2\times 1}, \ldots, K_{2\times 1}, K_{n\times m}) = \lceil \frac{m}{s}\rceil n$.

3. Some basic properties and some bounds

3.1. Growth properties

Proposition 4. *Given positive integers $s \geq 1$, $m_i \geq 1$, $t_i \geq 1$, $p_i \geq 2$, $n_i \geq 2$ for $1 \leq i \leq k$, the relations below hold:*

1. $M_s(K_{n_1\times m_1}, \dots, K_{n_k\times m_k}) \leq M_s(K_{p_1\times t_1}, \dots, K_{p_k\times t_k})$ *if* $n_i \leq p_i$, $m_i \leq t_i$ *for every* i, $1 \leq i \leq k$. *The inequality is strict if* $n_i < p_i$ *for some* i, $1 \leq i \leq k$.
2. $M_s(K_{n_1\times m_1}, \dots, K_{n_k\times m_k}) \leq M_q(K_{n_1\times m_1}, \dots, K_{n_k\times m_k})$ *if* $q \leq s$.

Proof. The proofs are similar to those given in [7, Proposition 2].

For part 1, let $w = M_s(K_{p_1\times t_1}, \dots, K_{p_k\times t_k})$. By the definition of w, any k-coloring of $K_{w\times s}$ contains a monochromatic copy of $K_{p_i\times t_i}$ for some color i, $1 \leq i \leq k$. The assumptions $n_i \leq p_i$ and $m_i \leq t_i$ yield that $K_{n_i\times m_i} \subseteq K_{p_i\times t_i}$. Then the subgraph of $K_{w\times s}$ induced by the color i contains a monochromatic copy of $K_{n_i\times m_i}$. We conclude that $M_s(K_{n_1\times m_1}, \dots, K_{n_k\times m_k}) \leq w$.

Without loss of generality, suppose that $n_1 < p_1$, $n_i \leq p_i$ for $2 \leq i \leq k$ and $m_i \leq t_i$ for $1 \leq i \leq k$. Suppose for a contradiction that $v = M_s(K_{n_1\times m_1}, \dots, K_{n_k\times m_k}) = M_s(K_{p_1\times t_1}, \dots, K_{p_k\times t_k})$. Given $G = K_{v\times s}$, the graph $K_{(v-1)\times s}$ is regarded as a subgraph of G.

By the definition of v, there is a k-coloring of $K_{(v-1)\times s}$ that contains no monochromatic copy of $K_{n_i\times m_i}$ for every i, $1 \leq i \leq k$.

It remains v copies of $K_{1\times s}$ in order to extent this coloring to G. We color all remaining edges with the color 1. Note that this coloring of G contains no copy of $K_{(n_1+1)\times m_1}$ (hence no copy of $K_{p_1\times t_1}$) at color 1. Moreover, G contains no copy of $K_{n_i\times m_i}$ (hence no copy of $K_{p_i\times t_i}$) for each color i, $2 \leq i \leq k$. The arguments above imply $M_s(K_{p_1\times t_1}, \dots, K_{p_k\times t_k}) > v$, a contradiction.

For part 2, let $w = M_q(K_{n_1\times m_1}, \dots, K_{n_k\times m_k})$. Consider an arbitrary coloring of $K_{w\times s}$ with k colors. The assumption $q \leq s$ forces $K_{w\times q} \subseteq K_{w\times s}$, so, this coloring induces a k-coloring of $K_{w\times q}$. By the definition of w, the graph $K_{w\times q}$ contains a monochromatic copy of $K_{n_i\times m_i}$ for some color i. Consequently, $K_{w\times s}$ contains a monochromatic copy of $K_{n_i\times m_i}$ at color i, proving the assertion. ■

Note that $M_s(K_{n_1\times 1}, \dots, K_{n_k\times 1}) = r(n_1, \dots, n_k)$ is derived from Theorems 2 and 3.

3.2. Gaps between Ramsey numbers

Gaps between Ramsey numbers can be investigated from a constructive approach. We now illustrate this method. Let $a = r(n_1 - 1, n_2)$. Thus, there exists a red–blue coloring of K_{a-1} that contains neither a red K_{n_1-1} nor a blue K_{n_2}. Consider the coloring of K_{a+n_2-2} by joining to K_{a-1} above a blue K_{n_2-1} and coloring all remaining (interconnecting) edges red. Therefore, this construction leads to $r(n_1, n_2) \geq r(n_1 - 1, n_2) + n_2 - 1$.

Burger and van Vuuren [7, Theorem 2] were able to extend this relation by means of a more refined construction, more precisely:

$$M_s(K_{n_1\times m_1}, K_{n_2\times m_2}) \geq M_s(K_{(n_1-1)\times m_1}, K_{n_2\times m_2}) + n_2\lceil m_2/s\rceil - 1. \tag{1}$$

In this spirit, a generalization of this result is stated below.

