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A modification of two graph-decomposition theorems based on a vertex-removing synchronised graph product

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Abstract

Recently, we have introduced two graph-decomposition theorems based on a new graph product, motivated by applications in the context of synchronising periodic real-time processes. This vertex-removing synchronised product (VRSP) is based on modifications of the well-known Cartesian product and is closely related to the synchronised product due to Wöhrle and Thomas. Here, we recall the definition of the VRSP and the two graph-decomposition theorems, we relax the requirements of these two graph-decomposition theorems and prove these two (relaxed) graph-decomposition theorems.

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1. Introduction

Recently, we have introduced two graph-decomposition theorems based on a new graph product [4], motivated by applications in the context of synchronising periodic real-time processes, in particular in the field of robotics. More on the background, definitions, and applications can be

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found in two conference contributions [3, 5], two journal papers [4, 6] and the thesis of the author [2]. In this contribution, we relax some of the requirements of these two graph-decomposition theorems. Also, we repeat some of the background, definitions, and theorems here for convenience.

The decomposition of graphs is well known in the literature. For example, decomposition can be based on the partition of a graph into edge-disjoint subgraphs. In our case, in the two graph-decomposition theorems we contract nonempty subsets of the vertex set V of the labelled acyclic directed multigraph G. The contraction of a nonempty set $X \subset V$ leads to a graph G/X where all the vertices of X are replaced by one vertex $\tilde{x}$ and the arcs with both ends in X are removed. In the first theorem, we have nonempty sets $X \subset V$ and $Y = V \backslash X$, giving G/X and G/Y. In the second theorem, we have nonempty sets $X_1 \subset V, X_2 \subset V$ and $Y = V \backslash X_1 \cup X_2$ giving $G/X_1/X_2$ and G/Y. Then, together with additional constraints given in the theorems, we have that G is isomorphic to the VRSP of G/X and G/Y in the first theorem and that G is isomorphic to the VRSP of $G/X_1/X_2$ and G/Y in the second theorem. In this paper, we recall the definition of the VRSP and the two graph-decomposition theorems given in [4] and we relax the requirements of these two graph-decomposition theorems. For the first theorem, the requirement was that for the arcs that have one end in X and the other end in Y (the set of arcs $[X, Y]$) the label of each arc is distinct. We relax this requirement in the following manner. The set of all arcs of $[X, Y]$ with the same label must arc-induce a complete bipartite graph. For the second theorem, the requirement was that for the arcs that have one end in X_1 and the other end in Y (the set of arcs $[X_1, Y]$), the arcs that have one end in Y and the other end in X_2 (the set of arcs $[Y, X_2]$) and the arcs that have one end in X_1 and the other end in X_2 (the set of arcs $[X_1, X_2]$) the label of each arc is distinct. We relax this requirement in the following manner. The set of all arcs of $[X_1, Y]$ with the same label must arc-induce a complete bipartite graph and the set of all arcs of $[Y, X_2]$ with the same label must arc-induce a complete bipartite graph. Furthermore, the only restriction on the labels of the arcs in $[X_1, X_2]$ is that the arcs of $[X_1, X_2]$ must not have a label identical to a label of any of the arcs of $A(G) \backslash [X_1, X_2]$.

The rest of the paper is organised as follows. In the next sections, we first recall the formal graph definitions (in Section 2), the definition of the VRSP as well as the graph-decomposition theorems, together with other relevant terminology and notation (in Section 3), the notions of graph isomorphism and contraction to labelled acyclic directed multigraphs (in Section 4), and the two graph theorems given in [4] (in Section 5). We relax the two theorems from [4] and we use the VRSP and the two relaxed decomposition theorems to state and prove two decomposition theorems (in Section 6).

2. Terminology and notation

We use the textbook of Bondy and Murty [1] for terminology and notation we do not specify here. Throughout, unless we specify explicitly that we consider other types of graphs, all graphs we consider are *labelled acyclic directed multigraphs*, i.e., they may have multiple arcs. Such graphs consist of a *vertex set* V (representing the states of a process), an *arc set* A (representing the actions, i.e., transitions from one state to another), a set of *labels* L (in our applications in fact a set of label pairs, each representing a type of action and the worst case duration of its execution), and two mappings. The first mapping $\mu : A \rightarrow V \times V$ is an incidence function that identifies the *tail* and *head* of each arc $a \in A$. In particular, $\mu(a) = (u, v)$ means that the arc a is directed from $u \in V$ to $v \in V$, where $tail(a) = u$ and $head(a) = v$. We also call u and v the *ends* of a. The second mapping $\lambda : A \to L$ assigns a label pair $\lambda(a) = (\ell(a), t(a))$ to each arc $a \in A$, where $\ell(a)$ is a string representing the (name of an) action and $t(a)$ is the *weight* of the arc a. This weight $t(a)$ is a real positive number representing the worst case execution time of the action represented by $\ell(a)$.

Let G denote a graph according to the above definition. An arc $a \in A(G)$ is called an *in-arc* of $v \in V(G)$ if $head(a) = v$, and an *out-arc* of v if $tail(a) = v$. The *in-degree* of v, denoted by $d^-(v)$, is the number of in-arcs of v in G; the *out-degree* of v, denoted by $d^+(v)$, is the number of out-arcs of v in G. The subset of $V(G)$ consisting of vertices v with $d^-(v) = 0$ is called the *source* of G, and is denoted by $S'(G)$. The subset of $V(G)$ consisting of vertices v with $d^+(v) = 0$ is called the *sink* of G, and is denoted by $S''(G)$.

For disjoint nonempty sets $X, Y \subseteq V(G)$, $[X, Y]$ denotes the set of arcs of G with one end in X and one end in Y. If the head of the arc $a \in [X, Y]$ is in Y, we call a a *forward arc* (of $[X, Y]$); otherwise, we call it a *backward arc*.

The acyclicity of G implies a natural ordering of the vertices into disjoint sets, as follows. We define $S^0(G)$ to denote the set of vertices with in-degree 0 in G (so $S^0(G) = S'(G)$), $S^1(G)$ the set of vertices with in-degree 0 in the graph obtained from G by deleting the vertices of $S^0(G)$ and all arcs with tails in $S^0(G)$, and so on, until the final set $S^t(G)$ contains the remaining vertices with in-degree 0 and out-degree 0 in the remaining graph. Note that these sets are well-defined since G is acyclic, and also note that $S^t(G) \neq S''(G)$, in general. If a vertex $v \in V(G)$ is in the set $S^j(G)$ in the above ordering, we say that v is *at* $level\ j$ in G.

A graph G is called *weakly connected* if all pairs of distinct vertices u and v of G are connected through a sequence of distinct vertices $u = v_0 v_1 \ldots v_k = v$ and arcs $a_1 a_2 \ldots a_k$ of G with $\mu(a_i) = (v_{i-1}, v_i)$ or (v_i, v_{i-1}) for $i = 1, 2, \ldots, k$. We are mainly interested in weakly connected graphs, or in the weakly connected components of a graph G. If $X \subseteq V(G)$, then the *subgraph of* G *induced by* X, denoted as $G[X]$, is the graph on vertex set X containing all the arcs of G which have both their ends in X (together with L, μ and λ restricted to this subset of the arcs). If $X \subseteq V$ induces a weakly connected subgraph of G, but there is no set $Y \subseteq V$ such that $G[Y]$ is weakly connected and X is a proper subset of Y, then $G[X]$ is called a *weakly connected component* of G. Also, the set of arcs of $G[X]$ is denoted as $A[X]$.

In the sequel, throughout we omit the words weakly connected, so a component should always be understood as a weakly connected component. In contrast to the notation in the textbook of Bondy and Murty [1], we use $\omega(G)$ to denote the number of components of a graph G.

We denote the components of G by G_i, where i ranges from 1 to $\omega(G)$. In that case, we use V_i, A_i and L_i as shorthand notation for $V(G_i)$, $A(G_i)$ and $L(G_i)$, respectively. The mappings μ and λ have natural counterparts restricted to the subsets $A_i \subset A(G)$ that we do not specify explicitly. We use $G = \sum_{i=1}^{\omega(G)} G_i$ to indicate that G is the disjoint union of its components, implicitly defining its components as G_1 up to $G_{\omega(G)}$. In particular, $G = G_1$ if and only if G is weakly connected itself. Furthermore, we use $\bigcup_{i=1}^{\omega(G)} G_i$ to denote the graph with vertex set $\bigcup_{i=1}^{\omega(G)} V_i$, arc set $\bigcup_{i=1}^{\omega(G)} A_i$ with the mappings $\mu_i(a_i) = (u_i, v_i)$ and $\lambda(a_i) = (\ell(a_i), t(a_i))$ for each arc $a_i \in A_i$.

