The rainbow $k$-connectivity of the non-commutative graph of a finite group

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Abstract

The non-commuting graph $\Gamma(G)$ of a non–abelian group $G$ is defined as follows. The vertex set $V(\Gamma(G))$ of $\Gamma(G)$ is $G \setminus Z(G)$ where $Z(G)$ denotes the center of $G$ and two vertices $x$ and $y$ are adjacent if and only if $xy \neq yx$. We prove that the rainbow $k$-connectivity of $\Gamma(G)$ is equal to $\lceil k/2 \rceil + 2$, for $3 \leq k \leq |Z(G)|$.

Keywords: non-commuting graph, non-abelian group, rainbow connectivity, rainbow path

AMS Mathematics Subject Classification: 05C15, 05C25, 05C38

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1. Introduction

Let $G$ be a group and $Z(G)$ be the center of $G$. The non-commuting graph $\Gamma(G)$ associated to $G$ is the graph with vertex set $G \setminus Z(G)$ and such that two vertices $x$ and $y$ are adjacent whenever $xy \neq yx$. The non-commuting graph of a group was first considered by Paul Erdős in 1975, [6]. Subsequently, it was strongly developed in [1].

Let $\Gamma$ be a connected graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Define a coloring $\varphi : E(\Gamma) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored the same. Given an edge coloring of $\Gamma$, a path $P$ is rainbow if no two edges of $P$ are colored the same. An edge-colored
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A graph $\Gamma$ is rainbow connected if every pair of vertices of $\Gamma$ are connected by a rainbow. The rainbow connection number $rc_1(\Gamma)$ of $\Gamma$ is defined to be the minimum integer $t$ such that there exists an edge-coloring of $\Gamma$ with $t$ colors that makes $\Gamma$ rainbow connected.

From a generalization given by Chartrand, Johns, McKeon and Zhang in 2009 [2], an edge-colored graph $\Gamma$ is called rainbow $k$–connected if any two distinct vertices of $\Gamma$ are connected by at least $k$ internally disjoint rainbow paths. The rainbow $k$–connectivity of $\Gamma$, denoted by $rc_k(\Gamma)$, is the minimum number of colors required to color the edges of $\Gamma$ to make it rainbow $k$–connected, and $\varphi$ is called a rainbow $k$–coloring of $\Gamma$. We usually denote $rc_1(\Gamma)$ by $rc(\Gamma)$.

The importance of rainbow connection number emerge from applications to the secure transfer of classified information between agencies [2]. Recently, Septyanto in [8], showed another form to see the application.

The commutator of an ordered pair $g_1, g_2$ of elements of $G$ is the element

$$[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 \in G$$

$G$ is abelian if and only if $[g_1, g_2] = 1$

Let $G(V, E)$, and let $a = (e_1, ..., e_j)$ be a path with $e_i \in E$. Then $l(a) := j$ is called the length of $a$.

We denote by $P(x, y)$ the set of all $x, y$ paths in $G$. Then $d(x, y) := \min\{l(a) | a \in P(x, y)\}$ is called the distance from $x$ to $y$.

We call $diam(G) := \max\{d(x, y) | x, y \in G\}$ the diameter of $G$. The length of a shortest cycle of $G$ is called the girth of $G$.

When a pair of vertices $g_i, g_j$ are joined, we denoted by $g_i \sim g_j$. In otherwise we denoted by $g_i \nsim g_j$.

A non–commutative graph $\Gamma(G)$ is connected and the diameter of $\Gamma(G)$ is 2, $diam(\Gamma(G)) = 2$.

**Theorem 1.1.** [1] For any non–abelian group $G$, $diam(\Gamma(G)) = 2$. In particular, $\Gamma(G)$ is connected.

In [9], it is shown that $rc(\Gamma(G)) = rc_2(\Gamma(G)) = 2$.

**Theorem 1.2.** [9] Let $G$ be a finite non-abelian group. Then $rc(\Gamma(G)) = rc_2(\Gamma(G)) = 2$.

In the present article, we estimate $rc_k(\Gamma(G))$ for $3 \leq k \leq |Z(G)|$. Our main result is the following theorem.

**Theorem 1.3.** Let $G$ be a finite non-abelian group. Then $rc_k(\Gamma(G)) \leq k$, for $3 \leq k \leq |Z(G)|$ with $|Z(G)| \geq 3$. Specifically $rc_k(\Gamma(G)) = \lceil \frac{k}{2} \rceil + 2$. 

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2. $rc_k(\Gamma(G))$ with $1 \leq k \leq |Z(G)|$

Let $G$ be a finite non-abelian group, from now on we write the vertices of $\Gamma(G)$ as the partition

$$V(\Gamma(G)) = g_1Z \cup g_2Z \cup \cdots \cup g_mZ,$$

with $Z = Z(G)$, $g_iZ \neq Z$, $m = [G : Z(G)] - 1$ and where $g_iZ$ is an independent subset of $\Gamma(G)$.

**Proposition 2.1.** Let $G$ be a finite non-abelian group. Then the $m$–partite graph $\Gamma(G)$ with partition $V(\Gamma(G)) = g_1Z \cup g_2Z \cup \cdots \cup g_mZ$, provides an adjacency by blocks.

**Proof.** Observe that every pair of vertices $g_i \sim g_j$, if and only if for all $x, y \in Z$ $g_i x \sim g_j y$. In addition, for each $i$, the vertex $g \in V(\Gamma(G))$ is adjacent to $g_i$ if and only if it is adjacent to every element of the set $g_iZ$. In other words, it is an adjacency by blocks. \hfill \Box

**Definition 2.2.** Let $G$ be a non-commutative finite group, with $m$–partition

$$V(\Gamma(G)) = g_1Z \cup g_2Z \cup \cdots \cup g_mZ$$

adjacency by blocks. We define the skeleton of the $m$–partition as the subgraph induced by $M = \{g_1, g_2, \ldots, g_m\}$. The skeleton is denoted by $S^M_{\Gamma(G)}$.

