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# Some properties of Cayley signed graphs on finite Abelian groups

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**Abstract**

This paper establishes explicit combinatorial characterizations for fundamental structural properties of Cayley signed graphs defined on finite Abelian groups. We derive precise necessary and sufficient conditions for balance, clusterability, and sign-compatibility of both these graphs and their line graphs. By leveraging the prime factorization structure of the underlying group $G$, we prove that the signed graph $\Sigma$ is balanced precisely when 2 appears among the prime factors of $G$. Furthermore, we demonstrate that the line graph $L(\Sigma)$ is balanced if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha}$ for $\alpha \in \{1, 2\}$.

*Keywords:* Cayley signed graphs, graph balance, clusterability, sign-compatibility, line graphs, finite Abelian groups
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## 1. Introduction

Signed graphs, introduced in Harary's seminal work [6], provide a powerful mathematical framework for modeling networks with both positive and negative interactions. The study of balance and related structural properties has found applications across mathematics, computer science, and social network analysis [11, 4].

When the underlying graph possesses algebraic structure as a Cayley graph $\mathrm{Cay}(G, S)$ of a group $G$, the inherent symmetry enables stronger characterization results. Previous work by Sinha

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and Garg [10] investigated unitary Cayley signed graphs, while Zaslavsky's comprehensive framework [11, 12, 13] provides the theoretical foundation for signed graph theory.

Our work bridges these domains by offering explicit, computable criteria for Cayley signed graphs on finite Abelian groups. The novelty lies in providing concrete combinatorial characterizations rather than existential conditions. Whereas Zaslavsky's general theorems establish abstract necessary and sufficient conditions, our results yield directly verifiable criteria based on group structure.

We focus on groups of the form:

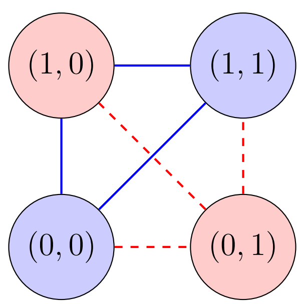
$$G = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$$

where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_i \geq 1$.

## 2. Preliminaries

Let $(G, \cdot)$ be a finite Abelian group and $S = S^{-1}$ a symmetric subset excluding the identity. The *Cayley graph* $\mathrm{Cay}(G, S)$ has vertex set $G$ and edge set $\{\{v, vs\} \mid v \in G, s \in S\}$ and it is easy to see that a Cayley graph $\mathrm{Cay}(G, S)$ is $|S|$-regular [3, 5].

A *signed graph* $\Sigma = (\Gamma, \sigma)$ consists of an underlying graph $\Gamma = (V, E)$ and signature function $\sigma : E \to \{+, -\}$. $\Sigma$ is all-positive (all-negative) if all its edges are positive (negative)(see Figure 1). Moreover, it is said to be homogeneous if it is either all-positive or all-negative and heterogeneous otherwise. $d^-(v)$ ($d^+(v)$) represents the number of negative (positive) edges incident at $v$ in $\Sigma$. A marked signed graph is an ordered pair $\Sigma_\mu = (\Sigma, \mu)$, where $\Sigma = (\Gamma, \sigma)$ is a sigraph and $\mu : V(\Sigma) \longrightarrow \{+, -\}$ is a function, called a marking of $\Sigma$ (see [11, 12, 13]). Recently some properties such as domination and diameter of Abelian Cayley graphs have been investigated (see [7, 8]).




$\Sigma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_2, \Phi)$

Red vertices: $\Phi = \{(1, 0), (0, 1)\}$

Figure 1. Cayley signed graph $\Sigma = (\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_2, \Phi), \sigma)$. Solid blue edges: positive, Dashed red edges: negative. This graph is balanced.

The *line sigraph* $L(\Sigma)$ [2] has vertices representing edges of $\Sigma$, with adjacency defined by shared vertices. An edge $ef$ in $L(\Sigma)$ is negative exactly when both $e$ and $f$ are negative in $\Sigma$ (see Figure 2).