Theorem 5. *Given $n_1 > k$, $n_i \geq 2$ for $2 \leq i \leq k$ and $s \geq 1$, $m_i \geq 1$ for $1 \leq i \leq k$, let $\alpha = \sum_{i=2}^{k} (n_i\lceil m_i/s\rceil - 1)$. Thus*

$$M_s(K_{n_1\times m_1}, \dots, K_{n_k\times m_k}) \geq M_s(K_{(n_1-k+1)\times m_1}, K_{n_2\times m_2}, \dots, K_{n_k\times m_k}) + \alpha.$$

Proof. Take the sequences of numbers:

- $w_1 = M_s(K_{(n_1-k+1)\times m_1}, K_{n_2\times m_2}, \dots, K_{n_k\times m_k})$
- $w_i = n_i \lceil m_i/s\rceil - 1$, for $2 \leq i \leq k$,

and let $w = (\sum_{i=1}^{k} w_i) - 1 = w_1 + \alpha - 1$. In order to prove the statement, it is enough to construct a k-coloring G of $K_{w\times s}$ without a monochromatic copy of $K_{n_i\times m_i}$ in color i, $1 \leq i \leq k$.

Consider the vertex set of G formed by the disjoint union of all vertex sets of $K_{w_i\times s}$, where $1 \leq i \leq k$.

By the choice of w_1, there exists a k-coloring of $K_{(w_1-1)\times s}$ which contains neither a monochromatic copy of $K_{(n_1-k+1)\times m_1}$ in color 1 nor a monochromatic $K_{n_i\times m_i}$ in color i, $2 \leq i \leq k$.

The coloring of $K_{(w_1-1)\times s}$ above is extended to G according to the rules:

- all edges of each $K_{w_i\times s}$ are colored with color i, where $2 \leq i \leq k$;
- all the remaining edges of G (interconnecting edges between two distinct $K_{w_i\times s}$ and $K_{w_l\times s}$, $1 \leq i < l \leq k$) are colored with color 1.

Suppose for a contradiction that G contains a monochromatic $K_{n_j\times m_j}$ for some color j.

We first analyze the case $j = 1$. Given i, $2 \leq i \leq k$, each $K_{w_i\times s}$ must contain at most one partite set of $K_{n_1\times m_1}$, because all edges receive color i. Then there are at least $n_1 - k + 1$ partite sets of $K_{n_1\times m_1}$ that belong to $K_{(w_1-1)\times s}$, that is, the subgraph of $K_{(w_1-1)\times s}$ induced by the color 1 contains a copy of $K_{(n_1-k+1)\times m_1}$, a contradiction.

The case where $j \neq 1$ remains. With a similar argument used in Theorem 3, the hypothesis $w_j < n_j \lceil m_j/s\rceil$ yields that $K_{n_j\times m_j}$ is not a subgraph of $K_{w_j\times s}$. Consequently, the graph G does not contain a monochromatic $K_{n_j\times m_j}$, where $2 \leq j \leq k$. ■

Inequality (1) can be extended in another way. For this purpose, we consider a slightly weaker condition $n_1 > 3$ instead of the hypothesis $n_1 > k$ of Theorem 5.

Theorem 6. *Given positive integers s, m_i for every $1 \le i \le k$, $n \ge 2$, $k \ge 2$, and $n_i \ge 3$ for every $1 \le i \le k-1$, let $\alpha = \sum_{i=2}^{k} (n_i \lceil m_i/s \rceil - 1)$. Thus*

$$M_s(K_{n_1 \times m_1}, \ldots, K_{n_k \times m_k}) \ge M_s(K_{(n_1-1)\times m_1}, \ldots, K_{(n_{(k-1)}-1)\times m_{(k-1)}}, K_{n_k \times m_k}) + \alpha.$$

Proof. The coloring is based on a slight modification of that given in the previous proof. Take the sequences of numbers:

- $w_1 = M_s(K_{(n_1-1)\times m_1}, \ldots, K_{(n_{(k-1)}-1)\times m_{(k-1)}}, K_{n_k \times m_k})$ and
- $w_i = n_i \lceil m_i/s \rceil - 1$ for every $2 \le i \le k$.

Let $w = (\sum_{i=1}^{k} w_i) - 1$.

Let the graph H isomorphic to $K_{w\times s}$ whose vertex set is formed by the disjoint union of all vertex sets of $K_{(w_1-1)\times s}$, and $K_{w_i \times s}$ where $2 \le i \le k$.

A k-coloring of H is defined as follows. By the choice of w_1, there is a k-coloring of $K_{(w_1-1)\times s}$ without any monochromatic copy of $K_{(n_i-1)\times m_i}$ in color i, for $1 \le i \le k-1$ and no copy of $K_{n_k \times m_k}$ with color k.

The coloring of $K_{(w_1-1)\times s}$ above is extended to H according to the rules:

- for each i, $2 \le i \le k$, all edges of $K_{w_i \times s}$ are colored with i;
- all incident edges joined two distinct graphs $K_{w_i \times s}$, $K_{w_j \times s}$, $2 \le i < j \le k$ are colored with 1;
- the edges between vertices in $K_{(w_1-1)\times s}$ and $K_{w_i \times s}$ are colored with $i-1$, where $2 \le i \le k$.

