Counting and labeling grid related graphs

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Abstract

In this work we explore some graphs associated with the grid $P_m \times P_n$. A fence is any subgraph of the grid obtained by deleting any feasible number of edges from some or all the copies of $P_m$. We present here a closed formula for the number of non-isomorphic fences obtained from $P_m \times P_n$, for every $m,n \geq 2$. A rigid grid is a supergraph of the grid, where for every square a pair of opposite vertices are connected; we show that the number of fences built on $P_m \times P_n$ is the same that the number of rigid grids built on $P_m \times P_{n+1}$. We also introduce a substitution scheme that allows us to substitute any interior edge of any $P_m$ in an $\alpha$-labeled copy of $P_m \times P_n$ to obtain a new graph with an $\alpha$-labeling. This process can be iterated multiple times on the $n$ copies of $P_m$; in this way we prove the existence of an $\alpha$-labeling for any graph obtained via these substitutions; these graphs form a quite robust family of $\alpha$-graphs where the grid is one of its members. We also show two subfamilies of disconnected graphs that can be obtained using this scheme, proving in that way that they are also $\alpha$-graphs.

Keywords: graceful, $\alpha$-labeling, fence, rigid grid
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1. Introduction

The grid $P_m \times P_n$ is the graph that results of the Cartesian product of the paths $P_m$ and $P_n$. Thus, the grid has vertex-set $V(P_m \times P_n) = V(P_m) \times V(P_n)$, and edge-set $E(P_m \times P_n) = E(P_m) \times V(P_n) \cup V(P_m) \times E(P_n)$. This graph has order $mn$ and size $2mn - (m + n)$. Given...
the simplicity of the structure of the paths, the grid also has a straightforward structure, being one of the first graphs studied in the context of graph labeling; in particular, its bipartite nature makes it an ideal candidate to be an \( \alpha \)-graph.

The following two definitions were introduced in the mid sixties by Rosa [13]. Let \( G \) be a graph of order \( m \) and size \( n \); an injective function \( f: V(G) \to \{0, 1, \ldots, n\} \) is said to be graceful when every edge \( uv \) of \( G \) has assigned a weight defined as \( |f(u) - f(v)| \) and the set of all weights induced on the edges of \( G \) is \( \{1, 2, \ldots, n\} \).

The function \( f \) is called an \( \alpha \)-labeling of \( G \) when \( G \) is a bipartite graph and there is an integer \( \lambda \) such that whenever \( f(u) \leq \lambda < f(v) \), \( u \) and \( v \) are in different stable sets of \( G \). The integer \( \lambda \) is called the boundary value of \( f \); when \( G \) admits an \( \alpha \)-labeling is named an \( \alpha \)-graph. This type of labeling is the most restrictive of the ones introduced by Rosa. Several other types of labeled graphs can be obtained by proving that they are of the \( \alpha \)-type. Rosa [13] used these labelings to study the problem of decomposing \( K_{2n+1} \) into copies of any tree of size \( n \). He proved that any tree of size \( n \) admitting a \( \rho \)-labeling decomposes (cyclically) the graph \( K_{2n+1} \). In a \( \rho \)-labeling of a tree of size \( n \), the labels are taken from \( \{0, 1, \ldots, 2n\} \) and the set of induced weights is \( \{x_1, x_2, \ldots, x_n\} \) where \( x_i = i \) or \( x_i = 2n+1-i \). Hence, a graceful labeling is also a \( \rho \)-labeling. Rosa proved that a cyclic \( T \)-decomposition of \( K_{2n+1} \) exists when \( T \) is a tree of size \( n \) that admits a \( \rho \)-labeling. These are some of the origins of the Graceful Tree Conjecture, that states that all trees are graceful. Several families of graceful trees are known; among them, we have the family of all trees of diameter up to five. Balbuena et al. [2], used the technique of edge switching to prove that all trees of diameter \( d \) are graceful, provided that all leaves are vertices of maximum eccentricity and all internal vertices (except a root) have even degree. They also showed that when the degrees of all vertices adjacent to a leaf are odd, the tree is also graceful. The same technique is used by Mishra et al. [11] to show the existence of a graceful labeling for each tree of diameter six such that \( \text{deg}(v) \) is even for every vertex \( v \) adjacent to the root with descendents in level 3. In Section 3, we use edge switching on \( \alpha \)-labeled graphs related to \( P_m \times P_n \). A detailed account of results on graph labelings can be found in [9].