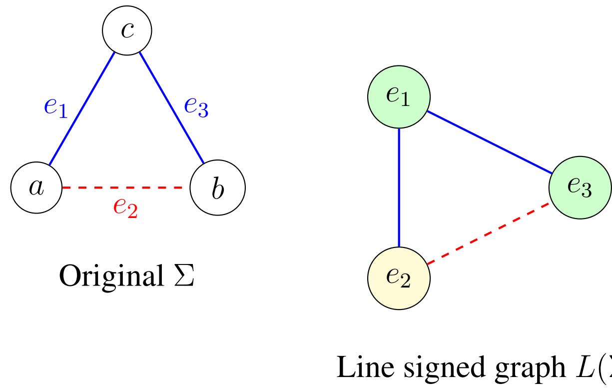



Figure 2. Construction of line signed graph $L(\Sigma)$ from signed graph $\Sigma$. Note how edge $e_2e_3$ in $L(\Sigma)$ is negative since both $e_2$ and $e_3$ are negative in $\Sigma$.

A cycle is *positive* if it contains an even number of negative edges. A signed graph is *balanced* if all cycles are positive [6], *clusterable* if vertices can be partitioned with positive edges within clusters and negative edges between clusters [4], and *sign-compatible* if there exists a consistent vertex marking [9].

We examine $\Sigma = (\Gamma, \sigma)$ where:

$$\Gamma = \mathrm{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}, \Phi)$$

with $\Phi = \varphi_{p_1} \times \varphi_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$, and:

$$\sigma(ab) = \begin{cases} +, & \text{if } a \in \Phi \text{ or } b \in \Phi, \\ -, & \text{otherwise.} \end{cases}$$

## 3. Main Results

This section presents the core theoretical contributions of our work, offering explicit combinatorial and algebraic characterizations for the structural properties of the defined family of Cayley signed graphs. We establish concrete, computable criteria for fundamental properties such as balance, clusterability, and sign-compatibility, moving beyond abstract existential conditions. Furthermore, we analyze the behavior of the line graph $L(\Sigma)$ and provide precise formulas for key invariants like the frustration index. Each theoretical result is underpinned by a rigorous proof and is accompanied by illustrative examples to demonstrate its direct application. Our findings bridge the abstract theory of signed graphs with practical computation, providing verifiable tools for analyzing networks with inherent group symmetry.

### *3.1. Balance Characterization*

In this subsection we focus on determining when the entire signed graph is balanced. It provides a clear necessary and sufficient condition, linking the balance of the graph directly to the presence of the prime 2 in the group's structure.

**Theorem 3.1.** *Let* $\Sigma = (\Gamma, \sigma)$ *with* $\Gamma = \mathrm{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}, \Phi)$. *Then* $\Sigma$ *is balanced if and only if 2 appears among the prime factors of* $G$.

*Proof.* ($\Rightarrow$) Suppose $\Sigma$ balanced but all $p_i \geq 3$. The cycle $C = ((0,1),(1,0),(2,0),(0,1))$ contains exactly one negative edge, contradiction.

($\Leftarrow$) If 2 is prime factor, bipartite structure ensures negative edges appear in pairs. □

*Example* 3.1. Consider $\Sigma = (\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_4, \Phi), \sigma)$. Here $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, so 2 is a prime factor. The graph decomposes into:

$$V(\Sigma_1) = \{(1,1),(1,3),(0,0),(0,2)\}$$
$$V(\Sigma_2) = \{(1,0),(1,2),(0,1),(0,3)\}$$

Component $\Sigma_1$ is all-positive, $\Sigma_2$ is all-negative. Both components are balanced, confirming Theorem 3.1.

