INTERLACE POLYNOMIALS OF LOLLIPOP AND TADPOLE GRAPHS

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Abstract. In this paper, we examine interlace polynomials of lollipop and tadpole graphs. The lollipop and tadpole graphs are similar in that they both include a path attached to a graph by a single vertex. In this paper we give both explicit and recursive formulas for each graph, which extends the work of Arratia, Bollóbas and Sorkin [3],[4], among others. We also give special values, examine adjacency matrices and behavior of coefficients of these polynomials.

Keywords: *graph polynomial, interlace polynomial, lollipop graph*

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1. Introduction

The interlace polynomial of a graph was introduced by Arratia, Bollóbas, and Sorkin, and it is a polynomial that contains and represents the information gained from performing the toggling process on the graph, see [3]. Interlace polynomials share similarities with other graph polynomials, such as Tutte and Martin polynomials, [6]. Some researchers have studied different types of graph polynomials, such as genus polynomials, [9].

Interlace polynomials give important information about the graph, for example, the number of k-component circuit partitions, for $k \in \mathbb{N}$. In [4], interlace polynomials for some simple graphs like paths, cycles, and complete graphs are given, although not all of the formulas are easy to use. Li, Wu and Nomani give recursive formulas for interlace polynomials of ladder and n-Claw graphs, see [11, 12]. In addition, Eubanks-Turner and Li give recursive and explicit formulas for friendship graphs, see [8]. Also, work has been done in utilizing bivariate and multi-variate interlace polynomials, see [2, 4, 13]. Here, we focus our attention on establishing the interlace polynomials of lollipop and tadpole graphs. Both graphs have the structure that each has a well-known subgraph which is connected to a path by a single vertex.

A lollipop graph is a simple graph that consists of a complete graph being joined to a path with a bridge. Lollipop graphs have applications to stochastic processes and spectral graph theory [14]. In particular, lollipop graphs have maximum possible hitting time, cover time and commute time [10]. In this work we consider the interlace polynomial of the lollipop graph. We give both the recursive and explicit formulas for the interlace polynomials of lollipop graphs. As Eubanks-Turner and Li did in [8], we consider natural questions regarding interlace polynomials of the lollipop graph, such as:

(1) What are the general forms of the coefficients of the interlace polynomial of a lollipop graph?

(2) What are the evaluations of the interlace polynomial of a lollipop graph at certain values and what does it describe about related graph properties?

We further consider graphs related to lollipop graphs called tadpole graphs. Tadpole graphs are simple graphs obtained by joining a cycle to a path via a bridge. Although the recursive and explicit formulas for interlace polynomials of tadpole graphs are given in [1], we give a different explicit formula and further results about interlace polynomials of tadpole graphs related to the latter mentioned questions.

For a simple graph G, to construct the interlace polynomial we first need the following definition. Note, $\forall a \in V(G), N(a) = \{\text{neighbors of } a\}$.

Definition 1.1. (Pivot) Let $G = (V(G), E(G))$ be any undirected graph without a loop and $a, b \in V(G)$ with $ab \in E(G)$. We first partition the neighbors of a or b into three classes:

(1) $N(a) \setminus (\{b\} \cup N(b))$;
(2) $N(b) \setminus (\{a\} \cup N(a))$;
(3) $N(a) \cap N(b)$.

The pivot graph G^{ab} of G, with respect to ab is the resulting graph of the toggling process: $\forall u, v \in V(G)$, if u, v are from different classes shown above, then $uv \in E(G) \iff uv \notin E(G^{ab})$.

Note that $G^{ab} = G^{ba}$. The pivot operation is only defined for an edge of G. The definition for the interlace polynomial of a simple graph G is given below. Here, $\forall u \in V(G)$, $G - u$ is the graph resulting from removing u and all the edges of G incident to u from G.

Definition 1.2. Let $\mathcal{G}$ be the class of finite undirected graphs having no loops nor multiple edges. There is a unique map $q : \mathcal{G} \to \mathbb{Z}[x]$, suhc that, $G \mapsto q(G)$. For any undirected graph G with n vertices that has no loops nor multiple edges, the interlace polynomial $q(G)$ of G is defined by

$$q(G) = \begin{cases} x^n & \text{if } E(G) = \emptyset \\ q(G-a) + q(G^{ab} - b) & \text{if } ab \in E(G). \end{cases}$$

This map is a well-defined polynomial on all simple graphs, see [2]. Next, we give some known results about interlace polynomials and results relating the interlace polynomials to structural components of graphs.