A graph G according to the above definition is called *bi-partite* if there exists a partition of non-empty sets V_1 and V_2 of $V(G)$ into two partite sets (i.e., $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$) such that every arc of G has its head vertex and tail vertex in different partite sets. Such a graph is called a *bipartite graph*, and we denote such a bipartite graph G by $B(V_1, V_2)$. A bipartite graph $B(V_1, V_2)$ is called complete if, for every pair $x \in V_1$, $y \in V_2$, there is an arc a met $\mu(a) = (x, y)$ or $\mu(a) = (y, x)$ in $B(V_1, V_2)$. We call $B(V_1, V_2)$ a trivial bipartite graph if $|V_1| = |V_2| = 1$.

In the next two sections, we recall some of the definitions that appeared in [4]. For the argumentation of these definitions we refer to [4].

3. Graph products

The *Cartesian product* $G_i \Box G_j$ of G_i and G_j is defined as the graph on vertex set $V_{i,j} = V_i \times V_j$, and arc set $A_{i,j}$ consisting of two types of labelled arcs. For each arc $a \in A_i$ with $\mu(a) = (v_i, w_i)$, an *arc of type* i is introduced between tail $(v_i, v_j) \in V_{i,j}$ and head $(w_i, w_j) \in V_{i,j}$ whenever $v_j = w_j$; such an arc receives the label $\lambda(a)$. Similarly, for each arc $a \in A_j$ with $\mu(a) = (v_j, w_j)$, an *arc of type* j is introduced between tail $(v_i, v_j) \in V_{i,j}$ and head $(w_i, w_j) \in V_{i,j}$ whenever $v_i = w_i$; such an arc receives the label $\lambda(a)$.

The *intermediate product* $G_i \boxtimes G_j$ of G_i and G_j is obtained from $G_i \Box G_j$ by first ignoring all except for the so-called *asynchronous* arcs, i.e., by only maintaining all arcs $a \in A_{i,j}$ for which $\mu(a) = ((v_i, v_j), (w_i, w_j))$, whenever $v_j = w_j$ and $\lambda(a) \notin L_j$, as well as all arcs $a \in A_{i,j}$ for which $\mu(a) = ((v_i, v_j), (w_i, w_j))$, whenever $v_i = w_i$ and $\lambda(a) \notin L_i$. Additionally, we add arcs that replace synchronising pairs $a_i \in A_i$ and $a_j \in A_j$ with $\lambda(a_i) = \lambda(a_j)$. If $\mu(a_i) = (v_i, w_i)$ and $\mu(a_j) = (v_j, w_j)$, such a pair is replaced by an arc $a_{i,j}$ with $\mu(a_{i,j}) = ((v_i, v_j), (w_i, w_j))$ and $\lambda(a_{i,j}) = \lambda(a_i)$. We call such arcs of $G_i \boxtimes G_j$ *synchronous* arcs.

The *vertex-removing synchronised product* (VRSP for short) $G_i \boxslash G_j$ of G_i and G_j is obtained from $G_i \boxtimes G_j$ by removing the vertices $(v_i, v_j) \in V_{i,j}$ and the arcs a with $tail(a) = (v_i, v_j)$, in the case that (v_i, v_j) has $level > 0$ in $G_i \Box G_j$ but $level\ 0$ in $G_i \boxtimes G_j$. This is then repeated in the newly obtained graph, and so on, until there are no more vertices at $level\ 0$ in the current graph that are at $level > 0$ in $G_i \Box G_j$.

However, for these results it is relevant to introduce counterparts of graph isomorphism and graph contraction that apply to our types of graphs. We define these counterparts in the next section.

4. Graph isomorphism and graph contraction

We assume that two different arcs with the same tail and head have different labels; otherwise, we replace such multiple arcs by one arc with that label, because these arcs represent exactly the same action at the same stage of a process.

An isomorphism from G to H is a bijection $\phi : V(G) \to V(H)$ such that there exists an arc $a \in A(G)$ with $\mu(a) = (u, v)$ if and only if there exists an arc $b \in A(H)$ with $\mu(b) = (\phi(u), \phi(v))$ and $\lambda(b) = \lambda(a)$. An isomorphism from G to H is denoted as $G \cong H$.

Let X be a nonempty proper subset of $V(G)$, and let $Y = V(G) \backslash X$. By *contracting* X we mean replacing X by a new vertex $\tilde{x}$, deleting all arcs with both ends in X, replacing each arc

$a \in A(G)$ with $\mu(a) = (u, v)$ for $u \in X$ and $v \in Y$ by an arc c with $\mu(c) = (\tilde{x}, v)$ and $\lambda(c) = \lambda(a)$, and replacing each arc $b \in A(G)$ with $\mu(b) = (u, v)$ for $u \in Y$ and $v \in X$ by an arc d with $\mu(d) = (u, \tilde{x})$ and $\lambda(d) = \lambda(b)$. We denote the resulting graph as G/X, and say that G/X is the *contraction of G with respect to X*.

5. Graph theorems from [4]

Finally, we recall the two decomposition theorems that were introduced in [4].

Theorem 5.1 ([4])**.** *Let G be a graph, let X be a nonempty proper subset of $V(G)$, and let $Y = V(G)\backslash X$. Suppose that all the arcs of $[X, Y]$ have distinct labels and that the arcs of G/X and G/Y corresponding to the arcs of $[X, Y]$ are the only synchronising arcs of G/X and G/Y. If $S'(G) \subseteq X$ and $[X, Y]$ has no backward arcs, then $G \cong G/Y \boxslash G/X$.*

Theorem 5.2 ([4])**.** *Let G be a graph, and let X_1, X_2 and $Y = V(G)\backslash(X_1 \cup X_2)$ be three disjoint nonempty subsets of $V(G)$. Suppose that all the arcs of $[X_1, Y]$ have distinct labels, all the arcs of $[Y, X_2]$ have distinct labels, all the arcs of $[X_1, X_2]$ have distinct labels, the arcs of $[X_1, X_2]$ have no labels in common with any arcs in $[X_1, Y] \cup [Y, X_2]$, and that the arcs of $G/X_1/X_2$ and G/Y corresponding to the arcs of $[X_1, Y] \cup [Y, X_2] \cup [X_1, X_2]$ are the only synchronising arcs of $G/X_1/X_2$ and G/Y. If $S'(G) \subseteq X_1$, and $[X_1, Y]$, $[Y, X_2]$ and $[X_1, X_2]$ have no backward arcs, then $G \cong G/Y \boxslash G/X_1/X_2$.*

6. The two graph-decomposition theorems revisited

We start with relaxing the requirement in Theorem 5.1 that states that all arcs of $[X, Y]$ have distinct labels in the following manner: each largest set of arcs of $[X, Y]$ with the same label arc-induces a complete bipartite subgraph of G. Furthermore, we relax the requirement in Theorem 5.2 that all arcs of $[X_1, Y]$, $[Y, X_2]$ and $[X_1, X_2]$ have distinct labels in the following manner: firstly, each largest set of arcs of $[X_1, Y]$ with the same label arc-induces a complete bipartite subgraph of G, secondly, each largest set of arcs of $[Y, X_2]$ with the same label arc-induces a complete bipartite subgraph of G and, thirdly, the labels of the arcs in $[X_1, X_2]$ do not have to be distinct.

The relaxed requirement of Theorem 5.1 and the first and second relaxed requirement of Theorem 5.2 are based on the decomposition of a complete bipartite graph where all arcs have the same label. If $B(X, Y)$ is a complete bipartite graph where all arcs have the same label and $[X, Y]$ does not contain backward arcs then $B(X, Y) \cong B(X, Y)/Y \boxslash B(X, Y)/X$, which we state and prove in Lemma 6.1. The third relaxed requirement of Theorem 5.2 is based on the observation that the contraction of X_1 and X_2, $G/X_1/X_2$, replaces the set of arcs $[X_1, X_2]$ by a set of arcs $[\{\tilde{x}_1\}, \{\tilde{x}_2\}]$. Then, the VRSP of the subgraph G' of G/Y arc-induced by the set of arcs $[X_1, X_2]$ of G/Y and the subgraph G'' of $G/X_1/X_2$ arc-induced by the set of arcs $[\{\tilde{x}_1\}, \{\tilde{x}_2\}]$ of $G/X_1/X_2$ is isomorphic to G', $G' \cong G' \boxslash G''$. We have depicted a simple example in Figure 1 which illustrates these three relaxed requirements. At the upper left of Figure 1, we show the graph G. The subgraph arc-induced by the arcs with label c contains two complete bipartite subgraphs. The arcs with label c are the only arcs in $[X_1, Y] \cup [Y, X_2]$. For all other sets of arcs of G with the same label we do not require that these sets arc-induce a complete bipartite graph as they are not in $[X_1, Y] \cup [Y, X_2]$. At the lower left and the upper right of Figure 1, we show the contracted graphs G/Y and $G/X_1/X_2$, respectively. At the lower right of Figure 1, we show the intermediate product of the graphs G/Y and $G/X_1/X_2$, $G/Y \boxtimes G/X_1/X_2$. The vertices in the set Z at the lower right of Figure 1 induce the graph $G/Y \boxslash G/X_1/X_2$ which is isomorphic to G.