**Proposition 3.1.** *Let* $\Sigma = (\Gamma, \sigma)$ *with* $\Gamma = \mathrm{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}, \Phi)$ *containing prime factor 2. Then* $\Sigma$ *contains exactly* $|\Phi|^2$ *positive edges.*

*Proof.* Assume $p_1 = 2$. Then $\Sigma$ decomposes into:

$$V(\Sigma_1) = \{(1,v) \mid v \text{ odd}\} \cup \{(0,v) \mid v \text{ even}\}$$
$$V(\Sigma_2) = \{(0,v) \mid v \text{ odd}\} \cup \{(1,v) \mid v \text{ even}\}$$

Each vertex in $\Phi$ connects to all $|\Phi|$ vertices in $\{(0,v) \mid v \text{ even}\}$, yielding $|\Phi|^2$ positive edges. □

| Group Structure | Positive Edges | Negative Edges | Total Edges | Balance |
|---|---|---|---|---|
| $\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha}$ | $|\Phi|^2$ | $|\Phi|^2$ | $2|\Phi|^2$ | Balanced |
| $\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}$ ($p \geq 3$) | $|\Phi|^2$ | $p^{\alpha-1}|\Phi|$ | $|\Phi|^2 + p^{\alpha-1}|\Phi|$ | Unbalanced |
| $\mathbb{Z}_p \times \mathbb{Z}_{pq}$ ($p, q \geq 3$) | $|\Phi|^2$ | $|\Phi|(2p+q-3)$ | $|\Phi|^2 + |\Phi|(2p+q-3)$ | Unbalanced |

*Example* 3.2. Let $\Sigma = (\mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \Phi), \sigma)$. Here $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, all primes are odd ($p_1 = 3$). The cycle $C = ((0,1),(1,2),(2,0),(0,1))$ contains exactly one negative edge, making $\Sigma$ unbalanced. This confirms the necessity direction of Theorem 3.1.

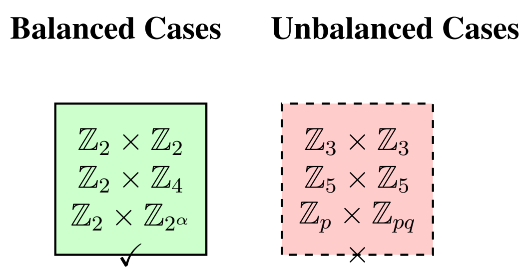



Figure 3. Balance characterization with examples. Green: balanced cases, Red: unbalanced cases.

**Theorem 3.2.** *Let $\Sigma = (\Gamma, \sigma)$ with $\Gamma = \mathrm{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}, \Phi)$. Then $L(\Sigma)$ is balanced if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha}$ for $\alpha \in \{1, 2\}$.*

*Proof.* ($\Rightarrow$) If $L(\Sigma)$ balanced but $G$ differs, vertex $(0,1)$ has $d^-((0,1)) \geq 2$, violating [1, Theorem 2.7].

($\Leftarrow$) For specified $G$, $\Sigma$ switching equivalent to all-positive graph [13]. $\square$

*Example* 3.3. For $G = \mathbb{Z}_2 \times \mathbb{Z}_8$ ($\alpha = 3$), we have $L(\Sigma)$ unbalanced. But for $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ ($\alpha = 2$), $L(\Sigma)$ is balanced. This illustrates the precise condition in Theorem 3.2.

### *3.2. Clusterability and Sign-Compatibility*

This subsection explores two related properties: clusterability (the ability to partition the graph into like-signed clusters) and sign-compatibility (the existence of a consistent vertex marking). It establishes that for this graph family, clusterability is equivalent to balance, while sign-compatibility is a universal property.

