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Enumeration for spanning trees and forests of join graphs based on the combinatorial decomposition

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Abstract

This paper discusses the enumeration for rooted spanning trees and forests of the labelled join graphs $K_m + H_n$ and $K_m + K_{n,p}$, where H_n is a graph with n isolated vertices.

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1. Introduction

In this paper we consider the enumeration problem of rooted spanning trees and forests of two labelled join graphs. In [2], the number of spanning forests of the labelled complete bipartite graph $K_{m,n}$ on m and n vertices has been enumerated by combinatorial method. In [1] and [3], it has been given the enumeration of spanning trees of the complete tripartite graph $K_{m,n,p}$ on m, n and p vertices and the complete multipartite graph, respectively. In [4], by using the multivariate Lagrange inverse, the number of spanning forests of the labelled complete multipartite graph was derived. And, in [5], it has been found the asymptotic number of labeled spanning forests of the complete bipartite graph $K_{m,n}$ as $m \to \infty$ when $m \leq n$ and $n = o(m^{6/5})$.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets, we let $G_1 + G_2$ denote the join of G_1 and G_2, that is, the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E(V_1, V_2))$ where $E(V_1, V_2) = \{(i, j) | i \in V_1, j \in V_2\}$, (i, j) denotes an edge between two vertices $i \in V_1, j \in V_2$.

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Clearly, by the definition of a join graph, the complete bipartite graph $K_{m,n}$ is a join graph $H_m + H_n$ and the complete tripartite graph $K_{m,n,p}$ is a join graph $H_m + H_n + H_p$, where H_m, H_n and H_p are graphs with m isolated vertices, n isolated vertices and p isolated vertices, respectively.

The goal of this paper first is to give a combinatorial proof of the enumeration for the spanning trees and forests of a labelled join graph $K_m + H_n$, where K_m is the complete graph on m vertices and H_n is the graph with n isolated vertices. Second, this paper also gives a combinatorial proof of the enumeration for the spanning trees and all forests of another labelled join graph $K_m + K_{n,p}$, where $K_{n,p}$ is the complete bipartite graph on n vertices and p vertices.

2. Enumeration for spanning trees and forests of a join graph $K_m + H_n$

Let $V(G)$ denote the vertex set of graph G. Throughout this paper, we will consider only the labelled graphs. In this section, we consider a join graph $K_m + H_n$ where K_m is the complete graph on the vertex set $\{x_1, x_2, \cdots, x_m\}$.

Lemma 2.1. *The number $f(m, l)$ of the labelled spanning forests of K_m with l roots is*

$$f(m, l) = \binom{m}{l} l m^{m-l-1}. \tag{1}$$

Proof Let $X = V(K_m) = \{x_1, x_2, \cdots, x_m\}$ be the vertex set of K_m and $\{x_{i_1}, x_{i_2}, \cdots, x_{i_l}\}$ be the given root set of K_m. There are $\binom{m}{l}$ ways to choose the l roots in $V(K_m)$. Also, let $X' = X \backslash \{x_{i_1}, x_{i_2}, \cdots, x_{i_l}\}$ be a subset of X, and X'' be another copy of X' and let $x'' \in X''$ denote copy of $x' \in X'$. Take the complete bipartite graph $K_{m,m-l}$ with the partition (X, X'') of its vertex set. Consider the subgraph G of $K_{m,m-l}$ that contains only the directed edges of the form $(x', x''), x' \in X', x'' \in X''$. The number of the components of G is equal to $m - l$ and G is a forest of $K_{m-l,m-l} = (X', X'')$. Let $D(m, |\{x_{i_1}, x_{i_2}, \cdots, x_{i_l}\}|; m - l, 0)$ be the set of the labelled spanning forests of $K_{m,m-l} = (X, X'')$ with l roots $x_{i_1}, x_{i_2}, \cdots, x_{i_l} \in X$ and $D^*(K_m; x_{i_1}, x_{i_2}, \cdots, x_{i_l})$ be the set of the labelled spanning forests of K_m with l roots $x_{i_1}, x_{i_2}, \cdots, x_{i_l} \in X$. Now any spanning forest in $D(m, |\{x_{i_1}, x_{i_2}, \cdots, x_{i_l}\}|; m - l, 0)$ containing G gives rise to a spanning forest in $D^*(K_m; x_{i_1}, x_{i_2}, \cdots, x_{i_l})$ by contracting the edges $(x', x''), x' \in X', x'' \in X''$.