Several graph operations have been studied under the perspective of graceful and \( \alpha \)-labelings. The Cartesian product and the join of two graphs have received special attention. Some results about the corona of two graphs have been published in the last years. In [3], Barrientos proved that if \( G \) is a graceful graph of order \( m \) and size \( m-1 \), then \( G \circ nK_1 \) is graceful. Recently, Mitra and Bhomik [12], showed that in the case of complete bipartite graphs, \( K_{2m,2m} \circ K_2 \) is graceful. As we mentioned before, \( P_m \times P_n \) is an ideal candidate to be an \( \alpha \)-graph, in fact Jungreis and Read [10] proved that the grid \( P_m \times P_n \) is an \( \alpha \)-graph, they also described a way to transform the \( \alpha \)-labeling of the grid into a harmonious labeling. Polyominoes form a robust family of subgraphs of the grid; Acharya [1] asked: Are all polyominoes arbitrarily graceful? Polyominoes can be described as a collection of a number of equal-sized squares arranged with coincident sides. Motivated by Acharya’s question we investigated [4] the existence of an \( \alpha \)-labeling for a subfamily of polyominoes, the one formed by the snake-polyominoes. Later [7] we used the idea of Jungreis and Read to prove that the \( \alpha \)-labeling of the snake polyominoes could also be transformed into a harmonious labeling, proving so that these snakes are also harmonious.

In the present work, we continue the search of subgraphs of the grid that admit \( \alpha \)-labelings. In [5] and [6] we show some results about \( \alpha \)-graphs associated with the cartesian product of paths and
caterpillars. We also defined a 2-link fence as the graph obtained with \( r \) copies of \( P_n \) by connecting two vertices of the \( i \)th copy to the corresponding two vertices in the \((i + 1)\)th copy. In Section 2 we extend this concept and define a fence as a subgraph of the grid \( P_m \times P_n \) that is formed by deleting \( k \) edges from the copies of \( P_n \). We enumerate these fences, discovering that even for quite small values of the parameters \( m, n, \) and \( k \), there is a big number of non-isomorphic fences. We close Section 2 with the introduction of a family of supergraphs of the grid \( P_m \times P_n \), called rigid grids; we establish a connection of these graphs and the fences, that allows us to determine its number as a function of \( m \) and \( n \). In Section 3 we study \( \alpha \)-labelings of grid-related graphs; we prove a general result that allows us to eliminate some edges of the grid, which originates a fence, and introduce new edges to replace the ones eliminated. This substitution of edges, done on an \( \alpha \)-labeled version of the grid, preserves the order and size of the grid and the resulting grid-like graph is also an \( \alpha \)-graph. In the last two results, we show the existence of an \( \alpha \)-labeling for two distinguishable disconnected graphs constructed using edge replacements on the grid.

In this paper, we follow the notation and terminology used in [8] and [9].

2. Subgraphs and supergraphs of the grid

2.1. The number of non-isomorphic fences

In this work, \( P_m \) denotes the path of order \( m \) with \( V(P_m) = \{v_1, v_2, \ldots, v_m\} \) and \( E(P_m) = \{v_iv_{i+1} : 1 \leq i \leq m - 1\} \). A fence is a subgraph of the grid \( P_m \times P_n \) obtained by deleting any subset of “horizontal” edges. Formally, suppose that \( P_{1m}, P_{2m}, \ldots, P_{nm} \) are disjoint copies of \( P_m \). For each \( j \in \{1, 2, \ldots, n - 1\} \), select a nonempty subset \( S_j \) of \( \{1, 2, \ldots, m\} \); a fence, of order \( m \times n \), is the graph obtained connecting \( P_{jm} \) and \( P_{j+1m} \) by drawing, for every \( i \in S_j \), an edge between the vertices \( v_{ij} \) and \( v_{i,j+1} \). Our goal is to determine the number of non-isomorphic fences of order \( m \times n \).