We first analyze the color k. Note that the edges with color k come from $K_{(w_1-1)\times s}$ or $K_{w_k \times s}$. From the choice of the k-coloring of $K_{(w_1-1)\times s}$ and the fact that we need at least $n_k \lceil m_k/s \rceil$ classes to produce a $K_{n_k \times m_k}$ (as explained in the proof of case 2 in Theorem 3), H does not contain any monochromatic $K_{n_k \times m_k}$ with the color k.

Given a color i, with $1 \le i \le k-1$, suppose for a contradiction that H contains a monochromatic subgraph $K_{n_i \times m_i}$ for some color i. If $i \ge 2$, the hypothesis $w_i < n_i \lceil m_i/s \rceil$ yields that $K_{n_i \times m_i}$ is not a subgraph of $K_{w_i \times s}$. If $K_{n_i \times m_i}$ is a subgraph of H, then only one class of $K_{n_i \times m_i}$ can be contained in $K_{w_{i+1} \times s}$. Therefore, we conclude that $K_{(n_i-1)\times m_i} \subseteq K_{(w_1-1)\times s}$, a contradiction. ∎

See applications of Theorems 5 and 6 in Example 18.

3.3. An upper bound

The upper bound given by Theorem 2 can be improved for a bipartite graph $K_{2\times m} = K_{m,m}$, as follows.

Theorem 7. *For every positive integers s, m, and k with $k \ge 2$, the following bound holds.*

$$M_s(K_{2\times m}; k) \le \left\lceil \frac{k(m-1)+1}{s} \right\rceil + \left\lceil \frac{k(m-1)\binom{k(m-1)+1}{m} + 1}{s} \right\rceil.$$

Proof. Let c be the bound mentioned in the statement. Given an arbitrary k-coloring of $K_{c\times s}$, take $S \cup T$ as a partition of the classes of $K_{c\times s}$ such that $|S| = \left\lceil \left(k(m-1)\binom{k(m-1)+1}{m} + 1\right)/s \right\rceil$ and $|T| = \lceil (k(m-1)+1)/s \rceil$.

Choose a subset U of $k(m-1)\binom{k(m-1)+1}{m} + 1$ vertices in S and a subset W of $k(m-1)+1$ vertices in T. Since a vertex $u \in U$ and a vertex $w \in W$ belong to distinct classes, the edge uw belongs to $K_{c\times s}$. Look at now the bipartite subgraph G induced by the vertex sets U and W.

For our purpose, the central vertex u in U of a star $K_{1,m}$ in G is adjacent to a set of m vertices in W, called base of this star.

For each vertex u in U, the pigeonhole principle asserts that there is a monochromatic copy of $K_{1,m}$ where the center is u and the base lies in W.

Since there are $\binom{k(m-1)+1}{m}$ subsets of cardinality m in W, the pigeonhole principle implies that there is at least $k(m-1)+1$ copies of these stars with the same base.

Again by the pigeonhole principle, there are m copies of $K_{1,m}$ with the same color where each central vertex belongs to U and these central vertices share the same base. Hence, the graph $K_{c\times s}$ contains a monochromatic copy of $K_{m,m} = K_{2\times m}$. ∎

The case where $k = 2$ of Theorem 7 corresponds to [6, Theorem 5].

4. A lower bound by the probabilistic method

Erdős proved an exponential lower bound for the classical Ramsey numbers in 1947, by using probabilistic arguments. Nowadays this method is a powerful tool to estimate bounds on extremal problems in combinatorics. By applying this method, Burger and van Vuuren [7] presented a lower bound on set multipartite Ramsey numbers. The method can be extended to an arbitrary number of colors, according to the next result.

Theorem 8. *The lower bound holds*

$$M_s(K_{n\times m};k) > \frac{1}{s}\left(n!(m!)^n k^{m^2\binom{n}{2}-1}\right)^{\frac{1}{nm}}.$$

Proof. Given a positive integer a (which will be estimated afterward), consider a random k-coloring of $K_{a\times s}$ where the color of each edge is determined by a uniform distribution. More precisely, the probability is determined by setting: for each edge e and for each color i, let

$$P[\text{ `` the edge } e \text{ is colored with } i\text{ ''}] = \frac{1}{k},$$

and make these probabilities mutually independent. Take the probability space formed by all k-colorings of $K_{a\times s}$. Let u denote the number of copies of the graph $K_{n\times m}$ in $K_{a\times s}$ and enumerate such copies, say, $K^1_{n\times m},\ldots,K^u_{n\times m}$. For a color i and $1\le j\le u$, denote the event "the jth copy $K^j_{n\times m}$ is monochromatic with color i" by $K^{(j,i)}_{n\times m}$.