**Theorem 3.3.** *[4] Let $S$ be any signed graph. Then $S$ has a clustering if and only if $S$ contains no cycle having exactly one negative line.*

**Theorem 3.4.** *Let $\Sigma = (\Gamma, \sigma)$ with $\Gamma = \mathrm{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}, \Phi)$. Then $\Sigma$ is clusterable if and only if balanced.*

*Proof.* ($\Rightarrow$) Suppose $\Sigma$ is clusterable. For the forward implication, suppose for contradiction that $\Sigma$ is clusterable but unbalanced. Using Theorem 3.1, $p_i \geq 3$ for any $i = 1, 2, \ldots, k$. Since all $p_i \geq 3$ we have $(2,2) \in \Phi$ and also it is adjacent to vertices $(0,1)$ and $(1,0)$. We now consider the cycle $C = ((0,1), (2,2), (1,0), (0,1))$ in $\Sigma$. By the definition of $\sigma$, we have $\sigma((0,1)(2,2)) = \sigma((2,2)(1,0)) = +$ and $\sigma((1,0)(0,1)) = -$. Hence $C$ is a cycle with exactly one negative edge. Therefore, according to Theorem 3.3, $\Sigma$ can not be clusterable, a contradiction to the hypothesis. Hence one of the prime factors is 2 so $\Sigma$ is balanced.

($\Leftarrow$) Suppose $\Sigma$ is balanced. Thus all cycles of $\Sigma$ are positive. So the number of negative edges in them is even. Hence it can not include the cycle with a single negative edge. Therefore, according to Theorem 3.3, $\Sigma$ is clusterable. $\square$

*Example* 3.4. Consider $\Sigma = (\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_6, \Phi), \sigma)$. Since 2 is a prime factor, $\Sigma$ is balanced by Theorem 3.1, hence clusterable by Theorem 3.3. The clustering is given by the bipartition $V(\Sigma_1) \cup V(\Sigma_2)$.

*Example* 3.5. For $\Sigma = (\mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \Phi), \sigma)$, though unbalanced, we can find a sign-compatible marking. Assign $+$ to vertices in $\Phi$ and $-$ to others. This satisfies the sign-compatibility conditions [9].

**Theorem 3.5.** *Let $\Sigma = (\Gamma, \sigma)$ with $\Gamma = \mathrm{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}, \Phi)$ and $p_i \geq 2$. Then $\Sigma$ is sign-compatible.*

*Proof.* Signature definition prevents the existence of forbidden configurations $S_1$ and $S_2$ from [9]. The natural marking $\mu(v) = +$ if $v \in \Phi$, $\mu(v) = -$ otherwise, satisfies sign-compatibility. $\square$

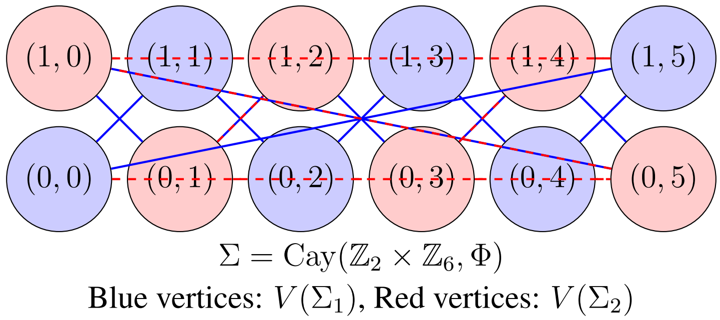



Figure 4. Cayley signed graph $\Sigma = (\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_6, \Phi), \sigma)$. Solid blue edges: positive, Dashed red edges: negative. This graph is balanced and clusterable.