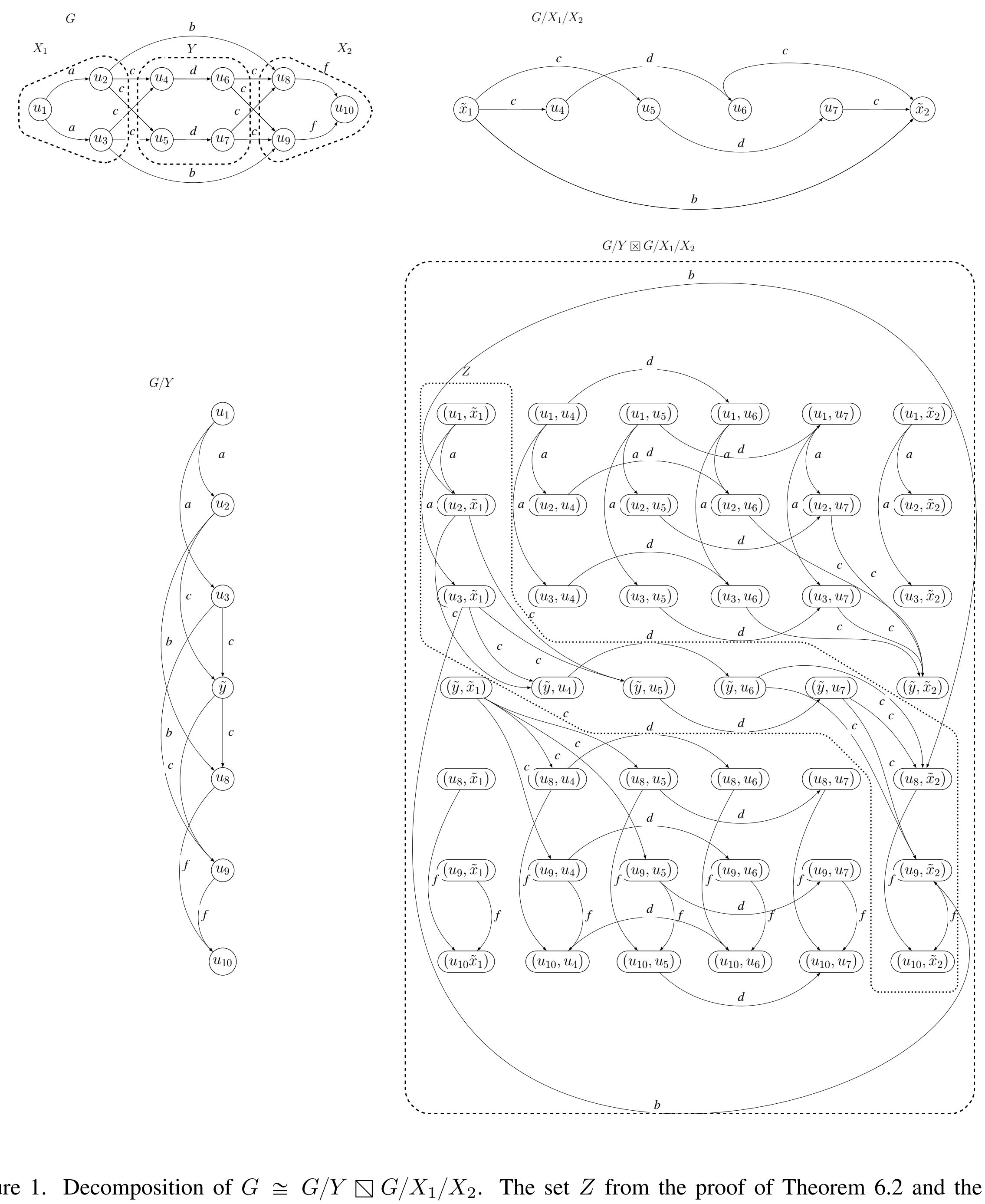

Figure 1. Decomposition of $G \cong G/Y \boxslash G/X_1/X_2$. The set Z from the proof of Theorem 6.2 and the graph isomorphic to G induced by Z in $G/Y \boxtimes G/X_1/X_2$ is indicated within the dotted region (apart from the arcs labelled b which are partially outside this region).

Before we can prove Theorem 6.1 and Theorem 6.2, we state and prove in Lemma 6.1 that a complete biparite graph $B(X,Y)$ where all arcs have the same label can be decomposed in such a manner that $B(X,Y) \cong B(X,Y)/Y \boxbslash B(X,Y)/X$. In Figure 2, we give a simple example of the decomposition of a bipartite graph where all arcs have the same label. Because all labels are identical, we have omitted these labels.

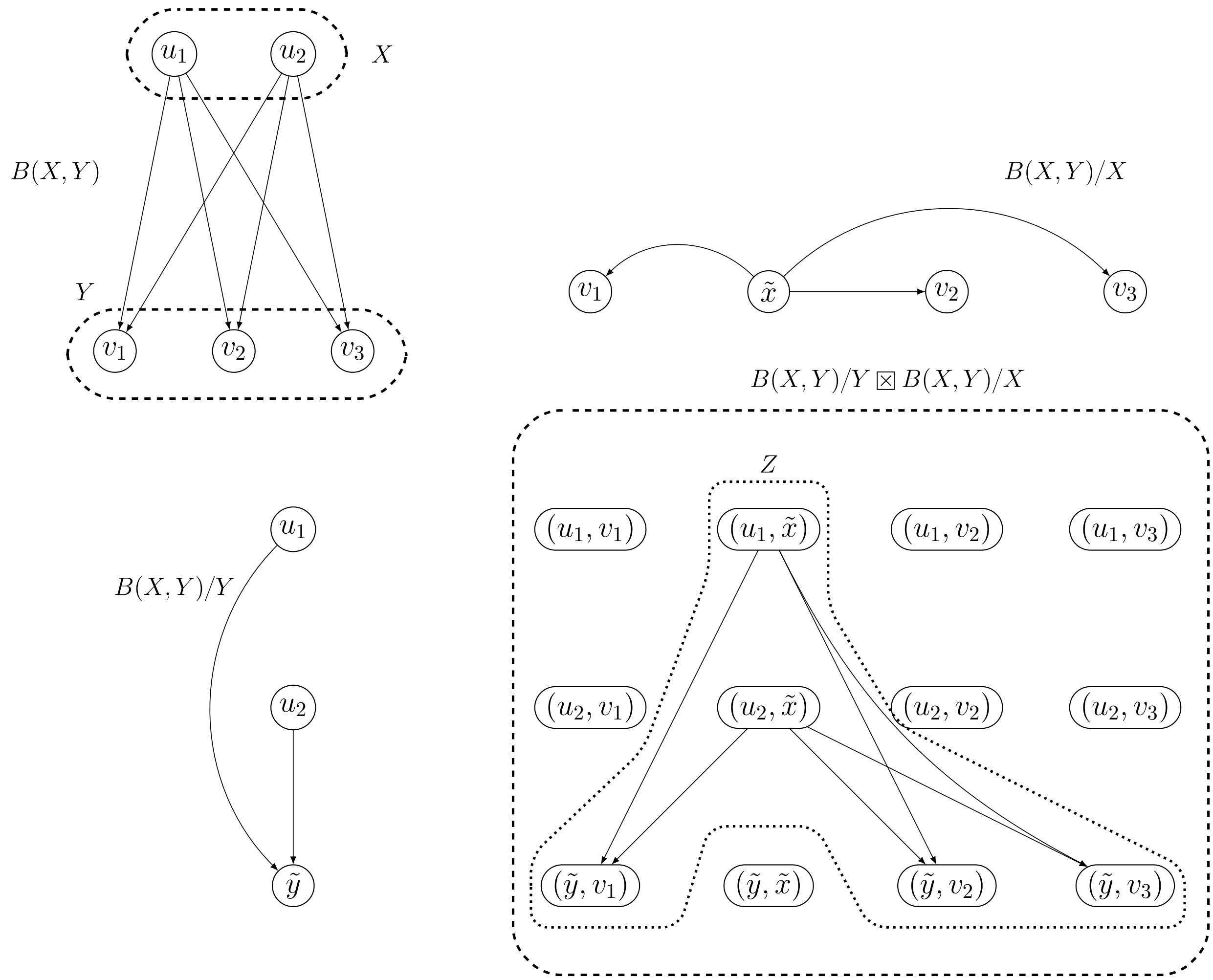

Figure 2. Decomposition of $B(X,Y) \cong B(X,Y)/Y \boxbslash B(X,Y)/X$. The set Z from the proof of Lemma 6.1 and the graph isomorphic to $B(X,Y)$ induced by Z in $B(X,Y)/X \boxtimes B(X,Y)/Y$ is indicated within the dotted region. Because all labels are identical, we have omitted these labels.

Lemma 6.1. *Let $B(X,Y)$ be a complete bipartite graph where all arcs have identical labels, $[X,Y] \neq \emptyset$, $[X,Y]$ has no backward arcs or $[X,Y]$ has no forward arcs. Then $B(X,Y) \cong B(X,Y)/Y \boxbslash B(X,Y)/X$.*

Proof. It suffices to define a mapping $\phi : V(B(X,Y)) \rightarrow V(B(X,Y)/Y \boxbslash B(X,Y)/X)$ and to prove that ϕ is an isomorphism from $B(X,Y)$ to $B(X,Y)/Y \boxbslash B(X,Y)/X$.