More generally, denote the event "some copy of $K_{n\times m}$ in $K_{a\times s}$ is monochromatic with color i" by $\cup_j K^{(j,i)}_{n\times m}$ (union over all copies). Since $\cup_j K^{(j,i)}_{n\times m} = K^{(1,i)}_{n\times m}\cup\cdots\cup K^{(u,i)}_{n\times m}$, the subadditivity of P yields

$$P\left[\cup_j K^{(j,i)}_{n\times m}\right] \le \sum_{j=1}^{u} P\left[K^{(j,i)}_{n\times m}\right] \le uP\left[K^{(j,i)}_{n\times m}\right] \le uk^{-m^2\binom{n}{2}}. \tag{2}$$

The event "some copy of $K_{n\times m}$ in $K_{a\times s}$ is monochromatic" can be denoted by $\cup_i\cup_j K^{(j,i)}_{n\times m}$. Combining the equality

$$\cup_i\cup_j K^{(j,i)}_{n\times m} = \cup_j K^{(j,1)}_{n\times m}\cup\cdots\cup_j K^{(j,k)}_{n\times m},$$

the subadditivity of P and Eq. (2), we obtain

$$P\left[\cup_i\cup_j K^{(j,i)}_{n\times m}\right] \le \sum_{i=1}^{k} P\left[\cup_j K^{(j,i)}_{n\times m}\right] \le kuk^{-m^2\binom{n}{2}}. \tag{3}$$

Let us now estimate u. There are at most $\binom{as}{nm}$ ways of choosing a vertex set with nm elements. There are at most $\binom{nm}{m}$ ways of choosing the first vertex class, and so on. Finally, there are at most $\binom{nm-(n-1)m}{m}$ ways of choosing the last class. Since the order of the classes does no matter,

$$u \le \binom{as}{nm}\binom{nm}{m}\cdots\binom{nm-(n-1)m}{m}\frac{1}{n!}.$$

Because

$$u \le \binom{as}{nm}\frac{(nm)!}{n!(m!)^n}.$$

Eq. (3) yields

$$P\left[\cup_i\cup_j K^{(j,i)}_{n\times m}\right] \le k\binom{as}{nm}\frac{(nm)!}{n!(m!)^n}k^{-m^2\binom{n}{2}}. \tag{4}$$

We now search the largest a as possible as such that the probability above is strictly less than 1. For this purpose, take a satisfying $as \le \left(n!(m!)^n k^{m^2\binom{n}{2}-1}\right)^{\frac{1}{nm}}$. Since $\frac{(as)!}{(as-nm)!} < (as)^{nm}$, the following bounds hold

$$\binom{as}{nm}(nm)! < (as)^{nm} \le n!(m!)^n k^{m^2\binom{n}{2}-1}. \tag{5}$$

By Eqs. (4) and (5), we derive

$$P\left[\cup_i\cup_j K^{(j,i)}_{n\times m}\right] < k\binom{as}{nm}\frac{(nm)!}{n!(m!)^n}k^{-m^2\binom{n}{2}} \le k\frac{1}{k} = 1.$$

Then the complement of the event $\cup_i\cup_j K^{(j,i)}_{n\times m}$ is not empty, that is, there is a k-coloring of $K_{a\times s}$ without any monochromatic copy of $K_{n\times m}$. ■

5. Upper bounds from density arguments

Density arguments have been a powerful method in exploring problems in extremal graph theory. In particular, this method yields many bounds on Ramsey numbers involving bipartite graphs (see [9,16] for instance) as well as bounds on bipartite Ramsey numbers arising from Zarankiewicz numbers (see [2,8,17] for instance). In order to improve some upper bounds for bipartite graphs, an adaptation of this approach for set multipartite Ramsey numbers is the main goal of this section.

5.1. The method

Lemma 9. *Let $s \geq 1$, $m \geq 2$, and $k \geq 2$ be positive integers. Let c be a positive integer such that*

$$cs \binom{\frac{2}{cs}\left\lceil \frac{c(c-1)s^2}{2k} \right\rceil}{m} > (m-1)\binom{cs}{m}.$$

Then $M_s(K_{2\times m}; k) \leq c$.

Proof. Given an arbitrary k-coloring of $K_{c\times s}$, the pigeonhole principle asserts that there is a color i with at least $\left\lceil \binom{c}{2} \frac{s^2}{k} \right\rceil$ edges. Let $H = (V, E)$ be the spanning subgraph of $K_{c\times s}$ formed by all edges with color i. Let Λ denote the number of stars of type $K_{1,m}$ in H. Each one of these stars corresponds to a center v and a basis A, where $v \in V$, $A \subseteq V$, $|A| = m$, and va is an edge of H for every $a \in A$.