Lemma 1.3. [4] Given the interlace polynomial $q(G)$ of any undirected graph G, the following results hold:

(1) The interlace polynomial of any simple graph has zero constant term.
(2) For any two disjoint graphs G_1, G_2, we have $q(G_1 \cup G_2) = q(G_1) \cdot q(G_2)$.
(3) The degree of the lowest-degree term of $q(G)$ is $k(G)$, the number of components of G.
(4) $\deg(q(G)) \geq \alpha(G)$, where $\alpha(G)$ is the independence number, i.e., the size of a maximum independent set.
(5) Let $\mu(G)$ denote the size of a maximum matching (maximum set of independent edges) in a graph G. If G is a forest with n vertices, then $\deg(q(G)) = n - \mu(G)$.

Lemma 1.4. [4] The interlace polynomials for the following graphs are as follows:

(1) (complete graph K_m) $q(K_m) = 2^{m-1}x$;
(2) (complete bipartite graph K_{mn})
$q(K_{mn}) = (1 + x + ... + x^{m-1})(1 + x + ... + x^{n-1}) + x^m + x^n - 1$;
(3) (path P_n with n edges) $q(P_1) = 2x$, $q(P_2) = x^2 + 2x$, and for $n \geq 3$, $q(P_n) = q(P_{n-1}) + xq(P_{n-2})$;
(4) (small cycles C_m) $q(C_3) = 4x$ and $q(C_4) = 2x + 3x^2$.

Definition 1.5. Let $m, n \in \mathbb{N}$, with $m \geq 3, n \geq 0$. The lollipop graph $L_{m,n}$ is the simple graph obtained by joining a complete graph K_m to a path graph P_n at one of the vertices of K_m. The lollipop graph $L_{m,n}$ has $m + n$ vertices and $\binom{m}{2} + n$ edges. Note, we hold $m \geq 3$ to disregard the case when $L_{m,n}$ is a path.

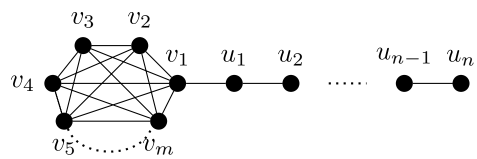

Diagram 1.5: The Graph $L_{m,n}$

2. Interlace Polynomial of Lollipop Graphs

Next, we develop interlace polynomials of lollipop graphs $L_{m,n}$, where $m, n \in \mathbb{N}$. We treat $L_{m,0}$ as the complete graph with m vertices. The lollipop graph has the same recursive formula as the path graph, see Lemma 1.4 (3).

Lemma 2.1. Let $m \geq 3$. For $n \geq 2$, $q(L_{m,n}) = q(L_{m,n-1}) + xq(L_{m,n-2})$, where $q(L_{m,0}) = q(K_m)$ and $q(L_{m,1}) = q(K_m) + xq(K_{m-1})$.

Proof. Clearly, $q(L_{m,0}) = q(K_m)$.
Also, $L_{m,1}$ is a graph consisting of a complete graph with m vertices and a path of length one. Labeling the edge of the path as ab, where $deg(a) = 1$, we have, $q(L_{m,1} - a) = q(K_m)$. Also note that $L_{m,1}^{ab} - b = K_{m-1} \cup \{a\}$, where $b \in V(K_m)$. So, $q(L_{m,1}) = q(K_m) + x \cdot q(K_{m-1})$, by Lemma 1.4 (2).
Now assume $n \geq 2$ and consider the graph $L_{m,n}$ with the edge ab, such that $deg(a) = 1$. See the following diagram.

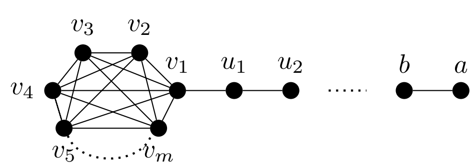

Diagram 2.1(a): $L_{m,n}$
Considering the interlace polynomial, we have that $L_{m,n} - a = L_{m,n-1}$.

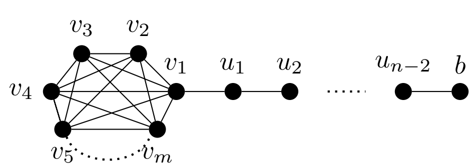

Diagram 2.1(b): $L_{m,n} - a$

Now toggling the graph, we have, $L^{ab}_{m,n} - b = L_{m,n-2} \cup \{a\}$, as illustrated in the following diagram, where $n \geq 2$.