Let $\tilde{x}$ and $\tilde{y}$ be the new vertices replacing the sets X and Y when defining $B(X,Y)/X$ and $B(X,Y)/Y$, respectively. Consider the mapping $\phi : V(B(X,Y)) \to V(B(X,Y)/Y \boxslash B(X,Y)/X)$ defined by $\phi(u) = (u, \tilde{x})$ for all $u \in X$, and $\phi(v) = (\tilde{y}, v)$ for all $v \in Y$. Then ϕ is obviously a bijection if $V(B(X,Y)/Y \boxslash B(X,Y)/X) = Z$, where Z is defined as $Z = \{(u, \tilde{x}) \mid u \in X\} \cup \{(\tilde{y}, v) \mid v \in Y\}$. We are going to show this later by arguing that all the other vertices of $B(X,Y)/Y \Box B(X,Y)/X$ will disappear from $B(X,Y)/Y \boxtimes B(X,Y)/X$. But first we are going to prove the following claim.

Claim 1. The subgraph of $B(X,Y)/Y \boxtimes B(X,Y)/X$ induced by Z is isomorphic to $B(X,Y)$.

Proof. Let $X = \{u_1, \ldots, u_m\}, Y = \{v_1, \ldots, v_n\}$ be the disjoint vertex sets of a complete bipartite graph $B(X,Y)$ with arcs with identical labels where $[X,Y]$ has no backward arcs. Because $B(X,Y)$ is a complete bipartite graph and $[X,Y] \neq \emptyset$, $B(X,Y)$ has the arc set $A = \{a \mid \mu(a) = (u_i, v_j) \in [X,Y]\}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, and $|A| = m \cdot n$. Any two arcs b with $\mu(b) = (u_i, \tilde{y})$ in $B(X,Y)/Y$ and c with $\mu(c) = (\tilde{x}, v_j)$ in $B(X,Y)/X$ are synchronising arcs, because $\lambda(b) = \lambda(c)$. Due to the VRSP, the arcs b in $B(X,Y)/Y$ and c in $B(X,Y)/X$ correspond to an arc d with $\mu(d) = ((u_i, \tilde{x}), (\tilde{y}, v_j))$ in $B(X,Y)/Y \boxtimes B(X,Y)/X$ with $\lambda(b) = \lambda(d)$. Because the arc set $A(B(X,Y)/Y) = \{b \mid \mu(b) = (u_i, \tilde{y})\}$ has cardinality m, the arc set $A(B(X,Y)/X) = \{c \mid \mu(c) = (\tilde{x}, v_j)\}$ has cardinality n and all arcs of $A(B(X,Y)/Y)$ and $A(B(X,Y)/X)$ have identical labels, it follows that the arc set $A' = \{d \mid \mu(d) = ((u_i, \tilde{x}), (\tilde{y}, v_j)), 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\} \subseteq A(B(X,Y)/Y \boxtimes B(X,Y)/X)$ has cardinality $m \cdot n$. Furthermore, ϕ maps vertices u_i and v_j onto vertices $(u_i, \tilde{x})$ and $(\tilde{y}, v_j)$, respectively, and therefore we have an arc a with $\mu(a) = (u_i, v_j)$ in $B(X,Y)$ which corresponds to the arc d with $\mu(d) = ((u_i, \tilde{x}), (\tilde{y}, v_j))$ in $B(X,Y)/Y \boxtimes B(X,Y)/X$, with $\lambda(a) = \lambda(d)$. Because, firstly, the vertices $(u_i, \tilde{x})$ and $(\tilde{y}, v_j)$ are in Z implies that the arc d is an arc of the graph induced by Z and, secondly, $|A| = |A'|$, we have the one-to-one relationship between the arcs d in $B(X,Y)/Y \boxtimes B(X,Y)/X$ and a in $B(X,Y)$. Therefore, because there are no other vertices in Z than $(u_i, \tilde{x})$ and $(\tilde{y}, v_j)$ and there are no other vertices in $B(X,Y)$ than (u_i, v_j), the subgraph of $B(X,Y)/Y \boxtimes B(X,Y)/X$ induced by Z is isomorphic to $B(X,Y)$. This completes the proof of Claim 1. □

It remains to show that ϕ is a bijection from $V(B(X,Y))$ to $Z' = V(B(X,Y)/Y \boxslash B(X,Y)/X)$. Now, we have $Z' \subseteq V(B(X,Y)/Y \boxtimes B(X,Y)/X) = \{(u_i, v_j)\} \cup \{(u_i, \tilde{x})\} \cup \{(\tilde{y}, v_j)\} \cup \{(\tilde{y}, \tilde{x})\}$. The arcs b with $\mu(b) = (u_i, \tilde{y})$ in $B(X,Y)/Y$ and c with $\mu(c) = (\tilde{x}, v_j)$ in $B(X,Y)/X$ are synchronising arcs. Therefore, the only vertices that are the tail of an arc in $B(X,Y)/Y \boxtimes B(X,Y)/X$ are $(u_i, \tilde{x})$ and the only vertices that are the head of an arc in $B(X,Y)/Y \boxtimes B(X,Y)/X$ are $(\tilde{y}, v_j)$. Next, the vertices u_i in $B(X,Y)/Y$ and the vertex $\tilde{x}$ in $B(X,Y)/X$ have $level\, 0$. All other vertices in $B(X,Y)/Y$ and $B(X,Y)/X$ have $level\, 1$. Therefore, the only vertices in $B(X,Y)/Y \Box B(X,Y)/X$ with $level\, 0$ are the vertices $(u_i, \tilde{x})$. It follows that the vertices (u_i, v_j) and $(\tilde{y}, \tilde{x})$ are removed from $V(B(X,Y)/Y \boxtimes B(X,Y)/X)$ because $level((u_i, v_j)) > 0$ in $B(X,Y)/Y \Box B(X,Y)/X$ but $level((u_i, v_j)) = 0$ in $B(X,Y)/Y \boxtimes B(X,Y)/X$ and $level((\tilde{y}, \tilde{x})) > 0$ in $B(X,Y)/Y \Box B(X,Y)/X$ but $level((\tilde{y}, \tilde{x})) = 0$ in $B(X,Y)/Y \boxtimes B(X,Y)/X$. Therefore, it follows that $Z' = \{(u_i, \tilde{x})\} \cup \{(\tilde{y}, v_j)\} = Z$, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Hence, ϕ is a bijection from $V(B(X,Y))$ to Z preserving the arcs and their labels and therefore $B(X,Y) \cong B(X,Y)/Y \boxslash B(X,Y)/X$. With similar arguments, it follows that $B(X,Y) \cong B(X,Y)/Y \boxbslash B(X,Y)/X$ if $[X,Y]$ contains no forward arcs. This completes the proof of Lemma 6.1. $\square$

In Figure 3, we give a bipartite graph where all arcs have identical labels which is not complete and, therefore, cannot be decomposed by Lemma 6.1. For the arc a with $\mu(a) = ((u_1,\tilde{x}),(\tilde{y},v_1))$ in $B(X,Y)/Y \boxtimes B(X,Y)/X$ there is no arc b with $\mu(b) = (u_1,v_1)$ in $B(X,Y)$.