Conversely, any forest in $D^*(K_m; x_{i_1}, x_{i_2}, \cdots\cdots, x_{i_l})$ can be extended to a forest in $D(m, |\{x_{i_1}, x_{i_2}, \cdots\cdots, x_{i_l}\}|; m - l, 0)$ containing G by inserting vertex $x'' \in X''$ after $x' \in X'$. Therefore, from G, we will construct the rooted spanning forests of $K_{m,m-l}$ with l roots in X as follows.

For any fixed integer $t \in [0, m - l - 1]$, add t edges consecutively to G as follows. At each step we add an edge of the form (v, x') between $x' \in X'$ and a (unique)vertex $v \in X''$ of out-degree zero in any component not containing x' in the graph already constructed. The number of components decreases by one each time such an edge is added.

Since $|X'| = m - l$ and the number of components not containing x' in the graph G is $m - l - 1$, there are $(m - l)(m - l - 1)$ choices for the first such edge. Similarly, there are $(m - l)(m - l - 2)$ choices for the second edge, $\cdots$, and $(m - l)(m - l - t)$ choices for the tth edge.

The order in which the t edges are added to G is immaterial, so it follows that there are

$$\frac{[(m - l)(m - l - 1)][(m - l)(m - l - 2)] \cdots [(m - l)(m - l - t)]}{t!} = \binom{m - l - 1}{t} (m - l)^t$$

ways.

Every graph we obtained will have $m-l-t$ (weakly) connected components each of which has a unique vertex in X'' of out-degree zero. Link edges from $m-l-t$ vertices of out-degree zero in these components to l given roots $x_{i_1}, x_{i_2}, \cdots\cdots, x_{i_l}$, there are l^{m-l-t} ways. Hence,

$$f(m,l) = \binom{m}{l} \sum_{t=0}^{m-l-1} \binom{m-l-1}{t} l^{m-l-t}(m-l)^t = \binom{m}{l} l m^{m-l-1}. \qquad \square$$

Let $D(m,l)$ be the set of the labelled spanning forests of K_m with l roots, i.e.,

$$f(m,l) = |D(m,l)|. \tag{2}$$

Theorem 2.1. *The number $g(m,n)$ of the labelled spanning trees of $K_m + H_n$ is*

$$g(m,n) = m^{n-1}(m+n)^{m-1}. \tag{3}$$

Proof Let $V(K_m) = \{x_1, x_2, \cdots, x_m\}$, $V(H_n) = \{y_1, y_2, \cdots, y_n\}$ be the vertex sets of K_m, H_n, respectively, and $y_1 \in V(H_n)$ be the given root of $K_m + H_n$. Let $D(m,0;n,|\{y_1\}|)$ be the set of the labelled spanning trees of K_m+H_n with root y_1 and $T(m,n)$ be the set of the labelled spanning trees of $K_m + H_n$. Clearly, $|T(m,n)| = |D(m,0;n,|\{y_1\}|)|$.

From every graph $F \in D(m,l)$, we will construct the rooted spanning trees of $K_m + H_n$ as follows. Link an edge (y,x) between every $y \in V(H_n)\backslash\{y_1\}$ and some $x \in V(F)$. There are m^{n-1} ways. Notice that the obtained graph G has l (weakly) connected components each of which has a unique vertex in $V(K_m)$ of out-degree zero.

Now, for any fixed integer t, let G' denote a graph obtained by adding t edges consecutively to G as follows. At each step we add an edge of the form (x,y) where y is any vertex of $y \in V(H_n)\backslash\{y_1\}$ and $x \in V(K_m)$ is a vertex of out-degree zero in any component not containing y in the graph already constructed. The number of components decreases by one each time such an edge is added.