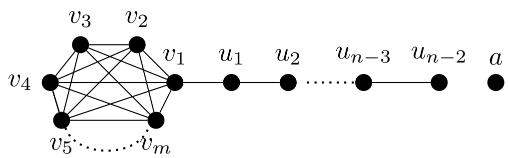

Diagram 2.1(c): $L^{ab}_{m,n} - b$

Hence, $q(L_{m,n}) = q(L_{m,n-1}) + xq(L_{m,n-2})$, for $n \geq 2$ as desired. □

The next result gives the explicit formula for the interlace polynomial of $L_{m,n}$ for $m \geq 3$ and $n \geq 0$.

Theorem 2.2. For $m \geq 3$ and $n \geq 0$,

$$q(L_{m,n}) = \frac{1}{\sqrt{1+4x}}\left[\frac{q(K_m) + \gamma x q(K_{m-1})}{\gamma^{n+1}} - \frac{(q(K_m) + \beta x q(K_{m-1}))}{\beta^{n+1}}\right]$$

where $\gamma = \dfrac{\sqrt{1+4x}-1}{2x}$ and $\beta = \dfrac{-\sqrt{1+4x}-1}{2x}$.

Proof. We utilize the recurrence relation in a generating function $G(y)$, where the coefficient of y^n is $q(L_{m,n})$:

$$G(y) = \sum_{n=0}^{\infty} q(L_{m,n})y^n$$

By Lemma 2.1,

$$G(y) = q(L_{m,0}) + y \cdot q(L_{m,1}) + \sum_{n=2}^{\infty} \left(q(L_{m,n-1}) + xq(L_{m,n-2})\right) y^n$$

$$= \frac{q(L_{m,0}) + y \cdot q(L_{m,1}) - y \cdot q(L_{m,0})}{1 - y - xy^2}$$

For convenience, let $\gamma = \dfrac{\sqrt{1+4x}-1}{2x}$ and $\beta = \dfrac{-\sqrt{1+4x}-1}{2x}$, the roots of the polynomial $1-y-xy^2$. Since $q(L_{m,0}) = q(K_m)$ and $q(L_{m,1}) = q(K_m) + xq(K_{m-1})$, we have the desired result.

□

Now we show that $\deg(q(L_{m,n})) = \alpha(L_{m,n})$, where α is the independence number of the graph $L_{m,n}$.

Proposition 2.3. For $m \geq 3$, $n \geq 0$, $\deg(q(L_{m,n})) = 1 + \lceil \frac{n}{2} \rceil$, where $1 + \lceil \frac{n}{2} \rceil = \alpha(L_{m,n})$, the independence number of $L_{m,n}$.

Proof. First to see that $1 + \lceil \frac{n}{2} \rceil = \alpha(L_{m,n})$, note one can construct an independent set of size $\frac{n}{2} + 1$ by taking the leaf and then choosing alternating non-adjacent vertices on the path of length n, and then select one vertex from the complete portion of the graph non-adjacent to any vertices already described. Including any additional vertex of the graph would result in a dependent vertex set. Therefore we have the desired result.

We prove that $\deg(q(L_{m,n})) = 1 + \lceil \frac{n}{2} \rceil$ by strong induction. First, the base cases:

$$deg(q(L_{m,0})) = 1 = 1 + \left\lceil \frac{0}{2} \right\rceil$$

$$deg(q(L_{m,1})) = 2 = 1 + \left\lceil \frac{1}{2} \right\rceil$$

Now assume that for all k with $0 \leq k \leq n-1$, $deg(q(L_{m,k})) = 1 + \lceil \frac{k}{2} \rceil$. Therefore,

$$deg(q(L_{m,n-1})) = 1 + \left\lceil \frac{n-1}{2} \right\rceil \leq 1 + \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$$

$$deg(q(L_{m,n-2})) = 1 + \left\lceil \frac{n-2}{2} \right\rceil = 1 + \left\lceil \frac{n}{2} - 1 \right\rceil = \left\lceil \frac{n}{2} \right\rceil$$

Since the addition of vertices can only increase the degree of the associated interlace polynomial,

$$\left\lceil \frac{n}{2} \right\rceil = deg(q(L_{m,n-2})) \leq deg(q(L_{m,n-1})) \leq \left\lceil \frac{n}{2} \right\rceil$$

$$\text{which implies, } deg(q(L_{m,n-1})) = \left\lceil \frac{n}{2} \right\rceil$$

Therefore,

$$\begin{aligned} deg(q(L_{m,n})) &= deg(q(L_{m,n-1}) + xq(L_{m,n-2})) \\ &= max\{deg(q(L_{m,n-1})), deg(xq(L_{m,n-2}))\} \\ &= max\{\left\lceil \frac{n}{2} \right\rceil, 1 + \left\lceil \frac{n}{2} \right\rceil\} \\ &= 1 + \left\lceil \frac{n}{2} \right\rceil \end{aligned}$$