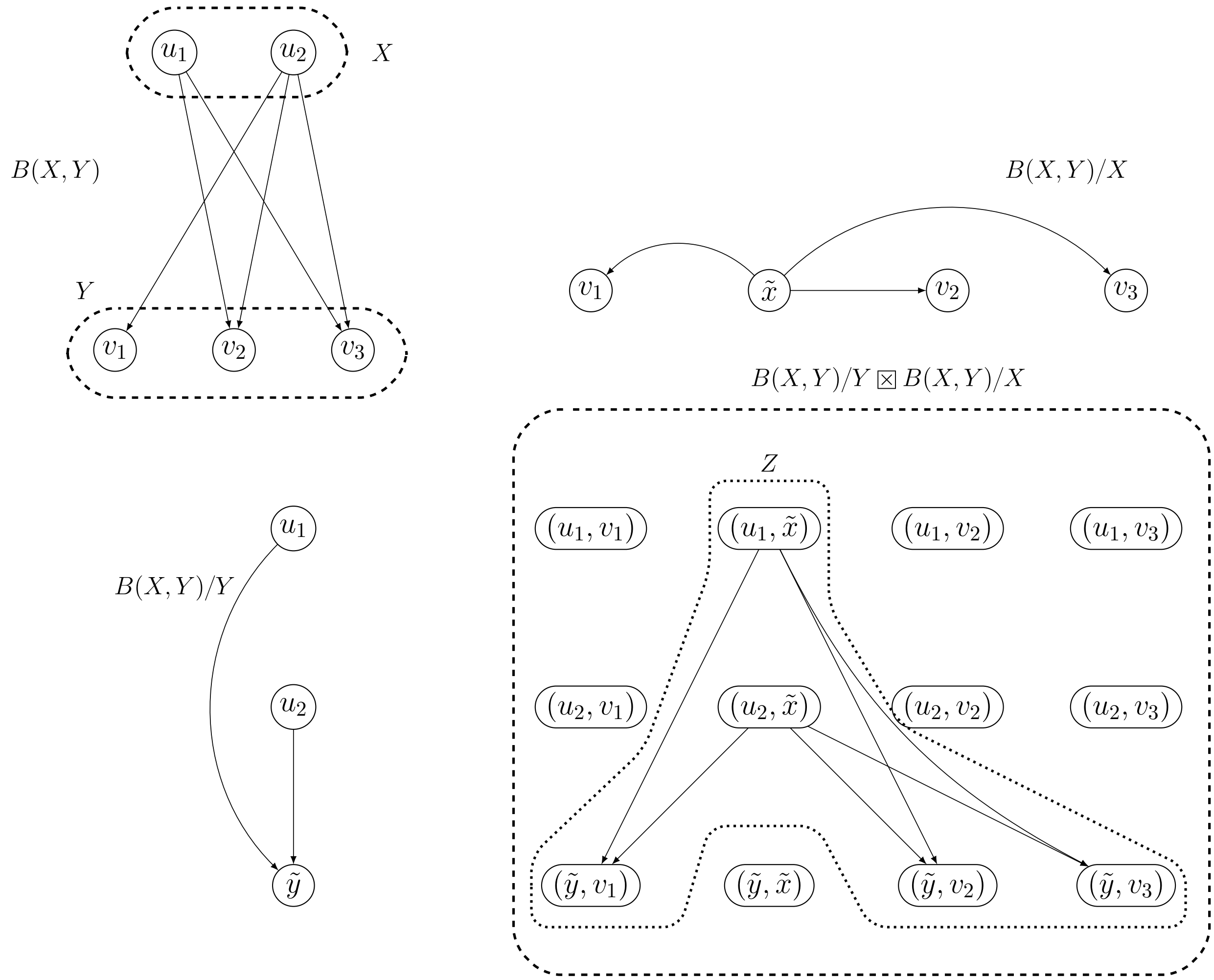

Figure 3. Decomposition of $B(X,Y)$ for which $B(X,Y) \ncong B(X,Y)/Y \boxbslash B(X,Y)/X$. Because all labels are identical, we have omitted these labels.

Using Lemma 6.1, we relax Theorem 5.1 and Theorem 5.2 leading to Theorem 6.1 and Theorem 6.2, respectively. We assume that the graphs we want to decompose are connected; if not, we can apply our decomposition results to the components separately. In Figure 4, we show the decomposition of a graph G that contains a complete bipartite subgraph $B(Z_1,Z_2)$ where all arcs of $B(Z_1,Z_2)$ have the label s.

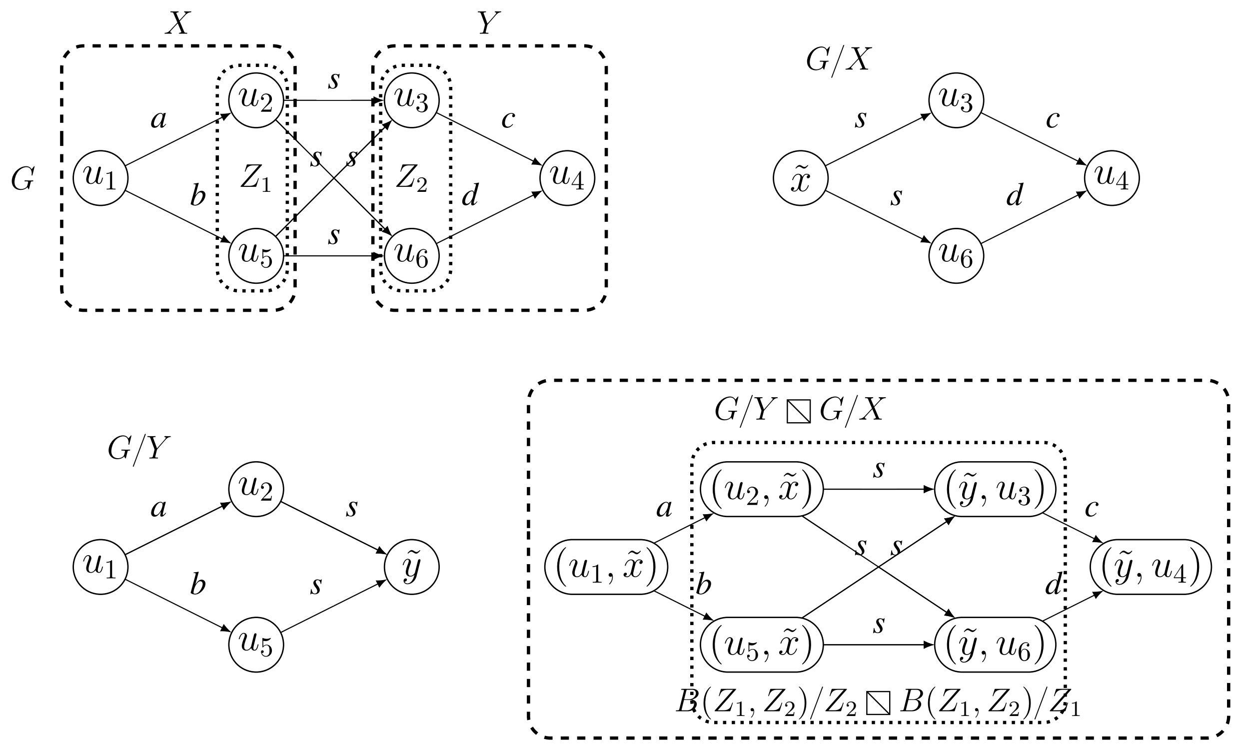

Figure 4. Decomposition of G into G/Y and G/X, where the arcs of $[X, Y]$ arc-induce a complete bipartite subgraph $B(Z_1, Z_2)$ of G with arcs with the same label. The dashed regions indicate the vertex sets X, Y and $V(G/Y \boxbslash G/X)$. The dotted regions indicate the vertex sets Z_1, Z_2 and $V(B(Z_1, Z_2)/Z_2 \boxbslash B(Z_1, Z_2)/Z_1$.

The only difference between Theorem 5.1 and Theorem 6.1 is that the arcs of $[X, Y]$ must have unique labels in Theorem 5.1, whereas this is not required in Theorem 6.1. To relax this requirement of Theorem 5.1, we require that any set of all arcs of $[X, Y]$ with identical labels must arc-induce a complete bipartite graph. By Lemma 6.1, these complete bipartite graphs are decomposable. Then we have that all arcs of a complete bipartite subgraph $B(X_1, Y_1), X_1 \subseteq X, Y_1 \subseteq Y$, of G with the same label are synchronous arcs. Furthermore, all other arcs of G have labels different from the labels of $B(X_1, Y_1)$. This means that $B(X_1, Y_1)/Y_1 \boxbslash G/X \subseteq G/Y \boxbslash G/X$ and $G/Y \boxbslash B(X_1, Y_1)/X_1 \subseteq G/Y \boxbslash G/X$, and, due to Lemma 6.1, $B(X_1, Y_1)/Y_1 \boxbslash B(X_1, Y_1)/X_1 \cong B(X_1, Y_1)$. Therefore, $G \cong G/Y \boxbslash G/X$, which we prove in Theorem 6.1. In Theorem 6.1, we use the proof of Theorem 5.1 given in [4] and modify this proof to support complete bipartite subgraph of G with arcs with the same label of $[X, Y]$.

Theorem 6.1. *Let G be a graph, let X be a nonempty proper subset of $V(G)$, and let $Y = V(G)\backslash X$. Suppose that each largest subset of arcs with the same label of $[X, Y]$ arc-induces a complete bipartite subgraph of G and that the arcs of G/X and G/Y corresponding to the arcs of $[X, Y]$ are the only synchronising arcs of G/X and G/Y. If $S'(G) \subseteq X$ and $[X, Y]$ has no backward arcs, then $G \cong G/Y \boxbslash G/X$.*

Proof. It clearly suffices to define a mapping $\phi : V(G) \to V(G/Y \boxbslash G/X)$ and to prove that ϕ is an isomorphism from G to $G/Y \boxbslash G/X$.

Let $\tilde{x}$ and $\tilde{y}$ be the new vertices replacing the sets X and Y when defining G/X and G/Y, respectively. Consider the mapping $\phi : V(G) \to V(G/Y \boxbslash G/X)$ defined by

$\phi(v) = (v, \tilde{x})$ for all $v \in X$ and $\phi(w) = (\tilde{y}, w)$ for all $w \in Y$.

Then ϕ is obviously a bijection if $V(G/Y \boxbslash G/X) = Z$, where Z is defined as $Z = \{(v, \tilde{x}) \mid v \in X\} \cup \{(\tilde{y}, w) \mid w \in Y\}$. We are going to show this later by arguing that all the other vertices of $G/Y \square G/X$ will disappear from $G/Y \boxtimes G/X$. But first we are going to prove the following claim.