Each vertex $v \in V$ is the center of $\binom{d(v)}{m}$ distinct stars. Since the binomial $\binom{x}{m}$ is a convex function, Jensen's inequality gives us the following lower bound

$$\Lambda = \sum_{v\in V} \binom{d(v)}{m} \geq cs \binom{\frac{\sum d(v)}{|V|}}{m}.$$

Euler's identity $\sum_{v\in V} d(v) = 2|E|$ and the hypothesis imply

$$\Lambda \geq cs \binom{\frac{2}{cs}\left\lceil \frac{c(c-1)s^2}{2k} \right\rceil}{m} > (m-1)\binom{cs}{m}.$$

By the pigeonhole principle again, there is a subset A that is the base of more than $m - 1$ stars, then H contains a copy of $K_{2\times m}$ with color i. ∎

We have weakened slightly Lemma 9 to facilitate applications, more precisely.

Lemma 10. *Let $s \geq 1$, $m \geq 2$, and $k \geq 2$ be positive integers. Let c be a positive integer such that*

$$cs \binom{\frac{(c-1)s}{k}}{m} > (m-1)\binom{cs}{m}.$$

Thus $M_s(K_{2\times m}; k) \leq c$.

Proof. Since $\frac{2}{cs}\left\lceil \frac{c(c-1)s^2}{2k} \right\rceil \geq \frac{(c-1)s}{k}$, the result follows as an immediate application of Lemma 9. ∎

5.2. Some applications

Several new bounds can be derived from Lemma 10. We discuss here some of these applications.

5.2.1. Bounds for $K_{2\times m}$

Chung and Graham [9, Theorem 1] proved that $M_1(K_{2\times m}; k) \leq (m-1)(k + k^{1/m})^m$. Hence Proposition 4.2 yields $M_2(K_{2\times m}; k) \leq (m-1)(k + k^{1/m})^m$. However the last bound can be improved as follows.

Theorem 11. *Given a positive integer $m \geq 2$, the bound below holds for any sufficiently large k*

$$M_2(K_{2\times m}; k) \leq (m-1)k^m.$$

Proof. Writing $c = (m-1)k^m$, it is enough to show the hypothesis of Lemma 10. Since $2(c-1)/k \geq 2(m-1)k^{m-1} - 1$, the hypothesis holds provided

$$2(m-1)k^m \binom{2(m-1)k^{m-1} - 1}{m} > (m-1)\binom{2(m-1)k^m}{m}. \tag{6}$$

By applying twice the relation $(a-m+1)^m/m! \le \binom{a}{m} \le a^m/m!$ (the first inequality when $a = 2(m-1)k^{m-1} - 1$ and the second one when $a = 2(m-1)k^m$, Eq. (6) follows if

$$2(m-1)k^m[2(m-1)k^{m-1} - m]^m > (m-1)[2(m-1)k^m]^m.$$

Simple calculation shows that the inequality above is equivalent to

$$(2^{1/m} - 1)2(m-1)k^{m-1} > m2^{1/m}, \tag{7}$$

and the result follows. ∎

Example 12. Since Eq. (7) holds for $m = 4$ and $k = 4$, we obtain $M_2(K_{2\times4}; 4) \le 768$, which improves significatively the previous upper bound $M_2(K_{2\times4}; 4) \le 4298$ from Theorem 7.

5.2.2. Bounds for $K_{2\times3}$

A closer look on the argument produces a slightly stronger bound for the case $m = 3$, more precisely:

Theorem 13. *Let $k \ge 2$ be a positive integer.*

1. $M_s(K_{2\times3}; k) \le \lceil (2k^3 + 6k)/s \rceil + 3$ *for every* s.
2. *For* $s = 2$, *the sharper bound* $M_2(K_{2\times3}; k) \le k^3 + 2k + 1$ *holds.*

Proof. For item 1, we need to find a positive integer c such that

$$cs \binom{\frac{(c-1)s}{k}}{3} > 2\binom{cs}{3}. \tag{8}$$

By applying the relation $(a-2)^3/3! \le \binom{a}{3} \le a^3/3!$, Eq. (8) can be derived from the following inequality

$$[cs - (s+2k)]^3 > 2k^3(cs)^2. \tag{9}$$

Note that both inequalities below

$$(xs)^3 - 3(s+2k)(xs)^2 \ge 2k^3(xs)^2$$
$$3(xs)(s+2k)^2 - (s+2k)^3 > 0$$

hold for any real number $x \ge (2k^3 + 6k)/s + 3$. In particular when $x = c = \lceil (2k^3 + 6k)/s \rceil + 3$, the sum of the inequalities above implies Eq. (9), and consequently Eq. (8) holds. Therefore the bound follows as an application of Lemma 10.

It remains the analysis of item 2. If

$$2(k^3 + 2k + 1)\binom{2k^2 + 4}{3} > 2\binom{2(k^3 + 2k + 1)}{3},$$

then Lemma 10 concludes the required bound. Elementary calculation shows us that the previous inequality is equivalent to $k^4 + 5k^2 + 6 > k^3 + 2k$, which holds for every $k \ge 2$. ∎

Table 1 presents some upper bounds on $M_s(K_{2\times3}; 2)$ for small s, which allows us a comparative analysis between [6, Theorem 5], Theorem 13.1 and Lemma 9.