In Figure 1 we show several fences, classified horizontally according to the parity of \( m \) and \( n \), and vertically according the symmetries within them.

On each of these graphs, the end-vertices of \( P_{1m} \) and \( P_{nm} \) have been highlighted. Any of these end-vertices can be placed in the lower left corner of the graphical representation of the graph. Thus the graphs of type I are the only one that have four different representations. In Figure 2 we show the four representations of a graph taken from Figure 1.

Graphs of type II only have two different representations because they are symmetric with respect to a vertical axis passing through the center of the figure. The graphs of type III also have two different representations, because they possess a symmetry that corresponds to a \( 180^\circ \) rotation around their center. Graphs of type IV are symmetric with respect to a horizontal axis passing by the center of the figure, so they also have only two different representations. In Figure 3 we show the different representations for these types of symmetric graphs. Since graphs of type V have all the above symmetries, they have a unique representation.

As we mentioned before, the edges connecting consecutive copies of \( P_m \) are associated to a binary string of length \( m \), so it is natural to represent a fence of order \( m \times n \) as a 0-1 matrix of order \( m \times (n - 1) \), where the \( i \)th column of this matrix corresponds to the binary string that determines the edges in between \( P_{im} \) and \( P_{i+1m} \). Thus, to count all the non-isomorphic fences of order \( m \times n \), we count special types of 0-1 matrices of order \( m \times (n - 1) \).
For all \(1 \leq i \leq m\) and \(1 \leq j \leq n\), let \(\mathcal{Z}\) be the set of all 0-1 matrices of order \(m \times (n - 1)\), \(\mathcal{V}\) be the subset of \(\mathcal{Z}\) containing all the matrices \(V = (v_{i,j})\) such that \(v_{i,j} = v_{i,n-j}\), that is, which associated fence has the vertical symmetry (fence of type II), \(\mathcal{C}\) be the subset of \(\mathcal{Z}\) containing all the matrices \(C = (c_{i,j})\) such that \(c_{i,j} = c_{m+1-i,n-j}\), i.e., those matrices which associated fence has the central symmetry (fence of type III), and \(\mathcal{H}\) be the subset of \(\mathcal{Z}\) containing all the matrices \(H = (h_{i,j})\) such that \(h_{i,j} = h_{m+1-i,j}\), in other terms, where the associated fence has the horizontal symmetry (fence of type IV). In the following theorem we prove that the number of non-isomorphic fences of order \(m \times n\) is determined by the cardinalities of these sets.

**Theorem 2.1.** The number \(f(m, n)\) of non-isomorphic fences obtained from \(P_m \times P_n\) is given by

\[
f(m, n) = \frac{1}{4}(t + v + c + h),
\]

where \(t, v, c,\) and \(h\) are the cardinalities of \(\mathcal{Z}\), \(\mathcal{V}\), \(\mathcal{C}\), and \(\mathcal{H}\), respectively.
Proof. Suppose that $\mathcal{A}$ is the set of all the matrices $A = (a_{i,j})$ in $\mathcal{Z}$ such that $a_{i,j} = a_{m+1-i,j}$ and $a_{i,j} = a_{i,n-j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. In other terms $\mathcal{A} = \mathcal{V} \cap \mathcal{C} \cap \mathcal{H}$. Let $a = |\mathcal{A}|$ and $v' = v - a$, $c' = c - a$, and $h' = h - a$; then, $v' + c' + h'$ is the number of 0-1 matrices that appear twice in $\mathcal{Z}$. Thus, $t - v' - c' - h' - a$ is the number of matrices that appear four times in $\mathcal{Z}$. Hence, the number $f(m,n)$ of non-isomorphic fences obtained from $P_m \times P_n$ is:

$$
\begin{align*}
f(m,n) & = \frac{t - v' - c' - h' - a}{4} + \frac{v'}{2} + \frac{c'}{2} + \frac{h'}{2} + a \\
 & = \frac{t}{4} - \frac{v'}{4} - \frac{c'}{4} - \frac{h'}{4} - \frac{a}{4} + \frac{v'}{2} + \frac{c'}{2} + \frac{h'}{2} + a \\
 & = \frac{t}{4} + \frac{v'}{4} + \frac{c'}{4} + \frac{h'}{4} + \frac{3a}{4} \\
 & = \frac{1}{4}(t + v' + a + c' + a + h' + a) \\
 & = \frac{1}{4}(t + v + c + h).
\end{align*}
$$

Now we use this equation to find a closed formula for $f(m,n)$. Let $T$ be a 0-1 matrix of order $m \times (n - 1)$; each column of $T$ is a binary string of length $m$. Since there are $2^m$ different binary strings of length $m$, there are $(2^m)^{n-1}$ different possible configurations for $T$, which implies that $|\mathcal{Z}| = t = (2^m)^{n-1}$.

Let $V \in \mathcal{V}$, since for every $1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$, the $j$th and $(n - j)$th columns of $V$ are identical, we conclude that there are $(2^m)^{\left\lfloor \frac{n}{2} \right\rfloor}$ different configurations for $V$; in other terms $|\mathcal{V}| = v = (2^m)^{\left\lfloor \frac{n}{2} \right\rfloor}$.
Let $C \in \mathcal{C}$, then $C$ is formed in such a way that for every $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, the $i$th column of $C$ is the reverse of its $(n-i)$th column. Note that when $n$ is even and $i = \frac{n}{2}$, we have that $i = n - i$, this implies that the $i$th column of $C$ is a symmetric binary string. Therefore, when $n$ is even $|\mathcal{C}| = c = (2^m)^{\frac{n}{2}} \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor}$; when $n$ is odd, $c = (2^m)^{\frac{n-1}{2}}$.

If $H \in \mathcal{H}$, then $H$ is formed in such a way that every column is a symmetric binary string. Since there are $2^{\left\lfloor \frac{m}{2} \right\rfloor}$ different symmetric binary strings, we conclude that $|\mathcal{H}| = h = \left(2^{\left\lfloor \frac{m}{2} \right\rfloor}\right)^{n-1}$.

Therefore, we have a closed formula for $f(m, n)$ and have proven the following theorem.

**Theorem 2.2.** For every $m \geq 2$ and $n \geq 2$, the number of non-isomorphic fences obtained from $P_m \times P_n$ is:

(i) $f(m, n) = 2^{m(n-1)-2} + 2 \cdot \frac{m}{2} - 2 + 2 \cdot \frac{m(n-1)-2}{2}$ when $m$ is even and $n$ is even,

(ii) $f(m, n) = 2^{m(n-1)-2} + 3 \cdot 2^{\frac{m(n-1)-4}{2}}$ when $m$ is even and $n$ is odd,

(iii) $f(m, n) = 2^{m(n-1)-2} + 2 \cdot \frac{m}{2} - 2 + 2 \cdot \frac{m(n-1)-3}{2} + 2 \cdot \frac{m(n-1)-4}{2}$ when $m$ is odd and $n$ is even,

(iv) $f(m, n) = 2^{m(n-1)-2} + 2 \cdot \frac{m(n-1)-2}{2} + 2 \cdot \frac{m(n-1)+3}{4} + 2 \cdot \frac{(m+1)(n-1)-4}{2}$ when $m$ is odd and $n$ is odd.

In Table 1 we show the first values of $f(m, n)$ where both $m$ and $n$ are in $\{2, 3, \ldots, 10\}$.

In Figure 4 we present an example, exhibiting all non-isomorphic fences obtained from $P_4 \times P_3$. The fences in brown and green correspond to matrices in $\mathcal{V}$, the ones in brown and red to the matrices in $\mathcal{C}$, and the ones in brown and blue to the matrices in $\mathcal{H}$; the ones in brown have matrices in $\mathcal{A}$, exclusively.