□

Arritia, Bollóbas and Sorkin evaluate interlace polynomials at certain small values of x, as these values give information relating to circuit partitions of the graphs, see [4]. They show that for any graph G, $q(G)(2) = 2^{|V(G)|}$ and if H is the interlace graph of an Euler circuit of a 2-in, 2-out digraph D, then $q(H)(1)$ is the number of Euler circuits of D. They also give a conjecture about the explicit formula of the interlace polynomial at $x = -1$, which Balister, Bollóbas, Cutler and Peabody prove in [5].

We now give some propositions which show the evaluations of the lollipop graph at 1 and -1. In [4], the following corollary was given.

Corollary 2.4. [4] For the path P_n, $q(P_n)(1) = F_{n+2}$, the $(n+2)$'nd Fibonacci number (with $F_0 = 0$ and $F_1 = 1$).

Since the lollipop has the same recursive formula as the path, we obtain similar results when evaluating at $x = 1$. For the evaluation at $x = -1$, we give our results in the same form as what was given in [1].

Proposition 2.5. For $m \geq 3, n \geq 0$,

(1) $q(L_{m,n})(1) = 2^{m-2}F_{n+3}$, the $(n+2)$'nd Fibonacci number (with $F_0 = 0$ and $F_1 = 1$).

$$(2)\ q(L_{m,n})(-1) = \begin{cases} -2^{m-1}, \text{ if } n \equiv 0 \mod 6 \\ -2^{m-2}, \text{ if } n \equiv 1,5 \mod 6 \\ 2^{m-2}, \text{ if } n \equiv 2,4 \mod 6 \\ 2^{m-1}, \text{ if } n \equiv 3 \mod 6 \end{cases}$$

Proof. We prove (1) by strong induction on n. Note, $q(L_{m,0})(1) = q(K_m)(1) = 2^{m-1} = 2^{m-2}(2) = 2^{m-2}F_3$, by Lemma 1.4 (1). Let $n \geq 1$ and assume $q(L_{m,n})(1) = 2^{m-2}(F_{k+3})$, for all k, $0 \leq k \leq n$. By Lemma 2.1,

$$\begin{aligned} q(L_{m,n+1})(1) &= q(L_{m,n})(1) + q(L_{m,n-1})(1) \\ &= 2^{m-2}(F_{n+3}) + 2^{m-2}(F_{n+2}) \\ &= 2^{m-2}(F_{n+3} + F_{n+2}) \\ &= 2^{m-2}(F_{n+4}), \text{by definition of the Fibonacci sequence,} \\ &= 2^{m-2}(F_{(n+1)+3}). \end{aligned}$$

For (2), also by Lemma 2.1,

$$\begin{aligned} q(L_{m,n})(-1) &= q(L_{m,n-1})(-1) - q(L_{m,n-2})(-1) \\ &= [q(L_{m,n-2})(-1) - q(L_{m,n-3})(-1)] - q(L_{m,n-2})(-1) \\ &= -q(L_{m,n-3})(-1) \\ &= -q(L_{m,n-4})(-1) + q(L_{m,n-5})(-1) \\ &= -q(L_{m,n-5})(-1) + q(L_{m,n-6})(-1) + q(L_{m,n-5})(-1) \\ &= q(L_{m,n-6})(-1). \end{aligned}$$

Therefore, $q(L_{m,n})(-1)$ is periodic of period 6. The first six values of $q(L_{m,n})(-1)$ are: $q(L_{m,0})(-1) = -2^{m-1}$, $q(L_{m,1})(-1) = -2^{m-2} = q(L_{m,5})(-1)$, $q(L_{m,2})(-1) = 2^{m-2} = q(L_{m,4})(-1)$, $q(L_{m,3})(-1) = 2^{m-1}$.

□

In the following Theorem Balister, Bollóbas, Cutler and Peabody give an explicit formula for the interlace at $x = -1$, therefore proving the conjecture in [5].