Claim 2. The subgraph of $G/Y \boxtimes G/X$ induced by Z is isomorphic to G.

Proof. Obviously, ϕ is a bijection from $V(G)$ to Z. It remains to show that this bijection preserves the arcs and their labels. By the definition of the Cartesian product, for each arc $a \in A(G)$ with $\mu(a) = (u, v)$ for $u \in X$ and $v \in X$, there exists an arc b in $G/Y \boxtimes G/X$ with $\mu(b) = ((u, \tilde{x}), (v, \tilde{x})) = (\phi(u), \phi(v))$ and $\lambda(b) = \lambda(a)$. This is because the arc $a \notin [X, Y]$, and hence a is not a synchronising arc of G/Y with respect to G/X (by hypothesis). Likewise, for each arc $a \in A(G)$ with $\mu(a) = (u, v)$ for $u \in Y$ and $v \in Y$, there exists an arc b in $G/Y \boxtimes G/X$ with $\mu(b) = ((\tilde{y}, u), (\tilde{y}, v)) = (\phi(u), \phi(v))$ and $\lambda(b) = \lambda(a)$. Next, each arc $a \in A(G)$ with $\mu(a) = (u_i, v_j)$, for $u_i \in X$ and $v_j \in Y$, is an arc of a complete bipartite subgraph of G (by hypothesis). Let $B(Z_1, Z_2)$ be any such complete bipartite subgraph of G with vertex sets $Z_1 = \{u_1, \ldots, u_m\} \subseteq X$ and $Z_2 = \{v_1, \ldots, v_n\} \subseteq Y$ and arc set $A = \{a_1, \ldots, a_{m \cdot n}\}$ with $\mu(a_k) = (u_i, v_j)$, $\lambda(a_k) = a$, $k = i + (j-1) \cdot m$, $i = 1, \ldots, m$, $j = 1 \ldots n$. According to Lemma 6.1, $B(Z_1, Z_2)$ can be decomposed in $B(Z_1, Z_2)/Y$ and $B(Z_1, Z_2)/X$ with $B(Z_1, Z_2) \cong B(Z_1, Z_2)/Y \boxbslash B(Z_1, Z_2)/X$. For $B(Z_1, Z_2)/Y$ and $B(Z_1, Z_2)/X$, we have the arc sets $A(B(Z_1, Z_2)/Y) = \{u_1\tilde{y}, \ldots, u_m\tilde{y}\}$ and $A(B(Z_1, Z_2)/X) = \{\tilde{x}v_1, \ldots, \tilde{x}v_n\}$, respectively. Because these arcs are the only arcs synchronising over label a, we have the arc set $\{(u_i, \tilde{x})(\tilde{y}, v_j) \mid i = 1, \ldots, m, j = 1, \ldots, n\}$ in $G/Y \boxbslash G/X$. Then the graph arc-induced by the arc set with label a of $B(Z_1, Z_2)$ is isomorphic to the graph arc-induced by the arc set with label a of $B(Z_1, Z_2)/Z_2 \boxtimes B(Z_1, Z_2)/Z_1$, where $B(Z_1, Z_2)/Z_2 \boxtimes B(Z_1, Z_2)/Z_1$ is a subgraph of $G/Y \boxtimes G/X$. This completes the proof of Claim 2. □

We continue with the proof of Theorem 6.1. It remains to show that all other vertices of $G/Y \square G/X$, except for the vertices of Z, disappear from $G/Y \boxtimes G/X$. This is clear for the vertex $(\tilde{y}, \tilde{x})$: all the arcs of $G/Y \square G/X$ corresponding to the arcs of $[X, Y]$ are synchronising arcs of G/Y and G/X, so they disappear from $G/Y \boxtimes G/X$. Hence, $(\tilde{y}, \tilde{x})$ has in-degree 0 (and out-degree 0) in $G/Y \boxtimes G/X$, while it has $level > 0$ in $G/Y \square G/X$. For the other vertices, the argument is as follows.

The vertex set of $G/Y \square G/X$ consists of $Z \cup \{(\tilde{y}, \tilde{x})\}$ and the vertex set $X \times Y$. We will argue that all vertices of $X \times Y$ will eventually disappear from $G/Y \boxtimes G/X$.

Therefore, we claim that all $(u, v) \in X \times Y$ have $level > 0$ in $G/Y \square G/X$. This is obvious if u has $level > 0$ in $G[X]$ or v has $level > 0$ in $G[Y]$. Now let $(u, v) \in X \times Y$ such that u has $level\, 0$ in $G[X]$ and v has $level\, 0$ in $G[Y]$. Then the claim follows from the fact that v has at least one in-arc from a vertex in X, since $S'(G) \subseteq X$. In fact, since v has only in-arcs from vertices in X and u has no in-arcs at all, (u, v) has $level\, 0$ in $G/Y \boxtimes G/X$. This is because all arcs $(u, v) \in A(G)$ are in $[X, Y]$, hence they correspond to synchronising arcs in G/Y with respect to G/X. Concluding, all vertices $(u, v) \in X \times Y$ such that u has $level\, 0$ in $G[X]$ and v has $level\, 0$ in

$G[Y]$ disappear from $G/Y \boxtimes G/X$, together with all the arcs with tail (u, v) for all such vertices $(u, v) \in X \times Y$. If after this first step there are still vertices of $X \times Y$ left in $G/Y \boxtimes G/X$, we can repeat the above arguments step by step for such remaining vertices $(u, v) \in X \times Y$ for which (u, v) has the lowest level in what has remained from $G/Y \boxtimes G/X$. Since $G/Y \boxtimes G/X$ is acyclic, it is clear that all vertices of $X \times Y$ disappear one by one from $G/Y \boxtimes G/X$. This completes the proof of Theorem 6.1. □

We continue with the proof of Theorem 6.2 which relaxes the requirement of Theorem 5.2 that all the arcs of $[X_1, Y]$ have distinct labels, all the arcs of $[Y, X_2]$ have distinct labels and all the arcs of $[X_1, X_2]$ have distinct labels. In Figure 1, containing a non-trivial complete bipartite subgraph of G for which all arcs have identical labels, we have shown in a simple example how a graph G can be decomposed into the graphs G/Y and $G/X_1/X_2$ such that $G \cong G/Y \boxbslash G/X_1/X_2$. In Theorem 6.2, we use the proof of Theorem 5.2 given in [4] and modify this proof to support complete bipartite subgraph of G with arcs with the same label of $[X_1, Y]$ and $[Y, X_2]$ and bipartite subgraphs of G with arcs with the same label of $[X_1, X_2]$. Note that a bipartite subgraph of G arc-induced by all arcs with the same label of $[X_1, X_2]$ does not have to be complete.

Theorem 6.2. *Let G be a graph, and let X_1, X_2 and $Y = V(G)\backslash(X_1 \cup X_2)$ be three disjoint nonempty subsets of $V(G)$. Suppose that each largest subset of arcs with the same label of $[X_1, Y]$ arc-induces a complete bipartite subgraph of G, each largest subset of arcs with the same label of $[Y, X_2]$ arc-induces a complete bipartite subgraph of G, the arcs of $[X_1, X_2]$ have no labels in common with any arc in $[X_1, Y] \cup [Y, X_2]$, and the arcs of $G/X_1/X_2$ and G/Y corresponding to the arcs of $[X_1, Y] \cup [Y, X_2] \cup [X_1, X_2]$ are the only synchronising arcs of $G/X_1/X_2$ and G/Y. If $S'(G) \subseteq X_1$, and $[X_1, Y]$, $[Y, X_2]$ and $[X_1, X_2]$ have no backward arcs, then $G \cong G/Y \boxbslash G/X_1/X_2$.*

Proof. It suffices to define a mapping $\phi : V(G) \to V(G/Y \boxbslash G/X_1/X_2)$ and to prove that ϕ is an isomorphism from G to $G/Y \boxbslash G/X_1/X_2$.

Let $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{y}$ be the new vertices replacing the sets X_1, X_2 and Y when defining $G/X_1/X_2$ and G/Y, respectively. Consider the mapping $\phi : V(G) \to V(G/Y \boxbslash G/X_1/X_2)$ defined by $\phi(u) = (u, \tilde{x}_1)$ for all $u \in X_1$, $\phi(v) = (v, \tilde{x}_2)$ for all $v \in X_2$ and $\phi(w) = (\tilde{y}, w)$ for all $w \in Y$. Then ϕ is clearly a bijection if $V(G/Y \boxbslash G/X_1/X_2) = Z$, where Z is defined as $Z = \{(u, \tilde{x}_1) \mid u \in X_1\} \cup \{(v, \tilde{x}_2) \mid v \in X_2\} \cup \{(\tilde{y}, w) \mid w \in Y\}$. We are going to show this later by arguing that all the other vertices of $G/Y \Box G/X_1/X_2$ will disappear from $G/Y \boxtimes G/X_1/X_2$. But first we are going to prove the following claim.