### 2.2. The number of non-isomorphic rigid grids

Consider the paths $P_m$ and $P_n$. A cell of $P_m \times P_n$ is any of the subgraphs induced by the vertices $v_{i,j}$, $v_{i,j+1}$, $v_{i+1,j}$, $v_{i+1,j+1}$, where $1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1$. A rigid grid is a graph obtained from $P_m \times P_n$ in such a way that, on each cell, either $v_{i,j}$ is connected to $v_{i+1,j}$ or $v_{i+1,j}$ is connected to $v_{i,j+1}$. In Figure 5 we show the six non-isomorphic rigid grids obtained from $P_3 \times P_3$.

Let $F$ be any fence constructed with $n$ copies of $P_m$. We may construct a rigid grid on $P_m \times P_{n+1}$ using the information contained in the binary strings associated with the fence $F$. If the vertices $v_{i,j}$ and $v_{i,j+1}$ are connected in $F$, in $P_m \times P_{n+1}$ we connect the vertices $v_{i,j}$ and $v_{i+1,j+1}$, otherwise we connect the vertices $v_{i+1,j}$ and $v_{i,j+1}$. Thus, there is a bijection between the set of all fences associated to $P_m \times P_n$ and the set of all rigid grids associated to $P_m \times P_{n+1}$. Therefore, the number of rigid grids constructed on $P_m \times P_{n+1}$ is the same that the number of fences constructed with $n$ copies of $P_m$. In Figure 6 we show an example of this bijection where the fence is built with 5 copies of $P_6$.

**Theorem 2.3.** The number of rigid grids constructed on $P_m \times P_{n+1}$ is $f(m, n)$.
Table 1. Initial values of $f(m, n)$

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3. α-labeling of grid-like graphs

In this section we prove that any graph obtained from the grid $P_m \times P_n$, by replacing some of its edges by new edges, specifically chosen, is also an α-graph. Before showing this new construction we present two, well-known, results about bipartite labelings of the path. The following result is due to Rosa [13]. Suppose that $v_1, v_2, \ldots, v_n$ are the consecutive vertices of $P_n$.

**Proposition 1.** For every $n \geq 1$, the function given below is an α-labeling of $P_n$.

$$g(v_j) = \begin{cases} 
\frac{(j - 1)}{2}, & \text{if } j \text{ is odd;} \\
\frac{n - j}{2}, & \text{if } j \text{ is even.}
\end{cases}$$

Before the next proposition we need some more definitions. Let $G$ be a bipartite graph of size $n$ with stable sets $A$ and $B$. A bipartite labeling of $G$ is an injection $f : V(G) \rightarrow \{0, 1, \ldots, t\}$ for which there is an integer $\lambda$, named the boundary value of $f$, such that $f(u) \leq \lambda < f(v)$ for every $(u, v) \in A \times B$, that induces $n$ different weights. The labeling $g : V(G) \rightarrow \{c, c + 1, \ldots, c + t\}$,
defined for every \( v \in V(G) \) and \( c \in \mathbb{Z} \) as \( g(v) = c + f(v) \), is the shifting of \( f \) in \( c \) units. Note that this labeling preserves the weights induced by \( f \). Suppose that \( f \) is a \( \alpha \)-labeling of \( G \) with boundary value \( \lambda \); the labeling \( h : V(G) \rightarrow \{0, 1, \ldots, t + d - 1\} \), defined for every \( v \in V(G) \) and \( d \in \mathbb{Z}^+ \) as \( h(v) = f(v) \) if \( f(v) \leq \lambda \) and \( h(v) = f(v) + d - 1 \) if \( f(v) > \lambda \), is the \( d \)-graceful labeling of \( G \) obtained from \( f \). The labels used by \( h \) are in the set \( \{0, 1, \ldots, \lambda\} \cup \{\lambda + d, \lambda + d + 1, \ldots, t + d - 1\} \) and the set of induced weight is \( \{d, d + 1, \ldots, t + d - 1\} \).