Theorem 2.6. [5] Let A be the adjacency matrix of G, $n = |V(G)|$ and let $r = rank(A + I)$ over $\mathbb{Z}_2$ and I be the $n \times n$ identity matrix. Then

$$q(G, -1) = (-1)^n(-2)^{n-r}.$$

Now we consider applications of lollipop graphs in solving a linear algebra problem. We determine the ranks of the adjacency matrices related to $L_{m,n}$ over $\mathbb{Z}_2$. For $m \geq 3$, $n \geq 0$, let $A_{m,n}$ denote the adjacency matrix of $L_{m,n}$. Then $A_{m,n}$ has the form

$$A_{m,n} = \begin{pmatrix} 0 & 1 & \dots & 1 & | & 0 & \dots & \dots & 0 \\ 1 & 0 & 1 & \vdots & | & \vdots & 0 & & \vdots \\ \vdots & 1 & \ddots & 1 & | & 0 & & 0 & \vdots \\ 1 & \dots & 1 & 0 & | & 1 & 0 & \dots & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & \dots & 0 & 1 & | & & & & \\ \vdots & 0 & & 0 & | & & A & & \\ 0 & & 0 & 0 & | & & & & \end{pmatrix}_{(m+n)\times(m+n)}$$

with $A_{n\times n} = [a_{ij}]$, where $a_{ij} = \begin{cases} 1, & \text{if } j = i+1 \text{ or } j = i-1 \\ 0, & \text{otherwise} \end{cases}$.

Proposition 2.7. Suppose $m \geq 3$, $n \geq 0$, let $C_{m,n} = A_{m,n} + I_{m+n}$, where I_{m+n} is the $(m+n)\times(m+n)$ identity matrix. Then, $rank(C_{m,n}) = \begin{cases} n+1, & \text{if } n \equiv 0,3 \mod 6 \\ n+2, & \text{otherwise} \end{cases}$

Proof. Using Theorem 2.6 and Proposition 2.5(2), we have the desired result. □

We end this section by giving information about coefficients of $L_{m,n}$, which gives us a way to describe the relations between coefficients.

Theorem 2.8. For $m \geq 3, n \geq 0$, write $q(L_{m,n})(x) = \sum_{k=0}^{1+\lceil \frac{n}{2} \rceil} l_{m,n,k}x^k$. Then the coefficients $l_{m,n,k} = 2^{m-2}\left(\binom{n-k+1}{k-2} + 2\binom{n-k+1}{k-1}\right)$ for all $k \geq 0$.

Proof. Note that from Lemma 2.1, we have, $q(L_{m,n}) = q(L_{m,n-1}) + xq(L_{m,n-2})$, which yields

$$l_{m,n,k} = l_{m,n-1,k} + l_{m,n-2,k-1}. \quad (1)$$

Fix $m \geq 3$. We proceed by multivariate strong induction on n and k. To show the base cases, note that by Lemma 1.3 (1), $l_{m,n,0} = 0$, for all $n \geq 0$ and so $l_{m,n,0} = 2^{m-2}\left(\binom{n+1}{-2} + 2\binom{n+1}{-1}\right)$, for all $n \geq 0$. By lemma 1.4 (1), $q(L_{m,0}) = q(K_m) = 2^{m-1}x$ and so $l_{m,0,k} = \begin{cases} 0, & k \neq 1 \\ 2^{m-1}, & k = 1. \end{cases}$

Now fix $n, k \geq 0$. Assume the following inductive hypotheses:

$l_{m,i,j} = 2^{m-2}\left(\binom{i-j+1}{j-2} + 2\binom{i-j+1}{j-1}\right)$, for all i, j, when $1 \leq i \leq n$ and $1 \leq j \leq k+1$.

$l_{m,i,j} = 2^{m-2}\left(\binom{i-j+1}{j-2} + 2\binom{i-j+1}{j-1}\right)$, for all i, j, when $1 \leq i \leq n+1$ and $1 \leq j \leq k$.

Consider $l_{m,n+1,k+1}$. From Equation (1), we have, $l_{m,n+1,k+1} = l_{m,n,k+1} + l_{m,n-1,k-1}$. By the first inductive hypothesis,

$l_{m,n,k+1} = 2^{m-2}\left(\binom{n-(k+1)+1}{(k+1)-2} + 2\binom{n-(k+1)+1}{(k+1)-1}\right) = 2^{m-2}\left(\binom{n-k}{k-1} + 2\binom{n-k}{k}\right)$.