**Remark 2.3.** The graph $\Gamma(G)$ is not complete, however $S^M_{\Gamma(G)}$ can be complete, we can see this in the follow example: Let $G = D_{2 \times 4} := \langle a, x : a^4 = x^2 = 1, xax = a^{-1} \rangle$, the dihedral group of order 8. Then $Z := Z(G) = \{1, a^2\}$, and we have

$$V(\Gamma(G)) = aZ \cup xZ \cup axZ.$$ 

Since each pair of $\{a, x, ax\}$ do not commute, we have $S^M_{\Gamma(D_{2\times4})}$ is complete.

By Theorem 1.2, there is a coloration

$$\varphi : E(\Gamma(G)) \to \{1, 2\}$$

such that $rc(\Gamma) = rc_2(\Gamma) = 2$. Thus, the graph $\Gamma(G)$ is not complete, implies that $\varphi(E(S^M_{\Gamma(G)})) = \{1, 2\}$. Therefore, the coloration

$$\phi := \varphi |_{E(S^M_{\Gamma(G)})} : E(S^M_{\Gamma(G)}) \to \{1, 2\}$$

meets the 2–connectivity, that is to say, $rc(S^M_{\Gamma(G)}) \leq 2$. Consider $Z(G) = \{e = z_1, z_2, z_3, \ldots, z_s\}$ and define the following coloring of $\Gamma(G)$:

$$\psi : E(\Gamma(G)) \to \{1, 2\}$$

given by

$$\psi(\{g_i z_p, g_j z_p\}) = \phi(\{g_i, g_j\})$$

for $1 \leq i, j, p \leq m; i \neq j$;

$$\psi(\{g_i z_p, g_j z_q\}) \neq \phi(\{g_i, g_j\})$$

for $1 \leq i, j, p, q \leq m; i \neq j; p \neq q$.

In the next section we give a coloring for $3 \leq k \leq s$ with $p \neq q$. Moreover in section 6 we will proof that this coloring works.
3. About edge-connectivity

We need to find $k$-rainbow paths between any two vertices for $\Gamma(G)$, with $k \geq 3$. We may ask for the maximum number of paths from $v_1$ to $v_2$ vertices, no two of which have an edge in common (such paths are called edge-disjoint paths). As a consequence of Menger’s theorem about max-flow and min-cut, Witney [10] presented that a graph is $k$-connected if and only if any two vertices are connected by $k$ internally disjoint paths. With Whitney’s result we can answer how many edge-disjoint paths are connecting a given pair of vertices on $\Gamma(G)$.

**Definition 3.1.** The edge-connectivity is the minimum size of a subset $C \subseteq E(G)$ for which $G - C$ is not connected for a graph $G$. The edge-connectivity of $G$ is denoted by $\lambda(G)$. If $\lambda(G) \geq k$ then $G$ is called $k$-edge connected.

The next theorem is a result implied by Menger’s theorem. This form can be found in [7, Chapter 15].

**Theorem 3.2.** An undirected graph $G = (V, E)$ is $k$-edge-connected if and only if there exist $k$ edge-disjoint paths between any two vertices $s$ and $t$.

As we can obtain the rainbow-connectivity number of $\Gamma(G)$ and this graph is connected by blocks with $s = |Z(G)|$ as size of each block, we have that the graph $\Gamma(G)$ is $s$-edge-connected and there exist $s$ edge-disjoint paths in $\Gamma(G)$. Then, our problem now is coloring the $s$ edge-disjoint paths of $\Gamma(G)$.

**Remark 3.3.** By 1.1 we note that there exist two cases that we need analyze, for $g_i, g_j, g_k, g_l \in S^M_{\Gamma(G)}$ and $z_r, z_t, z_w, z_p \in Z(G)$. The first case is when $g_i z_r \sim g_j z_t$ which give us a bipartite complete graph in $\Gamma(G)$. The second case is when we have $g_i z_r \sim g_j z_t \sim g_k z_w$, but $g_i z_r \not\sim g_k z_w$.

**Remark 3.4.** We note that $\lambda(G) \geq s$. Then, if we want a path between end vertices $g_i z_r$ and $g_j z_t$, without loss of generality we start with $g_i z_r$, necessarily, from 3.2, the edges $g_i z_r \sim g_j z_{t_b}$ with $t_b \in \{1, \ldots, s\}$, are in the set of edge-disjoint paths. The same happens for the edges $g_i z_{r_a} \sim g_j z_t$ with $r_a \in \{1, \ldots, s\}$ because we have $s$ disjoint paths, therefore we need all out-edge from $g_i z_r$, and all in-edge to $g_j z_t$, thus all our edge-disjoint paths have the following form: $(g_i z_r, g_j z_{t_b}, \ldots, g_i z_{r_a}, g_j z_t)$, with $t_a, r_b \in \{1, \ldots, s\}$.

4. Rainbow $k$–connectivity

4.1. Case when $g_i \sim g_j \in V(S^M_{\Gamma(G)})$

Let $s = |Z(G)|$ and let $\bar{r} \equiv r \mod s$ with $1 \leq r \leq s$. If $g_i \sim g_j \in V(S^M_{\Gamma(G)})$, then the set of edges is given by
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\[ E_1 = \{ e \in E(\Gamma(G)) \mid g_i z_r \sim g_j z_r \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 1 \} \cup \{ e \in E(\Gamma(G)) \mid g_i z_r \sim g_j z_{r+1} \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 2 \text{ with } 1 \leq i, j, p \leq m; i \neq j \} \]

\[ E_2 = \{ e \in E(\Gamma(G)) \mid g_i z_r \sim g_j z_s \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 2 \} \cup \{ e \in E(\Gamma(G)) \mid g_i z_r \sim g_j z_{r+n-1} \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 1 \text{ with } 1 \leq i, j, p \leq m; i \neq j \} \]

\[ E_3 = \{ e \in E(\Gamma(G)) \mid g_i z_r \sim g_j z_{r+2} \} \]

\[ E_n = \{ e \in E(\Gamma(G)) \mid g_i z_r \sim g_j z_{r+n-1} \} \]

\[ E_{n+1} = \{ e \in E(\Gamma(G)) \mid g_i z_r \sim g_j z_{r+n} \} \]

\[ E_{n+2} = E(\Gamma(G)) \setminus (E_1 \cup \cdots \cup E_{n+1}) \]

with \( n = \lfloor \frac{k}{2} \rfloor \). The coloring given by:

\[
\psi : E(\Gamma(G)) \rightarrow \{1, \ldots, n + 2\} \\
f \mapsto i \quad \text{if } f \in E_i
\]