Claim 3. The subgraph of $G/Y \boxtimes G/X_1/X_2$ induced by Z is isomorphic to G.

Proof. Obviously, ϕ is a bijection from $V(G)$ to Z. It remains to show that this bijection preserves the arcs and their labels. By the definition of the Cartesian product, for each arc $a \in A(G)$ with $\mu(a) = (u, v)$ for $u \in X_1$ and $v \in X_1$, there exists an arc b in $G/Y \boxtimes G/X_1/X_2$ with $\mu(b) = ((u, \tilde{x}_1), (v, \tilde{x}_1)) = (\phi(u), \phi(v))$ and $\lambda(b) = \lambda(a)$. Likewise, for each arc $a \in A(G)$ with $\mu(a) = (u, v)$ for $u \in Y$ and $v \in Y$, there exists an arc b in $G/Y \boxtimes G/X_1/X_2$ with $\mu(b) = ((\tilde{y}, u), (\tilde{y}, v)) = (\phi(u), \phi(v))$ and $\lambda(b) = \lambda(a)$, and for each arc $a \in A(G)$ with $\mu(a) = (u, v)$ for $u \in X_2$ and $v \in X_2$, there exists an arc b in $G/Y \boxtimes G/X_1/X_2$ with $\mu(b) = ((u, \tilde{x}_2), (v, \tilde{x}_2)) = (\phi(u), \phi(v))$

and $\lambda(b) = \lambda(a)$. Next, we distinguish two cases, the arcs of $[X_1, Y]$ and $[Y, X_2]$, and the arcs of $[X_1, X_2]$. Firstly, consider the arcs of $[X_1, Y]$ and $[Y, X_2]$. By hypothesis, the arcs with identical labels of $[X_1, Y]$ arc-induce a complete bipartite subgraph of G and the arcs with identical labels of $[Y, X_2]$ arc-induce a complete bipartite subgraph of G. Let $B(Z_1, Z_2)$ be a complete bipartite subgraph of G with vertex sets $Z_1 = \{u_1, \ldots, u_m\} \subseteq X_1$ and $Z_2 = \{v_1, \ldots, v_n\} \subseteq Y$ and let $B(Z_3, Z_4)$ be a complete bipartite subgraph of G with vertex sets $Z_3 = \{u'_1, \ldots, u'_{m'}\} \subseteq Y$ and $Z_4 = \{v'_1, \ldots, v'_{n'}\} \subseteq X_2$. Let all arcs of $B(Z_1, Z_2)$ and $B(Z_3, Z_4)$ have the same label a.

According to Lemma 6.1, $B(Z_1, Z_2)$ can be decomposed in $B(Z_1, Z_2)/Y$ and $B(Z_1, Z_2)/X_1/X_2$ with $B(Z_1, Z_2) \cong B(Z_1, Z_2)/Y \boxslash B(Z_1, Z_2)/X_1/X_2$ and $B(Z_3, Z_4)$ can be decomposed in $B(Z_3, Z_4)/Y$ and $B(Z_3, Z_4)/X_1/X_2$ with $B(Z_3, Z_4) \cong B(Z_3, Z_4)/Y \boxslash B(Z_3, Z_4)/X_1/X_2$. For $B(Z_1, Z_2)/Y$ and $B(Z_1, Z_2)/X_1/X_2$, we have the arc sets $A(B(Z_1, Z_2)/Y) = \{a_i \mid \mu(a_i) = (u_i, \tilde{y}), i = 1, \ldots, m\}$ and $A(B(Z_1, Z_2)/X_1/X_2) = \{b_j \mid \mu(b_j) = (\tilde{x}_1, v_j), j = 1, \ldots, n\}$, respectively and for $B(Z_3, Z_4)/Y$ and $B(Z_3, Z_4)\ /X_1/X_2$, we have the arc sets $A(B(Z_3, Z_4)/Y) = \{c_i \mid \mu(c_i) = (\tilde{y}, v'_i), i = 1, \ldots, m'\}$ and $A(B(Z_3, Z_4)\ /X_1/X_2) = \{d_j \mid \mu(d_j) = (u'_j, \tilde{x}_2), j = 1, \ldots, n'\}$, respectively. Because these arcs are the only arcs synchronising over label a, we have the arc set $\{e_{i,j} \mid \mu(e_{i,j}) = ((u_i, \tilde{x}_1), (\tilde{y}, v_j)), i = 1, \ldots, m, j = 1, \ldots, n\} \cup \{f_{j',j} \mid \mu(f_{j',j}) = ((\tilde{y}, \tilde{x}_1), (v'_{j'}, v_j)), j = 1, \ldots, n, j' = 1, \ldots, n'\} \cup \{g_{i,i'} \mid \mu(g_{i,i'}) = ((u_i, u'_{i'}), (\tilde{y}, \tilde{x}_2)), i = 1, \ldots, m, i' = 1, \ldots, m'\} \cup \{h_{i',j'} \mid \mu(h_{i',j'}) = ((\tilde{y}, u'_{i'}), (v'_{j'}, \tilde{x}_2)), i' = 1, \ldots, m', j' = 1, \ldots, n'\}$ in $G/Y \boxtimes G/X$. Therefore, for each arc $a \in A(G)$ with $\mu(a) = (u_i, v_j)$ for $u_i \in Z_1$ and $v_j \in Z_2$, there exists an arc $b \in G/Y \boxtimes G/X_1/X_2$ with $\mu(b) = ((u_i, \tilde{x}_1)(\tilde{y}, v_j)) = (\phi(u_i), \phi(v_j))$ and $\lambda(b) = \lambda(a)$, and for each arc $c \in A(G)$ with $\mu(c) = (u'_{i'}, v'_{j'})$ for $u'_{i'} \in Z_3$ and $v'_{j'} \in Z_4$, there exists an arc $d \in G/Y \boxtimes G/X_1/X_2$ with $\mu(d) = ((\tilde{y}, u'_{i'})(v'_{j'}, \tilde{x}_2)) = (\phi(u'_{i'}), \phi(v'_{j'}))$ and $\lambda(c) = \lambda(a)$.

It is sufficient to prove the preservation of the arcs with the same label for $B(Z_1, Z_2)$ and $B(Z_3, Z_4)$. If $B(Z_3, Z_4)$ does not exist, we do not have the subgraphs $B(Z_1, Z_2)/Y \boxslash B(Z_3, Z_4)/X_1/X_2$, $B(Z_3, Z_4)/Y \boxslash B(Z_1, Z_2)/X_1/X_2$ and $B(Z_3, Z_4)/Y \boxslash B(Z_3, Z_4)/X_1/X_2$ of $G/Y \boxslash G/X_1/X_2$ and if $B(Z_1, Z_2)$ does not exist, we do not have the subgraphs $B(Z_1, Z_2)/Y \boxslash B(Z_3, Z_4)/X_1/X_2$, $B(Z_3, Z_4)/Y \boxslash B(Z_1, Z_2)/X_1/X_2$ and $B(Z_1, Z_2)/Y \boxslash B(Z_1, Z_2)/X_1/X_2$ of $G/Y \boxslash G/X_1/X_2$. Therefore, this observation reduces the proof for $B(Z_1, Z_2)$ and $B(Z_3, Z_4)$ with arcs with identical labels to the proof for $B(Z_1, Z_2)$ with arcs with identical labels and the proof for $B(Z_3, Z_4)$ with arcs with identical labels.

Secondly, let $Z_1 \subseteq X_1$ and $Z_2 \subseteq X_2$. Let $B(Z_1, Z_2)$ be a bipartite subgraph of G with vertex sets $Z_1 = \{u_1, \ldots, u_m\} \subseteq X_1$ and $Z_2 = \{v_1, \ldots, v_n\} \subseteq Y$ where each arc $a \in A(B(Z_1, Z_2))$ has the same label a. Then the contraction G/Y will leave all arcs a of $B(Z_1, Z_2)$ with $\mu(a) = (u_i, v_j)$ and $\lambda(a) = a$ unchanged, therefore these arcs a correspond to arcs b of $B(Z_1, Z_2)/Y$ with $\mu(b) = (u_i, v_j)$ and $\lambda(b) = \lambda(a)$. The contraction $G/X_1/X_2$ will replace all vertices u_i of X_1 by one vertex $\tilde{x}_1$ and all vertices v_j of X_2 by one vertex $\tilde{x}_2$, and therefore, all the arcs a of $B(Z_1, Z_2)$ with $\mu(a) = (u_i, v_j)$ and $\lambda(a) = a$ are replaced by one arc c with $\mu(c) = (\tilde{x}_1, \tilde{x}_2)$ and $\lambda(c) = \lambda(a)$ of $G/X_1/X_2$. Because all arcs b of $B(Z_1, Z_2) \subseteq G/Y$ are synchronous arcs with respect to the arc c of $G/X_1/X_2$, we have that each pair of arcs b and c correspond with an arc d of $B(Z_1, Z_2)/Y \boxslash B(Z_1, Z_2)/X_1/X_2$ with $\mu(d) = ((u_i, \tilde{x}_1), (v_j, \tilde{x}_2))$ and $\lambda(d) = \lambda(a)$. Since there are no backward arcs in $[X_1, Y]$, $[Y, X_2]$ and $[X_1, X_2]$, the above arcs are the only arcs in $G/Y \boxtimes G/X_1/X_2$ induced by the vertices of Z. This completes the proof of Claim 3. $\square$

We continue with the proof of Theorem 6.2. It remains to show that all other vertices of $G/Y \boxtimes G/X_1/X_2$, except for the vertices of Z, disappear from $G/Y \boxtimes G/X_1/X_2$. This is clear for the vertex $(\tilde{y}, \tilde{x}_1)$: all the arcs of $G/Y \Box G/X_1/X_2$ corresponding to the arcs of $[X_1, Y]$ are synchronising arcs of G/Y and $G/X_1/X_2$, so they disappear from $G/Y \boxtimes G/X_1/X_2$. Hence, $(\tilde{y}, \tilde{x}_1)$ has in-degree 0 in $G/Y \boxtimes G/X_1/X_2$, while it has $level > 0$ in $G/Y \Box G/X_1/X_2$. For the other vertices, the argument is as follows.