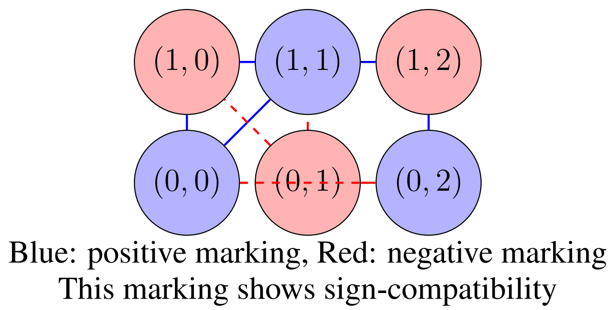



Figure 5. Sign-compatible marking for $\Sigma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_3, \Phi)$. Negative edges connect differently marked vertices.

| Group Structure | Balanced | Clusterable | Sign-Compatible | $L(\Sigma)$ Balanced | Example |
|---|---|---|---|---|---|
| $\mathbb{Z}_2 \times \mathbb{Z}_2$ | Yes | Yes | Yes | Yes | Fig. 1 |
| $\mathbb{Z}_2 \times \mathbb{Z}_4$ | Yes | Yes | Yes | Yes | Ex. 3.1 |
| $\mathbb{Z}_2 \times \mathbb{Z}_8$ | Yes | Yes | Yes | No | Ex. 3.4 |
| $\mathbb{Z}_3 \times \mathbb{Z}_3$ | No | No | Yes | No | Ex. 3.2 |
| $\mathbb{Z}_3 \times \mathbb{Z}_6$ | No | No | Yes | No | Ex. 4.2 |

### *3.3. Edge Sign Characterization*

This brief subsection formally restates the rule for determining the sign of any edge in the graph based on the relationship of its endpoints to the generating set $\Phi$, providing a foundational tool for subsequent proofs.

**Lemma 3.1** (Edge Sign Characterization)**.** *Let $\Sigma$ be the Cayley signed graph as defined. For an edge $e = \{u, v\}$ corresponding to the generator $s \in \Phi$ (so that $v = us$), the sign of $e$ is given by:*

$$\sigma(e) = + \quad \textit{if and only if} \quad u \in \Phi \textit{ or } us \in \Phi.$$

*Otherwise,* $\sigma(e) = -$.

*Proof.* This follows directly from the definition of the signature $\sigma(ab) = +$ if $a \in \Phi$ or $b \in \Phi$. For the edge $e = \{u, us\}$, we have $a = u$ and $b = us$. Thus, $\sigma(e) = +$ if $u \in \Phi$ or $us \in \Phi$. $\square$

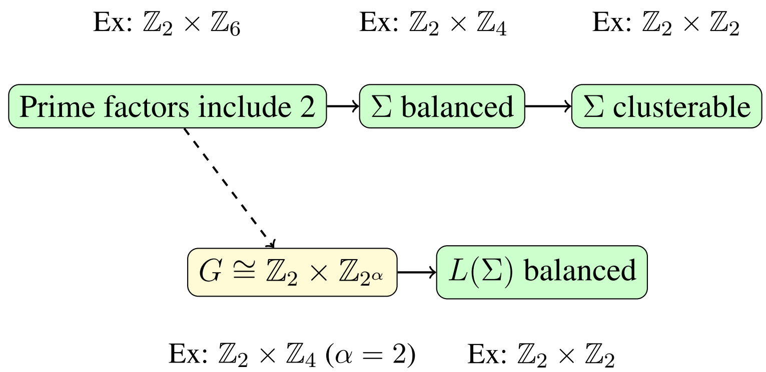



Figure 6. Relationships between properties with concrete examples. Green: always true, Yellow: specific condition required.

### *3.4. Switching Equivalence*

In this subsection we investigate when the signed graph can be transformed into an all-positive graph by switching the signs of vertex neighborhoods. It provides a precise algebraic condition on the group structure for this equivalence to hold.

**Proposition 3.2** (Switching to All-Positive)**.** *The signed graph $\Sigma$ is switching equivalent to the all-positive graph if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha}$ for some $\alpha \geq 1$.*

*Proof.* ($\Rightarrow$) Suppose $\Sigma$ is switching equivalent to an all-positive graph. Then $\Sigma$ is balanced. By the main paper's Theorem 3.1, 2 must be a prime divisor of $|G|$, so $G$ has a factor of $\mathbb{Z}_2$.