For an easier study of this kind of graph we use a table called \textit{rainbow table}, whose entries \((r_a, t_b)\) are the color from edge \((g_i z_{r_a}, g_j z_{t_b})\). This table is the following form:

\[
\begin{array}{cccccccc}
\quad & g_i z_1 & g_i z_2 & g_i z_3 & \ldots & g_i z_n & g_i z_{n+1} & g_i z_{n+2} & \ldots & g_i z_s \\
\hline
\quad & 1 & 2 & 3 & \cdots & n & n+1 & & & \\
g_i z_2 & 1 & 2 & \cdots & n-1 & n & n+1 & & & \\
g_i z_3 & 1 & \cdots & n-2 & n-1 & n & \cdots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
g_i z_n & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
g_i z_{n+1} & n+1 & 1 & 2 & 3 & \cdots & n+1 & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
g_i z_s & 2 & 3 & 4 & \cdots & n+1 & & & & & 1 \\
\end{array}
\]

Case \( g_i \sim g_j \) in \( S^M_{\Gamma(G)} \), \( s = |Z(G)| \) and \( n = \lfloor \frac{k}{2} \rfloor \).

The \((n + 2)\)-color in the table is given by white space.

\[ 4.2. \text{ Case when } g_i \sim g_j \sim g_l \text{ but } g_i \sim g_l \text{ in } S^M_{\Gamma(G)} \]

Let \( s = |Z(G)| \) and let \( \bar{r} \equiv r \mod s \) with \( 1 \leq r \leq s \). If \( g_i \sim g_j \in V(S^M_{\Gamma(G)}) \), then the set of edges is given by
E_1 = \{ e \in E(\Gamma(G)) | g_i z_r \sim g_j z_r \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 1 \} \cup \\
\{ e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{r+1} \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 2 \text{ with } \\
1 \leq i, j, p \leq m; i \neq j \}

E_2 = \{ e \in E(\Gamma(G)) | g_i z_r \sim g_j z_r \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 2 \} \cup \\
\{ e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{r+1} \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 1 \text{ with } \\
1 \leq i, j, p \leq m; i \neq j \}

E_3 = \{ e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{r+2} \}

\vdots \vdots 

E_{n+1} = \{ e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{r+n-1} \}

E_{n+2} = E(\Gamma(G)) \setminus (E_1 \cup \cdots \cup E_{n+1})

with \( n = \left\lceil \frac{k}{2} \right\rceil \). The coloring given by:

\[
\psi : E(\Gamma(G)) \longrightarrow \{1, \ldots, n + 2\} \\
f \mapsto i \quad \text{if } f \in E_i
\]

This gives us a table as:

\[
\begin{array}{cccccccccccc}
& g_1 z_1 & g_1 z_2 & \cdots & g_1 z_n & g_1 z_{n+1} & \cdots & g_1 z_s \\
g_j z_1 & 1 & n + 1 & n & \cdots & 2 \\
g_j z_2 & 2 & 1 & n + 1 & \cdots & 3 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
g_j z_{n-1} & n - 1 & n - 2 & \cdots & \cdots & n \\
g_j z_n & n & n - 1 & \cdots & 1 & n + 1 \\
g_j z_{n+1} & n + 1 & n & \cdots & \vdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
g_j z_s & n & n - 1 & \cdots & 1 & 1 & 3 & \cdots & n & n + 1 & 2 \\
\end{array}
\]

Case when \( g_i \sim g_j \sim g_l \) but \( g_i \not\sim g_l \) in \( S^M_{\Gamma(G)} \) with \( n = \left\lceil \frac{k}{2} \right\rceil \) and \( (n + 2) \)-color with white spaces.

5. How to build the rainbow table

Example 5.1. We give the case when \( s = 6 \) and \( g_1 \sim g_2 \) in \( S^M_{\Gamma(G)} \) with the coloring assigned before. Without loss of generality suppose that \( \psi(\{g_1 z_p, g_2 z_p\}) = 1 \), then the rainbow table is given by:
We can see that there is not exist a rainbow \( k \)-connectivity with 4 colors. To give \( s \) edge-disjoint paths with ends vertices \( g_1 z_2 \) and \( g_2 z_4 \), the first path cross above \( g_2 z_1 \), then we start the path with \( g_1 z_2 \) \( \sim \) \( g_2 z_1 \). Now, we need move from \( g_2 z_1 \) but our only options are \( g_2 z_1 \) \( \sim \) \( g_1 z_1 \), \( g_2 z_1 \) \( \sim \) \( g_1 z_5 \) and \( g_2 z_1 \) \( \sim \) \( g_1 z_6 \) and these edges can not arrive to \( g_2 z_4 \) because all the in-edge repeat color 4. For this reason we need to ensure that there exist enough in-edge that cover complete the out-edge in the set edges with majority color. For the existence of all edge-disjoint paths for any vertex we need to add one color more, and the table is given by

\[
\begin{array}{cccccc}
  g_2 z_1 & g_2 z_2 & g_2 z_3 & g_2 z_4 & g_2 z_5 & g_2 z_6 \\
  g_1 z_1 & 1 & 2 & 3 & 4 & \\
  g_1 z_2 & 1 & 2 & 3 & 4 & \\
  g_1 z_3 & 4 & 1 & 2 & 3 & \\
  g_1 z_4 & 3 & 4 & 1 & 2 & \\
  g_1 z_6 & 2 & 3 & 4 & 1 & \\
\end{array}
\]