By the second inductive hypothesis,

$l_{m,n-1,k} = 2^{m-2}\left(\binom{(n-1)-k+1}{k-2} + 2\binom{(n-1)-k+1}{k-1}\right) = 2^{m-2}\left(\binom{n-k}{k-2} + 2\binom{n-k}{k-1}\right).$

Hence,

$$l_{m,n+1,k+1} = l_{m,n,k+1} + l_{m,n-1,k-1}$$
$$= 2^{m-2}\left(\binom{n-k}{k-1} + \binom{n-k}{k-2} + 2\binom{n-k}{k} + 2\binom{n-k}{k-1}\right)$$

Applying Pascal's Identity yields:

$$l_{m,n+1,k+1} = 2^{m-2}\left(\binom{n-k}{k-1} + 2\binom{n-k}{k}\right)$$

Thus, $l_{m,n,k} = 2^{m-2}\left(\binom{n-k+1}{k-2} + 2\binom{n-k+1}{k-1}\right)$ for all $m \geq 3$ and $n, k \geq 0$.

□

3. Interlace Polynomial of Tadpole Graphs

Now we turn our attention to tadpole graphs. Tadpole graphs have structure similar to lollipop graphs as they are cycles connected to a path at a vertex of the cycle.

Definition 3.1. Let $m, n \in \mathbb{N}$, with $m \geq 3, n \geq 0$. The tadpole graph $T_{m,n}$ is the simple graph obtained by joining a cycle C_m to a path graph P_n at one of the vertices of C_m. The tadpole graph $T_{m,n}$ has $m+n$ vertices and $m+n$ edges. Note, we hold $m \geq 3$ to disregard the case when $T_{m,n}$ is a path.

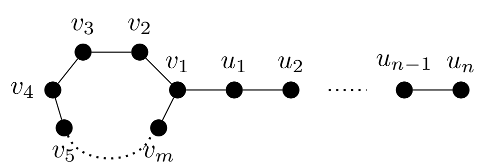

Diagram 3.1: The Graph $T_{m,n}$

Next, we show that the recursive formula for the interlace polynomial of the tadpole graph is the same as the interlace polynomial of the path and lollipop graphs. The explicit formula for the interlace polynomial of the tadpole graph has a structure similar to the interlace polynomial of the lollipop graph.

Lemma 3.2. For $m \geq 3, n \geq 2$, $q(T_{m,n}) = q(T_{m,n-1}) + xq(T_{m,n-2})$, where $q(T_{m,0}) = q(C_m)$ and $q(T_{m,1}) = q(C_m) + xq(P_{m-2})$.

We utilize the following explicit formulas given in [4].

Lemma 3.3. [4] Let P_m be the path of length m (with $m+1$ vertices and m edges) and C_m denote the cycle with m vertices.

(1) For $m \geq 2$, the interlace polynomial of the path P_m satisfies

$$q(P_m) = \frac{(3+y)(y-1)}{4y}\left(\frac{1+y}{2}\right)^{m+1} + \frac{(3-y)(y+1)}{4y}\left(\frac{1-y}{2}\right)^{m+1}$$

where $y = \sqrt{1+4x}$.

(2) For $m \geq 4$, with $y = \sqrt{1+4x}$,

$$q(C_m) = \Big(\frac{1-y}{2}\Big)^m + \Big(\frac{1+y}{2}\Big)^m + \frac{y^4 - 10y^2 - 7}{16} \quad \text{for } m \text{ even},$$

$$q(C_m) = \Big(\frac{1-y}{2}\Big)^m + \Big(\frac{1+y}{2}\Big)^m + \frac{y^2 - 5}{4} \quad \text{for } m \text{ odd}.$$

The next theorem gives an explicit form of $q(T_{m,n})(x)$ for $m \geq 4$ and $n \geq 0$. In [1] a general explicit function for $q(T_{m,n})$ is given for $m \geq 6, n \geq 5$. Here we provide an alternative form of the explicit function for $T_{m,n}$ and give results for $m \geq 3, n \geq 0$.

Theorem 3.4. For $m \geq 3$ and $n \geq 0$,

$$q(T_{m,n}) = \frac{q(C_m) + xq(P_{m-2})\gamma}{x(\beta-\gamma)}\Big(\frac{1}{\gamma}\Big)^{n+1} - \frac{q(C_m) + xq(P_{m-2})\beta}{x(\beta-\gamma)}\Big(\frac{1}{\beta}\Big)^{n+1}$$

where $\gamma = \dfrac{\sqrt{1+4x}-1}{2x}$, $\beta = \dfrac{-\sqrt{1+4x}-1}{2x}$ and $q(C_m)$ and $q(P_m)$ are the explicit forms given in Lemma 3.3.

Proposition 3.5. For $m \geq 3$, $n \geq 1$, $\deg(q(T_{m,n})) = 1 + \lceil \frac{n}{2} \rceil$, when n is odd and $\deg(q(T_{m,n})) = 2 + \lceil \frac{n}{2} \rceil$, when n is even.