Since $|V(H_n)\backslash\{y_1\}| = n-1$ and the number of components not containing y in the graph G already constructed is $l-1$, there are $(n-1)(l-1)$ choices for the first such edge. Similarly, there are $(n-1)(l-2)$ choices for the second edge, $\cdots$, and $(n-1)(l-t)$ choices for the tth edge, where, $0 \le t \le l-1$, because the number of components in the graph G is l. The graph G' thus constructed has $l-t$ components each of which has a unique vertex in $V(K_m)$ of out-degree zero and the remaining vertices all have out-degree one; if we add edges from these vertices of out-degree zero to y_1, we obtain a tree T' in $D(m,0;n,|\{y_1\}|)$ that contains G and in which the in-degree of y_1 equals to $l-t$. The order in which the t edges are added to G to form G' is immaterial, so it follows that there are

$$\frac{[(n-1)(l-1)][(n-1)(l-2)]\cdots[(n-1)(l-t)]}{t!} = \binom{l-1}{t}(n-1)^t$$

rooted spanning trees T' for fixed integer t. This implies that there are

$$\sum_{t=0}^{l-1} \binom{l-1}{t}(n-1)^t = n^{l-1}$$

spanning trees T in $D(m,0;n,|\{y_1\}|)$ that contain G. Hence, by (2) and Lemma 2.1, we have

$$\begin{aligned} g(m,n) &= |D(m,0;n,|\{y_1\}|)| = \sum_{l=1}^{m} |D(m,l)| n^{l-1} m^{n-1} \\ &= \sum_{l=1}^{m} \binom{m}{l} l m^{m-l-1} n^{l-1} m^{n-1} = m^{n-1}(m+n)^{m-1} \end{aligned}$$

as desired. □

Theorem 2.2. *The number $g(m,l;n,k)$ of the labelled spanning forests of $K_m + H_n$ with l roots in K_m and k roots in H_n is*

$$g(m,l;n,k) = \binom{m}{l}\binom{n}{k} m^{n-k-1}(m+n)^{m-l-1}(lm+mk+ln-kl). \tag{4}$$

Proof Let $V(H_n) = \{y_1, y_2, \cdots, y_n\}$ be the vertex set of H_n and $\{y_{i_1}, y_{i_2}, \cdots, y_{i_k}\}$ be the given root set of H_n. There are $\binom{n}{k}$ ways to choose the k roots in $V(H_n)$. Let $V(K_m) = \{x_1, x_2, \cdots, x_m\}$ be the vertex set of K_m and $Y' = V(H_n)\backslash\{y_{i_1}, y_{i_2}, \cdots, y_{i_k}\}$ be a subset of $V(H_n)$.

From every graph $F \in D(m,s)(s \geq l)$, we will construct the rooted spanning forests of $K_m + H_n$ with l roots in K_m and k roots in H_n as follows. Link an edge (y,v) between every $y \in Y'$ and some $v \in V(F)$. There are m^{n-k} ways. Notice that the obtained graph G has s (weakly) connected components each of which has a unique vertex in $V(K_m)$ of out-degree zero and the remaining vertices all have out-degree one.

As in the proof of former theorem, link an edge (v,y) between $y \in Y'$ and a vertex $v \in V(K_m)$ of out-degree zero in any component not containing y in the graph already constructed, we repeat this procedure i times, where, $0 \leq i \leq s-l$, because the required forests have l roots in $V(K_m)$.

There are

$$\frac{[(n-k)(s-1)][(n-k)(s-2)]\cdots[(n-k)(s-i)]}{i!} = \binom{s-1}{i}(n-k)^i \tag{5}$$

ways.

Every graph G' we obtained will have $s-i$ components each of which has a unique vertex in $V(K_m)$ of out-degree zero. Now, choose the $s-i-l$ vertices of out-degree zero in these $s-i$ components and link edges from these $s-i-l$ vertices to k roots $y_{i_1}, y_{i_2}, \cdots, y_{i_k}$. There are

$$\binom{s-i}{s-i-l} k^{s-i-l} = \binom{s-i}{l} k^{s-i-l} \tag{6}$$

ways.