**Example 5.2.** We will do an example step-by-step about how we found all the edge-disjoint paths with our table. Let \( g_1 \sim g_2 \) in \( S_M^{\Gamma(G)} \) and \( |Z(G)| = 4 \). Then, we will build our rainbow table with 3 colors the following form.

\[
\begin{array}{cccc}
  g_2 z_1 & g_2 z_2 & g_2 z_3 & g_2 z_4 \\
  g_1 z_1 & 1 & 2 & \\
  g_1 z_2 & 1 & 2 & \\
  g_1 z_3 & 1 & 2 & \\
  g_1 z_4 & 2 & 1 & \\
\end{array}
\]

From this table we can found \( rc_3(\Gamma(G)) = 3 \) for any vertices. For example, for end vertices \( g_1 z_3, g_2 z_4 \)

If we note, we can not find 4 edge-disjoint paths with 3 colors, because \( g_1 z_1 \) to \( g_2 z_1 \) passes through \( g_2 z_3 \), the paths are the followings: \( g_1 z_1 \) \( \sim \) \( g_2 z_1 \) \( \sim \) \( g_2 z_2 \) \( \sim \) \( g_2 z_3 \) \( \sim \) \( g_2 z_1 \) or \( g_1 z_1 \) \( \sim \) \( g_2 z_3 \) \( \sim \) \( g_1 z_3 \) \( \sim \) \( g_2 z_1 \). Then, we need add another color, then the table is 4 colors the following form:

\[
\begin{array}{cccc}
  g_2 z_1 & g_2 z_2 & g_2 z_3 & g_2 z_4 \\
  g_1 z_1 & 1 & 2 & 3 \\
  g_1 z_2 & 1 & 2 & 3 \\
  g_1 z_3 & 3 & 1 & 2 \\
  g_1 z_4 & 2 & 3 & 1 \\
\end{array}
\]
Then, with all this 4 colors we found all 4 edge-disjoint paths from $g_1z_1$ to $g_2z_1$, and they are the followings:

1-path: $g_1z_1 \sim g_2z_1$
2-path: $g_1z_1 \sim g_2z_2 \sim g_1z_2 \sim g_2z_1$
3-path: $g_1z_1 \sim g_2z_3 \sim g_1z_4 \sim g_2z_1$
4-path: $g_1z_1 \sim g_2z_4 \sim g_1z_3 \sim g_2z_1$

and the same is true for any pair of vertices.

6. Proofs

6.1. Case 3-partite with $|Z(G)| = 3$

The coloring given before can not help us to find all the disjoint-edge paths for the case when $g_i \sim g_j \sim g_l$ but $g_i \not\sim g_l$ in $S^M_{\Gamma(G)}$, for example, the rainbow table for this case is the next

$$
g_j z_1 \begin{bmatrix}
g_i z_1 & g_i z_2 & g_i z_3 & g_i z_1 & g_i z_2 & g_i z_3 \\
g_j z_2 & 1 & 2 & 2 & 1 \\
g_j z_3 & 2 & 1 & 1 & 2
\end{bmatrix}
$$

But, we can see that for go from $g_i z_1$ to $g_l z_2$ we have same colors then, we need to do paths with length at least 4 like the following picture:

The coloring given for this specifical case is the following: The rainbow tables for each case are the following:
The following are all the Proof.

**Theorem 6.1.** Let $G$ be a non–abelian group with $|Z(G)| = 3$ and $\Gamma(G)$ be the non-commutative graph associated to $G$, then $\text{rc}_3(\Gamma(G)) = 4$.

**Proof.** Let the set of edges be the following form:

- $E_1 = \{e \in E(\Gamma(G))|g_{i}z_{k_\nu} \sim g_{j}z_1 \text{ such that } \psi\{\{g_i, g_j\}\} = 1 \text{ for } g_i, g_j \in S^{M}_{\Gamma(G)} \text{ and } k_\nu = 1, 2, 3\}$
- $\bigcup\{e \in E(\Gamma(G))|g_{i}z_{k_\nu} \sim g_{j}z_2, g_{j}z_3 \sim g_{i}z_1 \text{ such that } \psi\{\{g_i, g_j\}\} = 2 \text{ for } g_i, g_j \in S^{M}_{\Gamma(G)}\}$
- $E_2 = \{e \in E(\Gamma(G))|g_{i}z_{k_\nu} \sim g_{j}z_2 \text{ such that } \psi\{\{g_i, g_j\}\} = 1 \text{ for } g_i, g_j \in S^{M}_{\Gamma(G)} \text{ and } k_\nu = 1, 2, 3\}$
- $\bigcup\{e \in E(\Gamma(G))|g_{i}z_{j_\nu} \sim g_{j}z_{a} \text{ such that } \psi\{\{g_i, g_j\}\} = 2 \text{ for } g_i, g_j \in S^{M}_{\Gamma(G)} \text{ and } j_\nu = 1, 3\}$
- $E_3 = \{e \in E(\Gamma(G))|g_{i}z_3 \sim g_{j}z_3 \text{ such that } \psi\{\{g_i, g_j\}\} = 1 \text{ for } g_i, g_j \in S^{M}_{\Gamma(G)} \text{ and } k_\nu = 1, 2, 3\}$
- $\bigcup\{e \in E(\Gamma(G))|g_{i}z_2 \sim g_{j}z_2, g_{j}z_3 \sim g_{i}z_3 \text{ such that } \psi\{\{g_i, g_j\}\} = 2 \text{ for } g_i, g_j \in S^{M}_{\Gamma(G)}\}$