Proof. Similar to Proposition 2.3. □

Proposition 3.6. For $m \geq 3$, $n \geq 1$, the independence number $\alpha(T_{m,n}) = \lceil \frac{n-1}{2} \rceil + \lfloor \frac{m}{2} \rfloor$.

Proof. To build an independent set, start by including the vertex of degree 1 and include alternating vertices along the path, which yield $\lceil \frac{n-1}{2} \rceil$ vertices in the independent set. Then add $\lfloor \frac{m}{2} \rfloor$ alternating vertices of the cycle to the independent set in such a way that the set remains independent. Note, that including any additional vertex of the graph results in a vertex set that is not an independent. This gives the independence number of $L_{m,n}$ since $\alpha(P_n) = \lceil \frac{n-1}{2} \rceil$ and $\alpha(C_m) = \lfloor \frac{m}{2} \rfloor$ □

Next, we give evaluations of $T_{m,n}$ at $x = 1$ and -1. Similar to the interlace polynomials of the path and lollipop graphs evaluated at $x = 1$, the interlace polynomial for the tadpole graph evaluated at 1 involves the Fibonacci sequence.

Theorem 3.7. For $m \geq 3, n \geq 0$,

$$q(T_{m,n})(1) = \begin{cases} (F_{n+1}\sqrt{5} + F_n)F_m + 2F_{n+1}\Big(\Big(\frac{1-\sqrt{5}}{2}\Big)^m - 1\Big), \text{ if m is even} \\ (F_{n+1}\sqrt{5} + F_n)F_m + 2F_{n+1}\Big(\frac{1-\sqrt{5}}{2}\Big)^m, \text{ if m is odd.} \end{cases},$$

where F_n is the n-th Fibonacci number with $F_0 = 0$ and $F_1 = 1$.

Proof. Fix $m \geq 3$. We proceed by induction on n. By Lemma 3.3(2),

$$q(T_{m,0})(1) = q(C_m)(1) = \begin{cases} \sqrt{5}F_m + 2\Big(\Big(\frac{1-\sqrt{5}}{2}\Big)^m - 1\Big), \text{ if m is even} \\ \sqrt{5}F_m + 2\Big(\frac{1-\sqrt{5}}{2}\Big)^m, \text{ if m is odd.} \end{cases}$$

Since, $F_0 = 0$ and $F_1 = 1$, we have,

$$q(T_{m,0})(1) = \begin{cases} (F_1\sqrt{5} + F_0)F_m + 2F_1\Big(\Big(\frac{1-\sqrt{5}}{2}\Big)^m - 1\Big), \text{ if m is even} \\ (F_1\sqrt{5} + F_0)F_m + 2F_1\Big(\frac{1-\sqrt{5}}{2}\Big)^m, \text{ if m is odd.} \end{cases}$$

Similarly, $q(T_{m,1})(1) = \begin{cases} (F_2\sqrt{5}+F_1)F_m + 2F_2\Big(\big(\frac{1-\sqrt{5}}{2}\big)^m - 1\Big), \text{ if m is even} \\ (F_2\sqrt{5}+F_1)F_m + 2F_2\big(\frac{1-\sqrt{5}}{2}\big)^m, \text{ if m is odd.} \end{cases}$

Suppose $n \geq 2$. By Lemma 3.2, $q(T_{m,n})(1) = q(T_{m,n-1})(1) + q(T_{m,n-2})(1)$. Consider the following cases: If m is even, then

$$\begin{aligned} q(T_{m,n+1})(1) &= \Big[(F_{n+1}+F_n)\sqrt{5} + (F_n + F_{n-1})\Big]F_m - 2(F_{n+1}+F_n)\Big(\Big(\frac{1-\sqrt{5}}{2}\Big)^m - 1\Big) \\ &= (F_{n+2}\sqrt{5} + F_{n+1})F_m - 2F_{n+2}\Big(\Big(\frac{1-\sqrt{5}}{2}\Big)^m - 1\Big). \end{aligned}$$

If m is odd, then

$$\begin{aligned} q(T_{m,n+1})(1) &= \Big[(F_{n+1}+F_n)\sqrt{5} + (F_n + F_{n-1})\Big]F_m + 2(F_{n+1}+F_n)\Big(\frac{1-\sqrt{5}}{2}\Big)^m, \\ &= (F_{n+2}\sqrt{5} + F_{n+1})F_m + 2F_{n+2}\Big(\frac{1-\sqrt{5}}{2}\Big)^m. \end{aligned}$$

□

Proposition 3.8. For $m \geq 3, n \geq 0$, let $C^m = \Big(\frac{1-\sqrt{-3}}{2}\Big)^m + \Big(\frac{1+\sqrt{-3}}{2}\Big)^m$. We have, $C^m = \begin{cases} 2, \text{ if } m \equiv 0 \mod 6 \\ 1, \text{ if } m \equiv 1,5 \mod 6 \\ -1, \text{ if } m \equiv 2,4 \mod 6 \\ -2, \text{ if } m \equiv 3 \mod 6 \end{cases}$.