**Proposition 2.** For every \( n \geq 1 \), there exists a bipartite labeling of \( P_n \) where the set of induced weights is \( \{d, d + 1, \ldots, d + n - 2\} \).

**Proof.** Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \) and \( E(P_n) = \{v_jv_{j+1} : 1 \leq j \leq n-1\} \). Let \( f : V(P_n) \rightarrow \mathbb{N} \) defined as:

\[
    f(v_j) = \begin{cases} 
    k + (j - 1)/2, & \text{if } j \text{ is odd;} \\
    d + k + n - (j + 2)/2, & \text{if } j \text{ is even.}
    \end{cases}
\]

Note that \( d + k + n - (j + 2)/2 = k + (d - 1) + n - j/2 \). Thus, \( f \) is a shifting in \( k \) units of
the \( d \)-graceful labeling of \( P_n \) obtained from the \( \alpha \)-labeling \( g \) in Proposition 1. Therefore, the set of induced weights is \{ \( d, d+1, \ldots, d+m-2 \} \).

**Proposition 3.** Let \( 2 \leq j \leq m-2 \) and \( x \leq \min\{ j-1, m-j-1 \} \). An \( \alpha \)-graph is obtained when the edge \( v_jv_{j+1} \) of \( P_m \) is replaced by the edge \( v_{j-x}v_{j+1+x} \).

**Proof.** Suppose that \( P_m \) has been labeled using the function \( f \) in Proposition 2.

If \( j \) is odd, then \( f(v_j) = k + (j - 1)/2 \) and \( f(v_{j+1}) = d + k + m - (j + 3)/2 \). So, the edge \( v_jv_{j+1} \) has weight \( f(v_{j+1}) - f(v_j) = d + k + m - (j + 3)/2 - k - (j - 1)/2 = d_m - j - 1 \).

When \( x \) is even,

\[
f(v_{j-x}) = k + (j - x - 1)/2 \\
= f(v_j) - x/2
\]

and

\[
f(v_{j+1+x}) = d + k + m - (j + 1 + x + 2) \\
= d + k + m - (j + 3) - x/2 \\
= f(v_{j+1}) - x/2.
\]

Thus,

\[
f(v_{j+1+x}) - f(v_{j-x}) = f(v_{j+1}) - x/2 - f(v_j) + x/2 \\
= f(v_{j+1}) - f(v_j).
\]

When \( x \) is odd,

\[
f(v_{j-x}) = d + k + m - (j - x + 2)/2 \\
= f(v_{j+1}) + (x + 1)/2
\]

and

\[
f(v_{j+1+x}) = k + (j + 1 + x - 1)/2 \\
= k + (j - 1)/2 + (x + 1)/2 \\
= f(v_j) + (x + 1)/2.
\]

Thus,

\[
f(v_{j-x}) - f(v_{j+1+x}) = f(v_{j+1}) + (x + 1)/2 - f(v_j) - (x + 1)/2 \\
= f(v_{j+1}) - f(v_j).
\]

If \( j \) is even, then \( f(v_j) = d + k + m - (j + 2)/2 \) and \( f(v_{j+1}) = k + j/2 \). So, the edge \( v_jv_{j+1} \) has weight \( f(v_j) - f(v_{j+1}) = d + k + m - (j + 2)/2 - k - j/2 = d + m - j - 1 \).
When $x$ is even,

$$f(v_{j-x}) = d + k + m - (j - x + 2)/2$$
$$= d + k + m - (j + 2) + x/2$$
$$= f(v_j) + x/2$$

and

$$f(v_{j+1+x}) = k + (j + 1 - x - 1)/2$$
$$= k + j/2 + x/2$$
$$= f(v_{j+1}) + x/2.$$

Hence,

$$f(v_{j-x}) - f(v_{j+1+x}) = f(v_j) + x/2 - f(v_{j+1}) - x/2$$
$$= f(v_j) - f(v_{j+1}).$$