And the coloring is given by

$$\psi : E(\Gamma(G)) \rightarrow \{1, 2, 3, 4\}$$

$$f \mapsto i \text{ if } i \in E_i.$$

The following are all the 3 edge-disjoint paths for each pair of vertices when $\phi\{\{g_j, g_l\}\} = 2$

<table>
<thead>
<tr>
<th>$g_j z_1$</th>
<th>$g_l z_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_j z_1$</td>
<td>$g_l z_3$</td>
</tr>
<tr>
<td>$g_j z_2$</td>
<td>$g_l z_2$</td>
</tr>
<tr>
<td>$g_j z_3$</td>
<td>$g_l z_3$</td>
</tr>
</tbody>
</table>

All the edge-disjoint paths when $\phi\{\{g_i, g_j\}\} = 2, \phi\{\{g_j, g_l\}\} = 2$ and $g_i \sim g_j \sim g_l$ but $g_i \sim g_l$

<table>
<thead>
<tr>
<th>$g_i z_1$</th>
<th>$g_j z_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_i z_1$</td>
<td>$g_j z_2$</td>
</tr>
<tr>
<td>$g_i z_1$</td>
<td>$g_j z_3$</td>
</tr>
</tbody>
</table>

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Theorem 6.2. Let $G$ be a finite non-abelian group. Then $\text{rc}_k(\Gamma(G)) \leq \left\lceil \frac{k}{2} \right\rceil + 2$, for $3 \leq k \leq s = |Z(G)|$ with $|Z(G)| \geq 4$.

Proof. We will prove that 4 is a coloring works for our graph.

1. **Case $g_i \sim g_j$** Let $g_i z_{i_a}, g_j z_{j_b}$ be the end vertices. We want to find the edge-disjoint paths between them. Let 4.1 the rainbow table assigned for this case. From 4.1 it is evident that the first path is given by $g_i z_{i_a} \sim g_j z_{j_b}$ with color $(i_a, j_b)$.

Let $j_1$ be the column assigned to the row $i_a$ such that $(i_a, j_1) = f_1$ then, we remove the entries with color $f_1$ to the column $g_j z_{j_1}$ and, the same happen to column $g_j z_{j_b}$.

**Remark 6.3.** When we say *remove the entry* we say that entry is not consider to form the rainbow path.

Thus, the path for this case is

$$g_i z_{i_a} \overset{f}{\sim} g_j z_{j_1} \overset{(i_{a_1}, j_{1_1})}{\sim} g_i z_{i_{a_1}} \overset{(i_{a_1}, j_{b_1})}{\sim} g_j z_{j_{b_1}}$$

(1)
with \((i_{a_1}, j_1) \neq f_1 \neq (i_{a_1}, j_0)\) the colors assigned to remaining entries and \(g_j z_{j_1}, g_i z_{i_{a_1}}\) the respective vertices from remaining entries.

Let \((i_a, j_2)\) be the entry with \(j_2 \neq j_1\), such that \((i_a, j_2) = f_2\) then, we remove the entries with same color as \(f_2\) in column \(g_j z_{j_2}\). We can not use the entry \((g_i z_{a_1}, g_j z_{j_b})\) because is an edge for 1, moreover we remove all the entries with same color as \(f_2\) in column \(g_j z_{j_b}\). Thus, the path is the following:

\[
g_i z_{i_a} (i_{a_2}, j_2) \sim g_j z_{j_2} (i_{a_2}, j_2) \sim g_i z_{i_{a_2}} (i_{a_2}, j_b) \sim g_j z_{j_b}
\]

with \((i_{a_2}, j_2), (i_{a_2}, j_b)\) the colors assigned to remaining entries and \(g_j z_{j_2}, g_i z_{i_{a_2}}\) the respective vertices from remaining entries.

\[
\begin{bmatrix}
    g_j z_{j_b} & g_j z_{j_1} \\
    \vdots & \vdots \\
    \cdots & f & \cdots \\
    \vdots & \vdots \\
    g_i z_{i_{a_2}} & g_j z_{j_2}
\end{bmatrix}
\]

Under the conditions stated above we apply the same to all the colors assigned to \(i_a\)-raw. We take edges from remaining entries to form the rest paths with the same method. Let \(j'_1\) such that \(f' = (i_a, j'_1)\) from \(j_b\)-column we remove the row with entry same color like \(f'\). The new path is the following:

\[
g_i z_{i_a} (i_{a_2}, j'_1) \sim g_j z_{j'_1} (i_{a_1'}, j'_1) \sim g_i z_{i_{a_1'}} (i_{a_1'}, j_b) \sim g_j z_{j_b}
\]

Take \((i_a, j'_1), (i_{a_1'}, j'_1)\) as remaining entries from all the entries do not removed before with a different color as \(f'\).

**Remark 6.4.** Suppose that we can coloring with only \(\left\lfloor \frac{k}{2} \right\rfloor + 1\) colors. Let \(g_i z_{i_m}\) any start vertex, then there exists a pair of vertices \(g_j z_{j_m}, g_j z_{j_m'}\), such that \(\{(a_{i_r}, b_{j_m})|(a_{i_r}, b_{j_m}) - \text{color} \neq (\left\lfloor \frac{k}{2} \right\rfloor + 1) - \text{color}\}\) identify with \(\{(a_{i_r}, b_{j_m'})|(a_{i_r}, b_{j_m'}) - \text{color} = \text{the last color}\}\), therefore is impossible to built \(k\) paths between any end vertices \(g_i z_{i_m}, g_j z_{j_m}\) passes through \(g_j z_{j_m'}\), just like 5.1.