Therefore, by (5) and (6), the number of the rooted spanning forests of $K_m + H_n$ which are obtained from F is equal to

$$\sum_{i=0}^{s-l} \binom{s-1}{i}\binom{s-i}{l}(n-k)^i k^{s-i-l} = \binom{s}{l} n^{s-l} - \binom{s}{l}\frac{s-l}{s} n^{s-l-1}(n-k). \tag{7}$$

Hence, by (2), (7) and Lemma 2.1, the number $g(m,l;n,k)$ of the labelled spanning forests of $K_m + H_n$ with l roots in K_m and k roots in H_n is as follows.

$$\begin{aligned} g(m,l;n,k) &= \binom{n}{k}\sum_{s=l}^{m}|D(m,s)|m^{n-k}\sum_{i=0}^{s-l}\binom{s-1}{i}\binom{s-i}{l}(n-k)^i k^{s-i-l} \\ &= \binom{n}{k}\sum_{s=l}^{m}\binom{m}{s}sm^{m-s-1}m^{n-k}\left[\binom{s}{l}n^{s-l} - \binom{s}{l}\frac{s-l}{s}n^{s-l-1}(n-k)\right] \\ &= \binom{m}{l}\binom{n}{k}m^{n-k-1}(m+n)^{m-l-1}(lm+mk+ln-lk). \end{aligned}$$

We get the required result. □

Corollary 2.1. *The number $S(m,n)$ of all spanning forests of the join graph $K_m + H_n$ is equal to*

$$S(m,n) = (m+n+1)^m(m+1)^{n-1}. \tag{8}$$

Proof By Theorem 2.2,

$$\begin{aligned} S(m,n) &= \sum_{l=0}^{m}\sum_{k=0}^{n} g(m,l;n,k) \\ &= \sum_{l=0}^{m}\sum_{k=0}^{n}\binom{m}{l}\binom{n}{k}m^{n-k-1}(m+n)^{m-l-1}(lm+mk+ln-kl) \\ &= (m+n+1)^m(m+1)^{n-1}. \end{aligned}$$

Thus, this corollary is true. □

3. Enumeration for spanning trees and forests of a join graph $K_m + K_{n,p}$

In this section, we consider another join graph $K_m + K_{n,p}$ where K_m is the complete graph and $K_{n,p}$ is the complete bipartite graph. We will show how to count the number of the spanning trees of a join graph $K_m + K_{n,p}$. Clearly, $K_m + K_{n,p} = (K_m + H_n) + H_p$. Let $D(m,l;n,k)$ be the set of the labelled spanning forests of $K_m + H_n$ with l roots in K_m and k roots in H_n, i.e.,

$$g(m,l;n,k) = |D(m,l;n,k)|. \tag{9}$$

Theorem 3.1. *The number $g(m,n,p)$ of the spanning trees of $K_m + K_{n,p}$ is equal to*

$$g(m,n,p) = (m+n)^{p-1}(m+p)^{n-1}(m+n+p)^m. \tag{10}$$

Proof Let $V(K_m + H_n) = \{x_1, x_2, \cdots, x_m; y_1, y_2, ..., y_n\}$ be the vertex set of $K_m + H_n$ and $V(H_p) = \{z_1, z_2, ..., z_p\}$ be the vertex set of H_p. Let $z_1 \in V(H_p)$ be the given roots of $K_m + K_{n,p}$ and $Z' = V(H_p)\backslash\{z_1\}$, $D(m,0;n,0;p,|\{z_1\}|)$ be the set of the labelled spanning trees of $K_m + K_{n,p}$ with root z_1. Clearly,

$$g(m,n,p) = |D(m,0;n,0;p,|\{z_1\}|)|.$$

We shall obtain the spanning trees in $D(m,0;n,0;p,|\{z_1\}|)$ from every graph $F\in D(m,l;n,k)$. As in the proof of former theorem, link an edge (z,v) between every $z\in Z'$ and some $v\in V(F)$. There are $(m+n)^{p-1}$ ways. Notice that the obtained graph G has $l+k$ (weakly) connected components each of which has a unique vertex in $V(K_m)\cup V(H_n)$ of out-degree zero and the remaining vertices all have out-degree one.

For any fixed integer t such that $0\le t\le l+k-1$, link an edge (v,z) between $z\in Z'$ and a vertex $v\in V(K_m)\cup V(H_n)$ of out-degree zero in any component not containing z in the graph already constructed, we repeat this procedure t times.

