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A note on the generator subgraph of a graph

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Abstract

Graphs considered in this paper are finite simple undirected graphs. Let $G = (V(G), E(G))$ be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$, for some positive integer m. The *edge space* of G, denoted by $\mathscr{E}(G)$, is a vector space over the field $\mathbb{Z}_2$. The elements of $\mathscr{E}(G)$ are all the subsets of $E(G)$. Vector addition is defined as $X + Y = X \Delta Y$, the symmetric difference of sets X and Y, for $X, Y \in \mathscr{E}(G)$. Scalar multiplication is defined as $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for $X \in \mathscr{E}(G)$. Let H be a subgraph of G. The *uniform set of* H with respect to G, denoted by $E_H(G)$, is the set of all elements of $\mathscr{E}(G)$ that induces a subgraph isomorphic to H. The subspace of $\mathscr{E}(G)$ generated by $E_H(G)$ shall be denoted by $\mathscr{E}_H(G)$. If $E_H(G)$ is a generating set, that is $\mathscr{E}_H(G) = \mathscr{E}(G)$, then H is called a *generator subgraph* of G. This study determines the dimension of subspace generated by the set of all subsets of $E(G)$ with even cardinality and the subspace generated by the set of all k- subsets of $E(G)$, for some positive integer k, $1 \leq k \leq m$. Moreover, this paper determines all the generator subgraphs of star graphs. Furthermore, it gives a characterization for a graph G so that star is a generator subgraph of G.

Keywords: edge-induced subgraph, edge space, even edge space, generator subgraph, uniform set
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1. Introduction

Many studies in graph theory use algebraic structures to develop new classes of graphs. Using the graph properties, the characteristics of the new developed graphs were obtained. This method

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leads to the development of many interesting results. For instance, to mention some, non-abelian group was used in [12], refer to [8] on the use of vector space, and [6] used commutative ring. There are several studies that can be found in the literature that are similar to the previous examples. However, some of them use different algebraic structures.

On the other hand, graph theory has been linked to algebra. An example is the concept of the edge space of a graph. The edge space of a graph G is a vector space over the field $\mathbb{Z}_2 = \{0, 1\}$. Researchers can investigate the subspaces of the edge space generated by some classes of subgraph of a graph. One interesting idea is to find a subgraph H of G such that the set of all subsets of the $E(G)$ that induces a subgraph isomorphic to H spans the edge space of G. Here, H is said to be a generator subgraph of G. Precise definitions of the edge space and the generator subgraph are given in the next section.

The problem on generator subgraph of a graph was introduced by Gervacio in 2008. Most of the previous researches on this problem focused on the determination and characterization of generator subgraphs of a particular graph. These graphs include path, cycle, complete graph [2], complete bipartite graph [10], fan, and wheel [5] graphs. It can be noted that among the special graphs being studied, only the generator subgraphs of the complete graphs were completely known.

Prior to the introduction of the generator subgraph problem, Gervacio and Mame [4], introduced the universal and primitive graphs. The study focused on the determination whether the given graph G is a universal graph or a primitive graph. It is related to the problem on generator subgraphs in the sense that the term universal graphs later became the generator graphs described in [2], and at present called the generator subgraph of complete graphs [3]. A characterization of the primitive graphs was found. There is no characterization for universal graphs but one significant result found was a necessary condition for universal graphs. It was shown that if G is universal then the size of G is odd. This result gives rise to the fundamental theorem on generator subgraph that any generator subgraph has an odd number of edges. Since then, in identifying generator subgraphs of a graph G, we only consider the subgraphs with odd number of edges.

The introduction of the generator subgraph of a graph is interesting. It may be applied to some problems to replace ordinary subgraphs so that new results may be obtained. For instance, this can be potentially applied to network science study; see [11].

In this paper, we introduce the even edge space of graph and determine its dimension. Also, we investigate the subspace of the edge space of a graph G generated by the set of all k - subset of the edge set of G. And, using these results, we determine all the generator subgraphs of stars and give a characterization for an arbitrary graph G so that the star graph is a generator subgraph of a graph G.