When $x$ is odd,

$$f(v_{j-x}) = k + (j - x - 1)/2$$
$$= f(v_{j+1})$$
$$= (x + 1)/2$$

and

$$f(v_{j+1+x}) = d + k + m - (j + 1 + x + 2)$$
$$= f(v_j) - (x + 1)/2.$$

Thus,

$$f(v_{j+1+x}) - f(v_{j-x}) = f(v_j) - (x + 1)/2 - f(v_{j+1}) + (x + 1)/2$$
$$= f(v_j) - f(v_{j+1}).$$

Therefore, independently of the parity of $j$ and $x$, the edges $v_jv_{j+1}$ and $v_{j-x}v_{j+1+x}$ have the same weight. Whence, if the edge $v_jv_{j+1}$ of $P_m$ is replaced by the new edge $v_{j-x}v_{j+1+x}$, the emerging graph is an $\alpha$-graph.

Now we turn our attention to the grid $P_m \times P_n$ and an $\alpha$-labeling of it. The grid $G = P_m \times P_n$ is a bipartite graph of order $mn$ and size $2mn - (m+n)$. If $V(G) = \{v_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$, then $E(G) = \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n - 1\} \cup \{v_{i,j}v_{i+1,j} : 1 \leq i \leq m - 1 \text{ and } 1 \leq j \leq n\}$.

Jungreis and Reid [10] proved that $G$ is an $\alpha$-graph for all the positive integers $m$ and $n$. We show below an $\alpha$-labeling of $G$. 

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Let

\[ f(v_{i,j}) = \begin{cases} 
(2n - 1)(i - 1)/2 + (j - 1)/2, & i \text{ odd and } j \text{ odd;} \\
(2n - 1)(2m - 1 - i)/2 + (n - 1) - (j - 1)/2, & i \text{ odd and } j \text{ even;} \\
(2n - 1)(2m - i)/2 + (j - 1)/2, & i \text{ even and } j \text{ odd;} \\
(2n - 1)i/2 - (n - 1) + (j - 1)/2, & i \text{ even and } j \text{ even.}
\end{cases} \]

Note that when \( i \) is odd (even), the sequence of consecutive integers formed by the labels \( f(v_{i,j}) \) is increasing for the odd (even) values of \( j \) and decreasing for the even (odd) values. That is, the labeling of the \( i \)th copy of \( P_m \) in \( G \) follows the same pattern than the labelings in propositions 2 and 3. That means that on any number of copies of \( P_m \) in \( G \) and for multiple values of \( j \), the edge \( v_{i,j}v_{i,j+1} \) can be replaced by the edge \( v_{i,j-x}v_{j+1+x} \) and the resulting graph is an \( \alpha \)-graph.

The process of replacing in \( G \) the edge \( v_{i,j}v_{i,j+1} \) by the new edge \( v_{i,j-x}v_{i,j+1+x} \) is called an elementary transformation. A graph \( H \) is said to be a grid-like graph if it is obtained from \( G \) via a sequence of elementary transformations. We claim that if \( H \) is a grid-like graph, then \( H \) is an \( \alpha \)-graph.

**Theorem 3.1.** If \( H \) is a grid-like graph, then \( H \) is an \( \alpha \)-graph.

**Proof.** Suppose \( H \) is a grid-like graph obtained from \( G = P_m \times P_n \). For any edge \( v_{i,j-x}v_{i,j+1+x} \) in \( H \), there exists an edge \( v_{i,j}v_{i,j+1} \) in the \( i \)th copy of \( P_m \) in \( G \). Since we can match every edge of \( H \) with an edge of \( G \) and the function \( f \) described before is an \( \alpha \)-labeling of \( G \), we have an \( \alpha \)-labeling of \( H \).

In Figure 7 we show the \( \alpha \)-labelings of all the graphs \( H \) obtained from \( P_5 \times P_2 \).

This theorem can be used to prove the following corollaries and to find, eventually, \( \alpha \)-labelings for many other families of graphs in an easy way.