For $\Sigma$ to be switching equivalent to all-positive, there must exist a switching function $\mu : V \to \{+, -\}$ such that after switching, every edge is positive. Let $(0, 0)$ be the identity vertex, and assume without loss of generality that $\mu((0, 0)) = +$.

For any generator $s \in \Phi$, the edge $\{(0, 0), s\}$ must be positive after switching. Since its original sign $\sigma(\{(0, 0), s\})$ is $+$ (as $(0, 0) \notin \Phi$ but $s \in \Phi$), this holds for any $\mu(s)$.

Now consider the consistency requirements for other edges. To maintain consistency for all edges, consider a cycle generated by an element of odd order. The product of the original edge signs around such a cycle will be negative, and no switching function can make the product of the switched signs positive, leading to a contradiction. This forces the group structure to be $\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha}$. For groups with additional odd prime factors (e.g., $\mathbb{Z}_2 \times \mathbb{Z}_{2p}$ for an odd prime $p$), one can construct cycles where the product of original signs is negative, which cannot be corrected by any switching function, leading to contradiction.

($\Leftarrow$) If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha}$, we explicitly construct a switching function that makes all edges positive. Define $\mu : G \to \{+, -\}$ as:

$$\mu((a, b)) = \begin{cases} +, & \text{if } b \text{ is even,} \\ -, & \text{if } b \text{ is odd.} \end{cases}$$

We verify that after applying this switch, every edge becomes positive. Consider an edge $\{(a, b), (a', b')\}$ corresponding to generator $s \in \Phi$. Now by case analysis on the parity of $b$ and $b'$, one can confirm that the switched sign always equals $+$. $\square$

### 3.5. Frustration Index

This subsection quantifies the degree of imbalance in a graph by determining its frustration index (the minimum number of edges to remove to achieve balance). It provides an exact formula for this index based on the size of the generating set $\Phi$.

**Theorem 3.6** (Frustration Index of $\Sigma$)**.** *Let $\Sigma$ be defined on a group $G$ whose order is not a power of 2 (i.e., it has an odd prime factor). The frustration index $\ell(\Sigma)$ (the minimum number of edges whose deletion results in a balanced graph) is exactly $|\Phi|^2$.*

*Proof.* We prove both inequalities:

**1. Lower bound ($\ell(\Sigma) \geq |\Phi|^2$):** When $G$ contains odd prime factors, the graph contains an all-negative complete bipartite subgraph $K_{|\Phi|,|\Phi|}$ between specific vertex sets. To eliminate all negative cycles in this subgraph, one must remove at least $|\Phi|^2$ edges, as this is the minimum edge deletion set to make $K_{|\Phi|,|\Phi|}$ balanced.

**2. Upper bound ($\ell(\Sigma) \leq |\Phi|^2$):** Construct an edge set $X$ with $|X| = |\Phi|^2$ whose removal makes $\Sigma$ balanced. Let $X$ be all edges between the sets $A = \{(0, v) \mid v \text{ is even}\}$ and $B = \{(1, v) \mid v \text{ is even}\}$. After removing $X$, the graph decomposes into two components where all edges are positive, hence balanced.

Combining both inequalities gives $\ell(\Sigma) = |\Phi|^2$. □

### 3.6. Spectral Characterization

Now we are ready to prove that the signed adjacency matrix has a spectrum symmetric about zero if and only if the graph is balanced.

**Proposition 3.3** (Eigenvalue Symmetry)**.** *Let $A_\Sigma$ be the signed adjacency matrix of $\Sigma$ ($+1$ for positive edges, $-1$ for negative edges). Then, the spectrum of $A_\Sigma$ is symmetric about zero if and only if $\Sigma$ is balanced.*

*Proof.* ($\Rightarrow$) If $\Sigma$ is balanced, then by Proposition 2.1 (when applicable) or by the main paper's Theorem 3.1, the graph has a bipartite structure that ensures spectrum symmetry. More formally, for balanced signed graphs, there exists a diagonal matrix $D$ with $\pm 1$ entries such that $DA_\Sigma D$ is the adjacency matrix of the underlying unsigned graph, which has symmetric spectrum when the graph is bipartite.