Then, $q(T_{m,n})(-1) = \begin{cases} 4, \text{ if } n, m \equiv 0 \mod 6 \\ -1, \text{ if } n \equiv 0 \mod 6; m \equiv 1,5 \mod 6 \\ 1, \text{ if } n \equiv 0 \mod 6; m \equiv 2,4 \mod 6 \\ -4, \text{ if } n \equiv 0 \mod 6; m \equiv 3 \mod 6 \\ \\ 2, \text{ if } n \equiv 1 \mod 6; m \text{ is even} \\ -2, \text{ if } n \equiv 1 \mod 6; m \text{ is odd} \\ \\ -C^m, \text{ if } n \equiv 2 \mod 6 \\ \\ -q(T_{m,0}), \text{ if } n \equiv 3 \mod 6 \\ \\ -q(T_{m,1}), \text{ if } n \equiv 4 \mod 6 \\ \\ -q(T_{m,2}), \text{ if } n \equiv 5 \mod 6 \end{cases}$.

Proof. For $m \geq 3$, we have, $C^m = \left(\frac{1-\sqrt{-3}}{2}\right)^m + \left(\frac{1+\sqrt{-3}}{2}\right)^m = \frac{\left(1-\sqrt{-3}\right)^m + \left(1+\sqrt{-3}\right)^m}{2^m} = \frac{\left(2e^{i\frac{\pi}{3}}\right)^m + \left(2e^{-i\frac{\pi}{3}}\right)^m}{2^m} = e^{i\frac{m\pi}{3}} + e^{-i\frac{m\pi}{3}} = 2\cos\frac{m\pi}{3}$. Note, that $\cos\frac{m\pi}{3} = \begin{cases} 1, \text{ if } m \equiv 0 \mod 6 \\ \frac{1}{2}, \text{ if } m \equiv 1,5 \mod 6 \\ -\frac{1}{2}, \text{ if } m \equiv 2,4 \mod 6 \\ -1, \text{ if } m \equiv 3 \mod 6 \end{cases}$.

Therefore, C^m is as claimed. Utilizing Lemma 3.3, the remainder of the proof is similar to Proposition 2.6.

□

Now we consider applications of tadpole graphs in solving a linear algebra problem. For $n \geq 1$, let B_{mn} denote the adjacency matrix of T_{mn}. Then

$$B_{mn} = \left(\begin{array}{cccc|cccc} & & & & 0 & \dots & \dots & 0 \\ & & & & \vdots & 0 & & \vdots \\ & \mathrm{B} & & & 0 & & 0 & \vdots \\ & & & & 1 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & 1 & & & & \\ \vdots & 0 & & 0 & & \mathrm{A} & & \\ 0 & & 0 & 0 & & & & \end{array}\right)_{(m+n)\times(m+n)}$$

with $B_{m\times m} = [b_{ij}]$, where $b_{ij} = \begin{cases} 1, \text{ if } j = i+1 \text{ or } j = i-1 \\ 1, \text{ if } i = 1, j = m \text{ or } i = m, j = 1 \\ 0, \text{ otherwise} \end{cases}$

$A_{n\times n} = [a_{ij}]$, where $a_{ij} = \begin{cases} 1, \text{ if } j = i+1 \text{ or } j = i-1 \\ 0, \text{ otherwise} \end{cases}$.

Proposition 3.9. Suppose $m \geq 3$, $n \geq 0$, let $D_{m,n} = B_{m,n} + I_{m+n}$, where I_{m+n} is the $(m+n)\times(m+n)$ identity matrix. Then,

$$rank(D_{m,n}) = \begin{cases} m+n-2, \text{ if } n, m \equiv 0,3 \mod 6 \\ m+n, \text{ if } n \equiv 0,2,3,5 \mod 6; m \equiv 1,2,4,5 \mod 6 \\ m+n-1, \text{ if } n \equiv 1,4 \mod 6 \text{ and } n \equiv 2,5 \mod 6; m \equiv 0,3 \mod 6 \end{cases}$$

Proof. Using Theorem 2.6 and Lemma 3.8, we have the desired result. □

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