**Corollary 1.** For positive integers \( t \) and \( n \), \( C_{4t} \times P_n \cup P_n \) is an \( \alpha \)-graph.
Proof. We need to prove that $C_{4t}$ can be obtained from $P_{4t+1}$ via a sequence of elementary transformations. Thus, suppose that $v_1, v_2, \ldots, v_{4t+1}$ are the consecutive vertices of $P_{4t+1}$. The edge $v_{2t+1}v_{2t+2}$ is replaced with $v_2 v_{4t+1}$. For every $j \in \{1, 2, \ldots, t\}$, the edge $v_{2j}v_{2j+1}$ is replaced with $v_{2j-1}v_{2j+2}$. Note that in the first transformation $x = 2t - 1$ and in all the remaining ones, $x = 1$. Moreover, both extreme vertices of $P_{4t+1}$ have now degree 2. In order to see that the resulting graph is actually $C_{4t+1} \cup P_1$ we must observe that now $v_1, v_4, v_3, v_6, v_5, v_8, v_7, \ldots, v_{2t-1}, v_{2t+2}$ are consecutive vertices. This implies that every vertex has degree 2 except $v_{2t+1}$ that has degree 0. Since the edges $v_{2t+2}v_{2t+3}, v_{2t+3}v_{2t+4}, \ldots, v_{4t}v_{4t+1}$ have not been touched, and $v_{2t+2}$ is connected to $v_{2t-1}$ as well as $v_2$ is connected to $v_{4t+1}$, we can see that our claim is true. Therefore, if these transformations are applied to every copy of $P_{4t+1}$ within $P_{4t+1} \times P_n$, the resulting graph is in fact $C_{4t} \times P_n \cup P_n$.

In Figure 8 we show an example of this result for $t = 3$ and $n = 4$.

Figure 8. $\alpha$-labeling of $C_{13} \times P_4 \cup P_4$

Figure 9. $\alpha$-labeling of $C_6 \times P_4 \cup P_2 \times P_4
Corollary 2. For every \( t \geq 2 \) and \( n \geq 1 \), the graph \( C_{2t} \times P_n \cup P_{t-1} \times P_n \) is an \( \alpha \)-graph.

Proof. Let \( t \geq 2 \) be an integer. Suppose that \( v_1, v_2, \ldots, v_{3t-1} \) are the consecutive vertices of \( P_{3t-1} \) and that they have been labeled using Proposition 1, in such a way that \( v_1 \) receives the label 0. If for every \( 2 \leq i \leq 3t - 3 \), the edge \( v_iv_{i+1} \) is replaced by the edge \( v_{i-1}v_{i+2} \), then the paths \( \langle v_1, v_4, \ldots, v_{3t-2} \rangle, \langle v_2, v_5, \ldots, v_{3t-1} \rangle, \) and \( \langle v_3, v_6, \ldots, v_{3t-3} \rangle \) are mutually disjoint.

Considering that the edges \( v_2v_3 \) and \( v_{3t-3}v_{3t-2} \) have been replaced, the vertices \( v_3 \) and \( v_{3t-3} \) have now degree one. In addition, the edges \( v_1v_2 \) and \( v_{3t-2}v_{3t-1} \) have not been replaced, which implies that all the vertices, except \( v_3 \) and \( v_{3t-3} \), have degree two and the graph induced by the vertices in \( \{ v_i : i \not\equiv 0(\text{mod } 3) \} \) is a cycle of length \( 2t \) with consecutive vertices \( v_1, v_4, \ldots, v_{3t-2}, v_{3t-1}, \ldots, v_5, v_2, v_1 \).

Since the \( \alpha \)-labeling of \( P_{3t-1} \) has not been affected by these substitutions, the resulting graph, \( C_{2t} \cup P_{t-1} \) is an \( \alpha \)-graph. When the same substitutions are made on each copy of \( P_{3t-1} \) in \( P_{3t-1} \times P_n \), we obtain an \( \alpha \)-labeling of the graph \( C_{2t} \times P_n \cup P_{t-1} \times P_n \).

In Figure 9 we show an example of this construction for the case \( t = 3 \) and \( n = 4 \).

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References