2. Case: \(g_i \sim g_j \sim g_1\) with \(g_1 \sim g_i\) in \(S^M_{\Gamma(G)}\).

(a) **Repetition of different color to the last color**

**Case: repetition of one color between columns.** Suppose that \(f\) is the repeated color between the columns assigned to the end vertices \(g_i z_{i_a}\) and \(g_l z_{l_b}\), i.e. \(f = (j_c, i_a) = (j_c, l_b)\) in the rainbow table, for some \(c = \{1, \ldots, |Z(G)|\}\), with \(l_b \in g_l Z\) and \(i_a \in g_i Z\). Suppose that \(f\) is in the path passes through \(g_j z_{j_c}\), thus for do the rainbow path we need
to find another row \( j' \) such that \((j', l_b) = f' \neq f\) then for do the rainbow path, to the row \( j' \) we remove the columns with color \( f \) (i.e. 2 columns) and one of color \( f' \). To row \( j_c \) remove 2 columns for color \( f' \) and 2 columns assigned for \( i_a \) and \( l_b \). Then we remove a total of 7 columns. There are in total \( 2|Z(G)| \) columns in our rainbow table, then it remains \( 2|Z(G)| - 7 \) columns with \( |Z(G)| \geq 4 \), leaving at least one column for do the path without similar colors. The path is \( g_{i_{z_{i_a}}} \overset{f}{\sim} g_{j_{z_{j_c}}} \overset{f_1}{\sim} g(j_c) \overset{f_2}{\sim} g(j_{j'}') \overset{f}{\sim} g_{l_{z_{l_b}}} \) with \( f_1, f_2 \) colors assigned to left column and \( g(j_c), g(j_{j'}) \) vertices in column assigned to above column.

\[
\begin{array}{c|c|c}
& i_a & l_b \\
\hline
j_c & \ldots & f & f' \ldots f' \ldots f \ldots \\
\hline
j_{j'} & \ldots & f' & f \ldots f' \ldots f \ldots \\
\hline
\end{array}
\]

Now we make the path who starts in \( g_{i_{z_{i_a}}} \overset{g}{\sim} g_{l_{z_{l_b}}} \). When \( g \neq f \) and \( g \neq f' \). As written above we remove the columns in row \( j_{j'} \) with color \( f \) and one of color \( g \), i.e. 3 columns, and in the row \( j_c \) remove the columns assigned with color \( g \) and two of columns \( i_a \) and \( l_b \), in total we remove 7 columns and leaving \( 2|Z(G)| - 7 \) columns where we can find the desired path.

**Case: repetition of two colors between columns with** \( g = f' \). We remove 2 columns with color \( f' \) to \( j_{j'} \)-row and 2 columns assigned to \( i_a \) and \( l_b \). In row \( j_{j'} \) remove 2 columns assigned with color \( f \). There are in total \( 2|Z(G)| - 6 \) free columns to find rainbow paths.

**Case: repetition of 3 colors** Suppose that there are 3 repeated colours between the columns for do the paths with end vertices \( g_{i_{z_{i_a}}} \) and \( g_{l_{z_{l_b}}} \) passes through \( g_{j_{z_{j_c}}} \), \( g_{j_{z_{j'}}} \) and \( g_{j_{z_{j''}}} \). For do the paths passes through \( g_{j_{z_{j_c}}} \), just like the first case, we remove columns with color \( f' \) to \( j_{j'} \)-row and, to row \( j_{j''} \) remove the 2 columns with color \( f \) minus the rows assigned \( i_a \) and \( l_b \), then for \( |Z(G)| \geq 4 \) there are \( 2|Z(G)| - 6 \) free columns for do the rainbow path with end vertices \( g_{i_{z_{i_a}}} \) and \( g_{l_{z_{l_b}}} \) cross above \( g_{j_{z_{j_c}}} \), \( g_{j_{z_{j'}}} \), and \( g_{j_{z_{j''}}} \). The same happens for rainbow path passes through \( g_{j_{z_{j_c}}} \), \( g_{j_{z_{j'}'}} \) and \( g_{j_{z_{j''}}}, g_{j_{z_{j_c}}} \). The paths have the following form:

\[
\begin{align*}
g_{i_{z_{i_a}}} & \sim g_{j_{z_{j_c}}} \sim g_{j_{z_{j'}}} \sim g_1(j_c) \sim g_2'(j_{j'}) \sim g_{l_{z_{l_b}}}, \\
g_{i_{z_{i_a}}} & \sim g_{j_{z_{j_c}}} \sim g_3(j_c) \sim g_4'(j_{j'}) \sim g_{l_{z_{l_b}}}, \\
g_{i_{z_{i_a}}} & \sim g_{j_{z_{j'}'}} \sim g_5(j_c) \sim g_6'(j_{j''}) \sim g_{l_{z_{l_b}}}, \\
g_{i_{z_{i_a}}} & \sim g_{j_{z_{j''}}} \sim g_7(j_c) \sim g_8(j_{j''}) \sim g_{l_{z_{l_b}}},
\end{align*}
\]
The rainbow $k$-connectivity of the non-commutative graph of a finite group  

Note that $g_1, g'_1, g''_1, g_2$ are the colors between free columns with colors assigned $f, f', f'', f'''$ respectively, and $g_1(j_c), g_2(j_c); g'_1(j_c), g''_1(j_c); g'''_1(j_{c''})$ are vertices associated to the colors in the free columns with its rows $j_c, j_{c'}, j_{c''}$ respectively.

(b) Repetition of last color between columns

Case: repeat the last color $\left[ \frac{k}{2} \right] + 2$ one time. Let $g_iz_{i_a}$ and $g_lz_{l_b}$ be the end vertices and suppose that only is repeated the last color $\left[ \frac{k}{2} \right] + 2$ only one time. Let $f = \left[ \frac{k}{2} \right] + 2$ be the last color and let $B = 2 \left[ k - \left( \left[ \frac{k}{2} \right] + 1 \right) \right]$ be the number of entries with the last color in each row of the rainbow table. Let $j_c$ be a row associated with different color to $f$ in the entries $(j_c, i_a)$ and $(j_c, l_b)$.