There are

$$\frac{[(p-1)(l+k-1)][(p-1)(l+k-2)]\cdots[(p-1)(l+k-t)]}{t!}=\binom{l+k-1}{t}(p-1)^t$$

ways. Therefore, the number of the spanning trees which are obtained from F is equal to

$$\sum_{t=0}^{l+k-1}\binom{l+k-1}{t}(p-1)^t=p^{l+k-1}.$$

Hence, by (9) and Theorem 2.2,

$$\begin{aligned}
g(m,n,p) &= |D(m,0;n,0;p,|\{z_1\}|)|\\
&= \sum_{l=0}^{m}\sum_{k=0}^{n}|D(m,l;n,k)|p^{l+k-1}(m+n)^{p-1}\\
&= \sum_{l=0}^{m}\sum_{k=0}^{n}\binom{m}{l}\binom{n}{k}m^{n-k-1}(m+n)^{m-l-1}(lm+km+ln-lk)p^{k+l-1}(m+n)^{p-1}\\
&= (m+n)^{p-1}(m+p)^{n-1}(m+n+p)^m.
\end{aligned}$$

Therefore, we get the required result. □

Theorem 3.2. *The number $S(m,n,p)$ of all spanning forests of the join graph $K_m+K_{n,p}$ is equal to*

$$S(m,n,p)=(m+n+p+1)^{m+1}(m+n+1)^{p-1}(m+p+1)^{n-1}. \tag{11}$$

Proof Let $B(p,r)$ denote the set of spanning forests of the join graph $K_m+K_{n,p}=(K_m+H_n)+H_p$ which r roots are in $V(H_p)$ and remaining roots are in $V(K_m)$ or $V(H_n)$.

From every graph $F\in D(m,l;n,k)$, we will construct the rooted spanning forests of $(K_m+H_n)+H_p$ with r roots in $V(H_p)$ as follows. Let $z_{i_1},z_{i_2},\cdots,z_{i_r}\in V(H_p)$ be root vertices. The number of ways to select r roots in $V(H_p)$ is equal to $\binom{p}{r}$. Let $Z'=V(H_p)\backslash\{z_{i_1},z_{i_2},\cdots,z_{i_r}\}$. Link an edge (z,v) between every $v\in Z'$ and some $v\in V(F)$. There are $(m+n)^{p-r}$ ways. Notice that the obtained graph G has $l+k$ (weakly) connected components each of which has a unique vertex in $V(K_m)\cup V(H_n)$ of out-degree zero and the remaining vertices all have out-degree one.

As in the proof of former theorem, for any fixed integer t such that $0\le t\le l+k-1$, link an edge (v,z) between $z\in Z'$ and a vertex $v\in V(K_m)\cup V(H_n)$ of out-degree zero in any component not containing z in the graph already constructed, we repeat this procedure t times. There are

$$\frac{[(p-r)(l+k-1)][(p-r)(l+k-2)]\cdots[(p-r)(l+k-t)]}{t!} = \binom{l+k-1}{t}(p-r)^t$$

ways.

The graph G' thus constructed has $l+k-t$ components each of which has a unique vertex in $V(K_m)\cup V(H_n)$ of out-degree zero and the remaining vertices all have out-degree one; if we add edges from some vertices of these vertices of out-degree zero to $z_{i_1}, z_{i_2}, \cdots, z_{i_r} \in Z$, we obtain a forest in $B(p,r)$ that contains G. There are $(r+1)^{l+k-t}$ ways. Therefore, this implies that there are

$$\sum_{t=0}^{l+k-1}\binom{l+k-1}{t}(p-r)^t(r+1)^{l+k-t} = (r+1)(p+1)^{l+k-1}$$

forests in $B(p,r)$ that contain G. Hence, by (9) and Theorem 2.2,

$$\begin{aligned}
S(m,n,p) &= \sum_{l=0}^{m}\sum_{k=0}^{n}|D(m,l;n,k)|\sum_{r=0}^{p}\binom{p}{r}(m+n)^{p-r}(r+1)(p+1)^{l+k-1}\\
&= \sum_{l=0}^{m}\sum_{k=0}^{n}\binom{m}{l}\binom{n}{k}m^{n-k-1}(m+n)^{m-l-1}(lm+mk+ln-lk)\\
&\quad \sum_{r=0}^{p}\binom{p}{r}(m+n)^{p-r}(r+1)(p+1)^{l+k-1}\\
&= (m+n+p+1)^{m+1}(m+n+1)^{p-1}(m+p+1)^{n-1}.
\end{aligned}$$

Thus, this theorem is true. □

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