The vertex set of $G/Y \Box G/X_1/X_2$ consists of the union of $Z \cup \{(\tilde{y}, \tilde{x}_1), (\tilde{y}, \tilde{x}_2)\}$ and the vertex sets $(X_1 \cup X_2) \times Y$, $X_1 \times \{\tilde{x}_2\}$ and $X_2 \times \{\tilde{x}_1\}$. We will argue that all vertices of $(X_1 \cup X_2) \times Y$, $X_1 \times \{\tilde{x}_2\}$ and $X_2 \times \{\tilde{x}_1\}$, as well as the vertex $(\tilde{y}, \tilde{x}_2)$ will eventually disappear from $G/Y \boxtimes G/X_1/X_2$.

Firstly, we claim that all $(u, v) \in X_1 \times Y$ have $level > 0$ in $G/Y \Box G/X_1/X_2$. This is obvious if u has $level > 0$ in $G[X_1]$ or v has $level > 0$ in $G[Y]$. Now let $(u, v) \in X_1 \times Y$ such that u has $level\,0$ in $G[X_1]$ and v has $level\,0$ in $G[Y]$. Then the claim follows from the fact that v has at least one in-arc from a vertex in X_1, since $S'(G) \subseteq X_1$. In fact, since v has only in-arcs from vertices in X_1 and u has no in-arcs at all, (u, v) has $level\,0$ in $G/Y \boxtimes G/X_1/X_2$. Hence, all vertices $(u, v) \in X_1 \times Y$ such that u has $level\,0$ in $G[X_1]$ and v has $level\,0$ in $G[Y]$ disappear from $G/Y \boxtimes G/X_1/X_2$, together with all the arcs with tail (u, v) for all such vertices $(u, v) \in X_1 \times Y$. If after this first step there are still vertices of $X_1 \times Y$ left in $G/Y \boxtimes G/X_1/X_2$, we can repeat the above arguments step by step for such remaining vertices $(u, v) \in X_1 \times Y$ for which (u, v) has the lowest level in what has remained from $G/Y \boxtimes G/X_1/X_2$. Since $G/Y \boxtimes G/X_1/X_2$ is acyclic, it is clear that all vertices of $X_1 \times Y$ disappear one by one from $G/Y \boxtimes G/X_1/X_2$. Now, since $(\tilde{y}, \tilde{x}_2)$ has possibly only in-arcs from vertices $(u, v) \in X_1 \times Y$, $(\tilde{y}, \tilde{x}_2)$ will disappear as well.

Next, we claim that all $(u, v) \in X_2 \times Y$ have $level > 0$ in $G/Y \Box G/X_1/X_2$. This is obvious if u has $level > 0$ in $G[X_2]$ or v has $level > 0$ in $G[Y]$. Now let $(u, v) \in X_2 \times Y$ such that u has $level\,0$ in $G[X_2]$ and v has $level\,0$ in $G[Y]$. Then the claim follows from the fact that u has at least one in-arc from a vertex in Y, since $[Y, X_2]$ has only forward arcs. In fact, since u has only in-arcs from vertices in Y and v has no in-arcs at all, (u, v) has $level\,0$ in $G/Y \boxtimes G/X_1/X_2$. Hence, all vertices $(u, v) \in X_2 \times Y$ such that u has $level\,0$ in $G[X_2]$ and v has $level\,0$ in $G[Y]$ disappear from $G/Y \boxtimes G/X_1/X_2$, together with all the arcs with tail (u, v) for all such vertices $(u, v) \in X_2 \times Y$. If after this first step there are still vertices of $X_2 \times Y$ left in $G/Y \boxtimes G/X_1/X_2$, we can repeat the above arguments step by step for such remaining vertices $(u, v) \in X_2 \times Y$ for which (u, v) has the lowest level in what has remained from $G/Y \boxtimes G/X_1/X_2$. Since $G/Y \boxtimes G/X_1/X_2$ is acyclic, it is clear that all vertices of $X_2 \times Y$ disappear one by one from $G/Y \boxtimes G/X_1/X_2$.

We continue with the claim that all $(u, \tilde{x}_1) \in X_2 \times \{\tilde{x}_1\}$ have $level > 0$ in $G/Y \Box G/X_1/X_2$. This is obvious if u has $level > 0$ in $G[X_2]$. Now let $(u, \tilde{x}_1) \in X_2 \times \{\tilde{x}_1\}$ such that u has $level\,0$ in $G[X_2]$. Then the claim follows from the fact that u has at least one in-arc from a vertex in Y, since $[Y, X_2]$ has only forward arcs. In fact, since u has only in-arcs from vertices in Y and $\tilde{x}_1$ has no in-arcs at all, $(u, \tilde{x}_1)$ has $level\,0$ in $G/Y \boxtimes G/X_1/X_2$. Hence, all vertices $(u, \tilde{x}_1) \in X_2 \times \{\tilde{x}_1\}$ such that u has $level\,0$ in $G[X_2]$ disappear from $G/Y \boxtimes G/X_1/X_2$, together with all the arcs with tail $(u, \tilde{x}_1)$ for all such vertices $(u, \tilde{x}_1) \in X_2 \times \{\tilde{x}_1\}$. If after this first step there are still vertices of $X_2 \times \{\tilde{x}_1\}$ left in $G/Y \boxtimes G/X_1/X_2$, we can repeat the above arguments step by step for such remaining vertices $(u, \tilde{x}_1) \in X_2 \times \{\tilde{x}_1\}$ for which $(u, \tilde{x}_1)$ has the lowest level in what has remained from $G/Y \boxtimes G/X_1/X_2$. Since $G/Y \boxtimes G/X_1/X_2$ is acyclic, it is clear that all vertices of $X_2 \times \{\tilde{x}_1\}$

disappear one by one from $G/Y \boxtimes G/X_1/X_2$.

Finally, we claim that all $(u, \tilde{x}_2) \in X_1 \times \{\tilde{x}_2\}$ have $level > 0$ in $G/Y \Box G/X_1/X_2$. This is obvious if u has $level > 0$ in $G[X_1]$. Now let $(u, \tilde{x}_2) \in X_1 \times \{\tilde{x}_2\}$ such that u has $level\, 0$ in $G[X_1]$. Then the claim follows from the fact that $\tilde{x}_2$ has at least one in-arc from a vertex in Y, since $[Y, X_2]$ has only forward arcs and $S'(G) \subseteq X_1$ by hypothesis. Noting that $\tilde{x}_2$ has only in-arcs from vertices in Y, and all $u \in S'(G) \subseteq X_1$ have no in-arcs at all, clearly for all $u \in S'(G) \subseteq X_1$, $(u, \tilde{x}_2)$ has $level\, 0$ in $G/Y \boxtimes G/X_1/X_2$. Hence, all vertices $(u, \tilde{x}_2) \in X_1 \times \{\tilde{x}_2\}$ such that u has $level\, 0$ in $G[X_1]$ disappear from $G/Y \boxtimes G/X_1/X_2$, together with all the arcs with tail $(u, \tilde{x}_2)$ for all such vertices $(u, \tilde{x}_2) \in X_1 \times \{\tilde{x}_2\}$. If after this first step there are still vertices of $X_1 \times \{\tilde{x}_2\}$ left in $G/Y \boxtimes G/X_1/X_2$, we can repeat the above arguments step by step for such remaining vertices $(u, \tilde{x}_2) \in X_1 \times \{\tilde{x}_2\}$ for which $(u, \tilde{x}_2)$ has the lowest level in what has remained from $G/Y \boxtimes G/X_1/X_2$. Since $G/Y \boxtimes G/X_1/X_2$ is acyclic, it is clear that all vertices of $X_1 \times \{\tilde{x}_2\}$ disappear one by one from $G/Y \boxtimes G/X_1/X_2$. This completes the proof of Theorem 6.2. □

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