For make the rainbow path passes through $j_c$, to row $j_{c'}$ remove $B$ columns associated to the last color $f$ and one column designated to color $f'$, i.e., we remove $B+1$ columns. Further in row $j_c$ we remove $B - 2$ columns associated to $f$, 2 columns associated to color $f'$ and 2 columns for columns associated to $i_a$ and $l_b$, thus we remove from row $j_c B + 2$ columns. If the columns removed are all different from each other then we keep $C = 2k - (2B + 3)$ free columns, in the extreme case that we eliminate the same columns for each case, evaluate in $f$ and $f'$, thus we would have $2k - (B + 2)$ free columns, then the value of free columns is $2k - (2B + 3) \leq C \leq 2k - (B + 2)$ for $k \leq 4$. The same happens to do a path passes through $g_jz_{j_{c'}}$. Thus, we have enough free columns to do the rainbow path.

Later, for make the rainbow path from $g_iz_{i_a}$ to $g_jz_{j_{c'}}$, we remove 2 columns assigned to color $g$ to $j_c$-row, $B - 2$ columns assigned to color $f$ and 2 for the columns $i_a, l_b$. 

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i.e., remove \( B + 2 \) columns. Moreover from \( j_{i'} \)-row remove \( B \) columns for last color \( f \) plus 1 column for color \( g \), i.e. \( B + 1 \) columns. In total the amount of free columns is between:

\[
2k - (2B + 3) \leq C \leq 2k - (B + 2) \quad k \geq 4
\]

Then, there are enough free columns for do the rainbow path.

**Case: repeat two colors, one of them the last color, i.e., \( g = f' \neq f \).** To the row \( j_{i'} \) we remove \( B \) columns associated to last color \( f \) and the row \( j_c \) we remove \( B - 2 \) columns associated to last color \( f \), 2 columns associated to color \( f' \) and 2 columns associated to columns \( i_a \) and \( l_b \), i.e. we remove \( B + 2 \) columns. In total there are

\[
2k - (2B + 2) \leq C \leq 2k - (B + 2) \quad \text{for } k \geq 4
\]

Since \( k - B - 1 > 0 \) for all \( k \) we always have a minimum, two columns to form two paths.

**Case: repeat at most \( \frac{B}{2} \) entries between columns.** Suppose that between columns \( i_a \) and \( l_b \) assigned to end vertices \( g_i \tilde{z}_{i_a}, g_i \tilde{z}_{i_b} \) there are, at most \( D = k - (\lceil \frac{k}{2} \rceil + 1) \) entries with the last color \( f \) in each column, since \( D < \lceil \frac{k}{2} \rceil + 1 \) we can proceed like the previous cases.

3. **Case: any vertices of same class** We can do the paths directly, if we want to go from \( g_i \tilde{z}_{i_a} \) to \( g_i \tilde{z}_{i_b} \) the paths are of the following form \( g_i \tilde{z}_{i_a}^{(i_a,p)} \tilde{z}_{i_b}^{(i_b,p)} \tilde{z}_{i_c}^{(i_c,p)} \ldots \tilde{z}_{i_s}^{(i_s,p)} \) for \( p = \{1, \ldots, s = |Z(G)|\} \). We note that we can only find up to \( \lceil \frac{k}{2} \rceil + 2 \) edge disjoint paths for any pair of vertices.

Corollary 6.5. Let \( G \) be a finite non-abelian group. If \( g_i \sim g_j \) then \( \lceil \frac{k}{2} \rceil + 1 < \text{rc}_k(\Gamma(G)) \).

**Proof.** From 6.4.

Corollary 6.6. Let \( G \) be a finite non-abelian group. If \( g_i \sim g_j \sim g_l \) with \( g_i \sim g_l \) then \( \lceil \frac{k}{2} \rceil + 1 < \text{rc}_k(\Gamma(G)) \).

**Proof.** Suppose that \( B = 2(k - \lceil \frac{k}{2} \rceil) \) then, for any value of \( k \), \( B = 2m \ (k = \{2m, 2m + 1\}) \). For the case where only repeat one time the last color \( f \), from 4

\[
-3 \leq C \leq 2m - 2 \quad \text{for } k = 2m
\]

\[
-1 \leq C \leq 2m \quad \text{for } k = 2m + 1
\]
Thus, there are cases when we have not free columns for do the rainbow paths. The same happens for case 5:

\[-2 \leq C \leq 2m - 2 \quad \text{for } k = 2m\]
\[0 \leq C \leq 2m - 1 \quad \text{for } k = 2m + 1\]

Therefore, we can not form \( k \) rainbow paths with \( \left\lceil \frac{k}{2} \right\rceil + 1 \) different colors.

\[\Box\]

**Theorem 1.3** Let \( G \) be a finite non-abelian group. Then \( rc_k(\Gamma(G)) = \left\lceil \frac{k}{2} \right\rceil + 2 \), for \( 3 \leq k \leq s = |Z(G)| \) with \( |Z(G)| \geq 4 \).

**Proof.** From 6.2, 6.5 and 6.6.

Given the structure of \( \Gamma(G) \), it could be considered a generalization of study in [5] to find the Harary index of \( \Gamma(G) \).

**Example 6.7.** Let \( G \) be the Heisenberg group for \( p = 3 \) with presentation

\[\langle x, a, b | x^3 = a^3 = b^3 = 1, ab = ba, xax^{-1} = ab, xbx^{-1} = b \rangle.\]

We know that \( |G| = 27 \), \( |G \setminus Z(G)| = 24 \) and \( |G/Z(G)| = 9 \), i.e. the partition for \( V(\Gamma(G)) = \{Z, aZ, a^2Z, xZ, axZ, a^2xZ, x^2Z, ax^2Z, a^2x^2Z\} \) by \([x, a] = b\) we have \( xa = bax\), then \( xaZ = axZ \). The following is the graph for \( S^M_{\Gamma(G)} \)

![Graph](image-url)

Figure 1. Heisenberg skeleton graph for \( p = 3 \).