As mentioned in Table 1, the bound $M_2(K_{2\times3}; 2) \le 24$ was obtained in [6]. Theorem 13.1 improves to $M_2(K_{2\times3}; 2) \le 17$. However, the sharper bound $M_2(K_{2\times3}; 2) \le 13$ holds from Theorem 13.2.

Table 1
Upper bounds on $M_s(K_{2\times3}; 2)$.

s	1	2	3	4	5	6	7	8
Theorem 5 of [6]	46	24	14	13	10	8	7	7
Theorem 13.(1)	31	17	13	10	9	8	7	7
Lemma 9	22	13	9	8	7	6	6	5

5.3. Turán number and bounds for $K_{2\times 2}$

Given graphs G and F, the *Turán number* $ex(G; F)$ denotes the maximum number of edges in a subgraph of G containing no copy of F. We refer to [4] for a survey on this mainstream problem of extremal graph theory.

Chvátal and Harary [10] established a simple but particularly important connection between Turán numbers and multicolored Ramsey numbers:

$$\text{If } ex(K_c; F) < \binom{c}{2} \Big/ k, \text{ then } r(F; k) \le c,$$

where $r(F; k)$ denotes the smallest n such that any k-coloring of the edges of K_n must contain a monochromatic copy of F. A natural extension to multipartite graphs is stated, more precisely.

Lemma 14. *If $ex(K_{c\times s}; F) < s^2 \binom{c}{2} /k$, then $M_s(F; k) \le c$.*

Let us focus on the case where $F = K_{2,2} = C_4$. Erdős–Rényi–Sós [13] (see also [9,16]) computed

$$ex(K_n; K_{2,2}) \le \frac{n}{4}(1 + \sqrt{4n-3}). \tag{10}$$

Determining Turán numbers even for this cycle has been a long-standing problem. The topic is so difficult that exact values on $ex(K_n; K_{2,2})$ are known only for a particular class, according to Füredi [14].

The well-known bound $r(K_{2\times 2}; k) \le k^2 + k + 1$ was obtained by Irving [16, Theorem 3.12] and, independently, by Chung and Graham [9, Corollary 1]. We extend this bound as follows.

Theorem 15. *Let $k \ge 2$ and s be positive integers. Thus*

$$M_s(K_{2\times 2}; k) \le \left\lceil \frac{k^2 + k + 2s - 1}{s} \right\rceil.$$

Proof. The proof is an application of Lemma 14 when $F = K_{2,2}$. Because the case $s = 1$ is already known, we only prove the case where $s \ge 2$. Let c be the upper bound of the statement above. Since $K_{c\times s}$ is a subgraph of K_{cs}, any $K_{2,2}$-free subgraph of $K_{c\times s}$ is also a $K_{2,2}$-free subgraph of $K_{c\times s}$, and consequently $ex(K_{c\times s}; K_{2,2}) \le ex(K_{cs}; K_{2,2})$. By using this previous inequality and the bound in Eq. (10), the hypothesis of Lemma 14 holds if the inequality below is satisfied for all k.

$$4sc - 3 < \frac{(2sc - 2s - k)^2}{k^2}. \tag{11}$$

Take now the real function $f(x) = (2sx - 2s - k)^2 - (4sx + 3)k^2$. Note that Eq. (11) can be reformulated as $f(c) > 0$. For this purpose, it is easy to see that its derivate $f'(x) = 4s(2sx - 2s - k) - 4sk^2$ is nonnegative for all $x \ge (k^2 + k + 2s)/2s$.

Put $c = \lceil A/s \rceil$, where $A = k^2 + k + 2s - 1$. Elementary calculation reveals that $f(A/s) > 0$. Combining the facts: f is an increasing function for all $x \ge A/s$, $f(A/s) > 0$, $c \ge A/s$, we conclude that $f(c) > 0$ and consequently Eq. (11) holds. Finally, apply Lemma 14 to obtain the desired bound. ■

6. A new connection

As mentioned above, the works [6,7] explore important connections between $M_s(K_{n_1\times m_1}, K_{n_2\times m_2})$ and $r(K_{u_1}, K_{u_2})$, Ramsey number that ensures the occurrence of a monochromatic copy of a complete graph on suitable u_1 and u_2 vertices. With this spirit, we aim to discuss now a connection between $M_s(K_{n_1\times m_1}, \dots, K_{n_k\times m_k})$ and $r(K_{2\times u_1}, \dots, K_{2\times u_k})$.