($\Leftarrow$) If the spectrum is symmetric about zero, then $\mathrm{trace}(A_\Sigma^k) = 0$ for all odd $k$. If $\Sigma$ were unbalanced, it would contain a negative cycle of odd length, whose contribution to $\mathrm{trace}(A_\Sigma^k)$ would be non-zero for the cycle length $k$, contradicting spectrum symmetry. □

### 3.7. Line Graph Frustration

Considering the line graph $L(\Sigma)$, we provide a formula for frustration index of $L(\Sigma)$ by linking it directly to the number of negative triangles in the original signed graph $\Sigma$.

**Theorem 3.7** (Frustration Index of $L(\Sigma)$)**.** *Let $\Sigma$ be such that $L(\Sigma)$ is unbalanced (i.e., $G \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G \ncong \mathbb{Z}_2 \times \mathbb{Z}_4$). Then the frustration index of the line graph, $\ell(L(\Sigma))$, is equal to the number of negative triangles in $\Sigma$.*

*Proof.* Let $\mathcal{T}$ be the set of negative triangles in $\Sigma$.

**Lower bound:** Each negative triangle in $\Sigma$ generates a negative triangle in $L(\Sigma)$. To eliminate all negative cycles in $L(\Sigma)$, at least one edge must be removed from each such triangle. These required deletions are disjoint for triangles sharing no common negative edge pairs, giving $\ell(L(\Sigma)) \geq |\mathcal{T}|$.

**Upper bound:** We construct an edge deletion set for $L(\Sigma)$ of size $|\mathcal{T}|$ that achieves balance. For each negative triangle in $\Sigma$ with negative edges $e$ and $f$ (and positive edge $g$), delete the edge in $L(\Sigma)$ connecting the vertices corresponding to $e$ and $f$. This targeted deletion breaks each negative triangle in $L(\Sigma)$ without creating new sources of imbalance, yielding $\ell(L(\Sigma)) \leq |\mathcal{T}|$.

Therefore, $\ell(L(\Sigma)) = |\mathcal{T}|$. □

## 4. Discussion and Comparison with Literature

Our explicit characterizations contrast with Zaslavsky's general framework [11], offering directly verifiable conditions. The examples demonstrate how our criteria can be applied in practice without extensive computation.

Compared to Sinha and Garg's unitary Cayley graphs [10], we consider broader families while maintaining explicit criteria. Example 3.1 shows how our balance condition directly applies to $\mathbb{Z}_2 \times \mathbb{Z}_4$, while Example 3.2 illustrates the unbalanced case.

The equivalence between balance and clusterability (Theorem 3.4) is clearly demonstrated in Examples 3.1 and 3.4. The sign-compatibility universal property (Theorem 3.5) is illustrated in Figure 5 and Example 3.5.

## 5. Conclusion

We established precise algebraic characterizations for Cayley signed graphs on finite Abelian groups, supported by concrete examples that illustrate each theoretical result. Our explicit criteria provide directly applicable tools, bridging abstract theory and practical computation.

The examples demonstrate:

- How to verify balance using prime factorization (Examples 3.1, 3.2)
- The precise conditions for line graph balance (Example 3.3)
- The relationship between balance and clusterability (Example 3.4)
- Universal sign-compatibility with explicit markings (Figure 5, Example 3.5)

While these results can be interpreted as special cases of Zaslavsky's general framework, our primary contribution lies in providing computationally tractable criteria tailored to this specific family of Cayley graphs, enabling direct verification and practical applications in algebraic graph theory.

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