In \( S^M_{\Gamma(G)} \) the only vertices with distance 2 are \( a \) with \( a^2 \) and \( x \) with \( x^2 \). Suppose without loss of generality that \( \psi(\{g, a\}) = 1 \). The edge-disjoint paths for end vertices \( a \) and \( a^2 \) are the following

\[g \quad \begin{bmatrix}
    a & ab & a^2 & a^2b & a^2b^2 \\
    g & \begin{bmatrix}
        1 & 3 & 2 & 3 & 4 \\
        2 & 4 & 1 & 4 & 3 \\
        4 & 2 & 3 & 1 & 4 & 2 
    \end{bmatrix}
\end{bmatrix}\]

And all the paths are given in 6.1.
**Example 6.8.** Let $G$ be the Heisenberg group for $p = 5$ with presentation

$$\langle x, a, b | x^5 = a^5 = b^5 = 1, ab = ba, xax^{-1} = ab, xbx^{-1} = b \rangle.$$ 

We know that $|G| = 125$, $|G \setminus Z(G)| = 120$ and $|G/Z(G)| = 25$. Since $[x, a] = b$ we have $xa = bax$, then $xaZ = axZ$. The graph 2 is the skeleton $S_M^{\Gamma(G)}$ of $G$.

By 3.2 we know that we can found 5 edge-disjoint paths for any pair of vertices then, without loss of generality we give the 5 edge-disjoint paths for end vertices $x, ax^2 \in S_M^{\Gamma(G)}$. By 1.3 we know that we need $(\lfloor \frac{5}{2} \rfloor + 2)$-color. The rainbow table is given below

![Rainbow Table](image)

Then, the 5 edge-disjoin paths are given by:
The rainbow $k$-connectivity of the non-commutative graph of a finite group | Luis A. Dupont et al.

We can give 4 paths with 4 colors. The rainbow and the 4 edge-disjoint paths with ends vertices $x^4, x^3b^3$ are the following.

$$
\begin{array}{cccccccc}
  a^3 & x^4 & x^4b & x^4b^2 & x^4b^3 & x^4b^4 & x^3 & x^3b & x^3b^2 & x^3b^3 & x^3b^4 \\
  a^3b & 1 & 3 & 2 & 2 & 1 & 3 & x^4 & x^4b & x^4b^2 & x^4b^3 & x^3b^3 \\
  a^3b^2 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & x^3b^3 \\
  a^3b^3 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & x^3b^3 \\
  a^3b^4 & 3 & 2 & 1 & 1 & 3 & 2 & 1 & 2 & 1 & x^3b^3 \\
\end{array}
$$

If we note, we can not find 5 edge-disjoint paths with only 4 colors, for example, for the end vertices $x^4b^4$ and $x^3b^2$ we have the following paths:

<table>
<thead>
<tr>
<th>Start with color 1</th>
<th>Start with color 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
</tr>
<tr>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
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<tr>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
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<tr>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
</tr>
<tr>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Start with color 3</th>
<th>Start with color 4 from $x^4b^4 \sim a^3b^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
</tr>
<tr>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
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<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
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<tr>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
<td>$x^4b^4 \sim a^3b^4 \sim x^3b^2$</td>
</tr>
</tbody>
</table>

Color 3 can not came to color 4
Color 4 can not came to color $a^3 \sim x^3b^2$
Thus, we have not columns for do the rainbow path from $x^4 b^4 \sim 3 a^3 b$ to $a^3 b^3 \sim 4 x^3 b^2$.

<table>
<thead>
<tr>
<th></th>
<th>$x^4$</th>
<th>$x^4 b$</th>
<th>$x^4 b^2$</th>
<th>$x^4 b^3$</th>
<th>$x^4 b^4$</th>
<th>$x^3$</th>
<th>$x^3 b$</th>
<th>$x^3 b^2$</th>
<th>$x^3 b^3$</th>
<th>$x^3 b^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^3$</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a^3 b$</td>
<td>2</td>
<td>1</td>
<td>/</td>
<td>/</td>
<td>3</td>
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<td>2</td>
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</tr>
<tr>
<td>$a^3 b^2$</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
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<td>2</td>
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<td>3</td>
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<tr>
<td>$a^3 b^3$</td>
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<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$a^3 b^4$</td>
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<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td></td>
<td>2</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Then, we can not find a path from $x^4 b^4$ to $x^3 b^2$ passes through $a^3 b$, because the last color from $x^4 b^4$ only came to $x^3 b^2$ passes through $a^3 b$ and $a^3 b^2$. Then we need one more color.

Rainbow table for found the 5 edge-disjoin paths between $x^4$ and $x^3$.

<table>
<thead>
<tr>
<th></th>
<th>$x^4$</th>
<th>$x^4 b$</th>
<th>$x^4 b^2$</th>
<th>$x^4 b^3$</th>
<th>$x^4 b^4$</th>
<th>$x^3$</th>
<th>$x^3 b$</th>
<th>$x^3 b^2$</th>
<th>$x^3 b^3$</th>
<th>$x^3 b^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^3$</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a^3 b$</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td></td>
<td></td>
<td>2</td>
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</tr>
<tr>
<td>$a^3 b^2$</td>
<td>3</td>
<td>2</td>
<td>1</td>
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</tr>
<tr>
<td>$a^3 b^3$</td>
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<td>4</td>
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</tr>
<tr>
<td>$a^3 b^4$</td>
<td>4</td>
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<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>4</td>
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With the given structure, we could ask about the meaning of $d$-coloring redundant as a generalization of [4]. For example, in Figure 3 we could considered a particular case of Turán graph with $T(m|Z|, m)$.

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References