Theorem 16. *Given positive integers, the following lower bound holds*

$$M_s(K_{n_1\times m_1}, \dots, K_{n_k\times m_k}) \ge \left\lfloor \frac{M_1(K_{n_1\times m_1}, \dots, K_{n_k\times m_k}) - 1}{s} \right\rfloor + 1.$$

Proof. Let $n = M_1(K_{n_1\times m_1}, \dots, K_{n_k\times m_k})$. By the choice of n, there is a k-coloring of the complete graph K_{n-1} without a monochromatic copy of $K_{n_i\times m_i}$ in color i, where $1 \le i \le k$. Let $c = \lfloor (n-1)/s \rfloor$. Since $cs \le n - 1$, select a subgraph H (isomorphic to $K_{c\times s}$) of K_{n-1} and consider the k-coloring restricted to H. It is clear that H contains no monochromatic copy of $K_{n_i\times m_i}$ in color i, where $1 \le i \le k$. ■

Example 17. Let us illustrate some numerical results:

1. Bialostocki and J. Schönheim [3] proved that $M_1(K_{2\times 2}; 3) = r(C_4; 3) = 11$. A combination of Theorems 15 and 16 yields the following numbers:

$$6 \le M_2(C_4; 3) \le 8 \quad 4 \le M_3(C_4; 3) \le 6 \quad 3 \le M_4(C_4; 3) \le 5$$
$$3 \le M_5(C_4; 3) \le 5 \quad 2 \le M_6(C_4; 3) \le 4.$$

2. The lower bound $27 \leq r(C_4; 5)$ is due to Lazebnik and Woldar [18]. Theorems 16 and 15 produce:

$$\begin{array}{ll} 14 \leq M_2(C_4; 5) \leq 17 & 9 \leq M_3(C_4; 5) \leq 12 \\ 7 \leq M_4(C_4; 5) \leq 10 & 6 \leq M_5(C_4; 5) \leq 8. \end{array}$$

Example 18. Table 2 presents some lower bounds on $M_s(K_{4\times 2}, K_{2\times 2}, K_{2\times 2})$ by using Theorem 5 and the lower bounds in Example 17.1. The same values and applications of Theorem 6 produce lower bounds on $M_s(K_{3\times 2}, K_{3\times 2}, K_{2\times 2})$ in Table 2.

Table 2
Some lower bounds.

s	2	3	4	5	6
$M_s(K_{4\times 2}, K_{2\times 2}, K_{2\times 2})$	8	6	5	5	4
$M_s(K_{3\times 2}, K_{3\times 2}, K_{2\times 2})$	9	7	6	6	5

We now discuss additional applications.

6.1. Extending a result by Burger et al.

Given a graph G, let $\chi(G)$ denote the chromatic number of G and let $c(G)$ denote the cardinality of the largest connected component of G. Chvátal and Harary [11] discovered a useful bound $r(H, G) \geq (\chi(G) - 1)(c(H) - 1) + 1$. As an application, Burger and van Vuuren [7, Corollary 1] proved that

$$M_1(K_{n_1\times m_1}, K_{n_2\times m_2}) \geq (n_2 - 1)(n_1m_1 - 1) + 1. \tag{12}$$

The authors pointed out that an extension to an arbitrary s would be interesting.

We attempt to extend this result by discussing firstly an adaptation of Eq. (12) to an arbitrary number of colors. This method is motivated by the recursive construction from the proof of $r(K_3; k) \geq 2^k$, see [15, page 145].

Proposition 19. *Given positive integers $m_i \geq 1$ and $n_i \geq 2$ for $1 \leq i \leq k$, let $r = r(K_{n_1\times m_1}, \ldots, K_{n_{k-1}\times m_{k-1}})$. For every positive integer s,*

$$M_s(K_{n_1\times m_1}, \ldots, K_{n_k\times m_k}) \geq \left\lfloor \frac{(n_k - 1)(r - 1)}{s} \right\rfloor + 1.$$

Proof. We firstly analyze the case $s = 1$. By the choice of r, there is a $k-1$-coloring of K_{r-1} without a monochromatic copy of $K_{n_i\times m_i}$ in color i, where $1 \leq i \leq k-1$. Let $c = (n_k - 1)(r - 1)$ and consider K_c to be made up of $n_k - 1$ copies of K_{r-1} with edges interconnecting all pairs of vertices in the distinct copies of K_{r-1}. Color all edges within a copy K_{r-1} by using the $k-1$ coloring above, and all remaining edges with color k. There is no copy of $K_{n_i\times m_i}$ in color i, where $1 \leq i \leq k-1$. Since the largest complete graph in K_c with color k has $n_k - 1$ vertices, no copy of $K_{n_k\times m_k}$ with color k can occur. The proof for an arbitrary s follows from Theorem 16. ■

We note that for 1-coloring, the degenerate case of the notation considers $r(K_{n_1\times m_1}; 1) = n_1m_1$, which is used in the lower bound of Proposition 19 when $s = 1$ and $k = 2$. Under this viewpoint, Proposition 19 can be regarded as an extension of Eq. (12).

Corollary 20. *Given positive integers $s \geq 1$, $n \geq 2$, $m \geq 1$, $k \geq 2$,*

$$M_s(K_{n\times m}; k) \geq \frac{m(n-1)^k + 1}{s}.$$

Proof. We analyze the case $s = 1$. Since $r(K_{n\times m}; 1) = nm$, Proposition 19 produces

$$M_1(K_{n\times m}; 2) \geq (M_1(K_{n\times m}; 1) - 1)(n - 1) + 1 = (nm - 1)(n - 1) + 1 \geq m(n-1)^2 + 1.$$

The argument follows by induction on k. Theorem 16 completes the proof to an arbitrary s. ■

The case where $s = 1$, $n = 3$, and $m = 1$ yields essentially the bound $r(K_3; k) \geq 2^k$, mentioned above.

6.2. Asymptotic bound for the four cycle

We conclude this work with some remarks about asymptotic results. Alon et al. [1] proved that $r(K_{2\times 3}; k) = (1+o(1))k^3$, improving a well-known construction based on certain finite geometries by Brown [5]. As an immediate application of Theorems 16 and 13, for a given s, the number $M_s(K_{2\times 3}; k)$ is bounded by $(1 + o(1))\frac{k^3}{s}$ and $(2 + o(1))\frac{k^3}{s}$.

An asymptotically sharp class can be established, more specifically.

Corollary 21. *For a positive integer s, we have*

$$\lim_{k\to\infty} M_s(K_{2\times 2}; k) = \frac{k^2}{s}.$$

Proof. The upper bound follows from Theorem 15. Irving [16, Theorem 3.2] proved that $M_1(K_{2\times 2}; k) > k^2 - k + 1$ for every $k - 1$ prime power (see also [9, Theorem 3]). The proof follows from this lower bound, Theorem 16, and the fact that prime powers are sufficiently dense. ∎

The case where $s = 1$ was obtained in [16,9]. As another consequence, it is worth mentioning that Theorem 15 is asymptotically sharp for an arbitrary s.

Acknowledgments

The first author is partially supported by CNPq/MCT Grant: 311518/2013-3. The second author is partially supported by Capes. The authors are grateful to Pablo H. Perondi and the anonymous referees for valuable suggestions and several remarks which lead to the improvement of the present version.

References

[1] N. Alon, L. Rónyai, T. Szabó, Norm-graphs: variations and applications, J. Combin. Theory Ser. B 76 (1999) 280–290.
[2] L.W. Beineke, A.J. Schwenk, On a bipartite form of the Ramsey problem, in: Proceedings of the Fifth British Combinatorial Conference, Aberdeen, 1975, pp. 17—22.
[3] A. Bialostocki, J. Schönheim, On some Turán and Ramsey numbers for C_4, in: B. Bollobás (Ed.), Graph Theory and Combinatorics, Academic Press, London, 1984, pp. 29–33.
[4] B. Bollobás, Extremal Graph Theory, Dover Publications, New York, 2004.
[5] W.G. Brown, On graphs that do not contain a Thomsen graph, Canad. Math. Bull. 9 (1966) 281–285.
[6] A.P. Burger, P.J.P. Grobler, E.H. Stipp, J.H. van Vuuren, Diagonal Ramsey numbers in multipartite graphs, Util. Math. 66 (2004) 137–163.
[7] A.P. Burger, J.H. van Vuuren, Ramsey numbers in complete balanced multipartite graphs Part I: Set numbers, Discrete Math. 283 (2004) 37–43.
[8] W.A. Carnielli, E.L. Monte Carmelo, On the Ramsey problem for multicolor bipartite graphs, Adv. Appl. Math. 22 (1999) 48–59.
[9] F.R. Chung, R.L. Graham, On multicolor Ramsey numbers for complete bipartite graphs, J. Combin. Theory Ser. B 18 (1975) 164–169.
[10] V. Chvátal, F. Harary, Generalized Ramsey theory for graphs I: diagonal numbers, Period. Math. Hungar. 3 (1973) 115–124.
[11] V. Chvátal, F. Harary, Generalized Ramsey theory for graphs III: small off-diagonal numbers, Pacific J. Math. 41 (1972) 335–345.
[12] D. Day, W. Goddard, M. Henning, H. Swart, Multipartite Ramsey numbers, Ars Combin. 58 (2001) 23–31.
[13] P. Erdős, A. Rényi, V.T. Sós, On a problem of graph theory, Studia Sci. Math. Hungar. 1 (1966) 215–235.
[14] Z. Füredi, Graphs without quadrilaterals, J. Combin. Theory Ser. B 34 (1983) 187–190.
[15] R. Graham, B. Rothschild, J. Spencer, Ramsey Theory, John Wiley and Sons, New York, 1990.
[16] R.W. Irving, Generalized Ramsey numbers for small graphs, Discrete Math. 9 (1974) 251–264.
[17] R.W. Irving, A bipartite Ramsey problem and the Zarankiewicz numbers, Glasg. Math. J. 19 (1978) 13–26.
[18] F. Lazebnik, A. Woldar, New lower bounds on the multicolor Ramsey numbers $r_k(C4)$, J. Combin. Theory Ser. B 79 (2000) 172–176.
[19] S.P. Radziszowski, Small Ramsey numbers, Electron. J. Combin. (2014) DS1. dynamical survey.