Results of this study are useful in determining the generator subgraph of a graph. The introduction of the even edge space of a graph opens a new method for finding the generator subgraph of a graph. In this sense, the authors believed that more results on this problem will be obtained in the future.

Graphs considered in this paper are finite simple undirected graphs. By a graph G, we mean an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite non-empty set of elements called *vertices* and $E(G)$ is a set of $2-$ subset of $V(G)$ whose elements are called *edges*. The sets $V(G)$ and $E(G)$ are called the *vertex set* and *edge set* of G, respectively. The *order* of G is the cardinality of $V(G)$, denoted by $|V(G)|$ and the *size* of G is the cardinality of $E(G)$, denoted by $|E(G)|$.

If $[x, y] \in E(G)$, we say that x is *adjacent* to y or y is *adjacent* to x. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. A mapping $\phi : V(G) \longmapsto V(H)$ is called *isomorphism* if the following conditions are satisfied: (i) ϕ is bijective, (ii) $[a, b] \in E(G)$ implies $[\phi(a), \phi(b)] \in E(H)$, and (iii) $[c, d] \in E(H)$ implies $[\phi^{-1}(c), \phi^{-1}(d)] \in E(G)$. Graph G_1 is *isomorphic* to a graph G_2, written as $G_1 \simeq G_2$, if there exists an isomorphism $\phi : V(G_1) \longmapsto V(G_2)$. The *degree* of a vertex x in a graph G is the number of edges incident with x and denoted by $\deg(x)$. A vertex in a graph is called an *isolated vertex* if its degree is zero. Sometimes isolated vertex is called a *trivial component* of a graph. A vertex in a graph with degree 1 is called a *pendant vertex* while an edge of the graph incident to a pendant vertex is called *pendant edge*. We use the usual notations for some special classes of graphs, K_n for complete graph of order n, P_n for path of order n, and S_n for star graph of order $n + 1$.

Other terms in graph theory whose definitions are not given here may be found in several graph theory books, e.g. Chartrand and Zhang [1]. For the vector spaces, reader may refer to the book written by E.D. Nering [9].

2. Preliminaries

Here, we give the definition of the edge space of a graph and the generator subgraph of a graph and discuss some of their properties. Some known results are also included in this section.

Let $G = (V(G), E(G))$ be a graph with $E(G) = \{e_1, e_2, \dots, e_m\}$, for some positive integer m. The *edge space* of G, denoted by $\mathscr{E}(G)$, is a vector space over the field $\mathbb{Z}_2 = \{0, 1\}$. The elements of $\mathscr{E}(G)$ are all the subsets of $E(G)$. Vector addition is defined as $X + Y = X \,\Delta\, Y$, the symmetric difference of sets X and Y, for $X, Y \in \mathscr{E}(G)$. Scalar multiplication is defined as $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for $X \in \mathscr{E}(G)$.

It can be verified that the set $\mathscr{A} = \{\{e_1\}, \{e_2\}, \dots, \{e_m\}\}$ forms a basis of $\mathscr{E}(G)$. Hence, $\dim \mathscr{E}(G) = m$, the size of G. Valdez, Gervacio and Bengo [5] called this set the natural basis for the edge space of G.

For a non-empty set $X \subseteq E(G)$, the smallest subgraph of G with edge set X is called the edge-induced subgraph of G, which we denote by $G[X]$. In this paper, when we say induced subgraph, we mean an edge-induced subgraph of a graph.

Let H be a subgraph of G. The *uniform set of* H with respect to G, denoted by $E_H(G)$, is the set of all elements of $\mathscr{E}(G)$ that induces a subgraph isomorphic to H. The subspace of $\mathscr{E}(G)$ generated by $E_H(G)$ shall be denoted by $\mathscr{E}_H(G)$. If $E_H(G)$ is a generating set, that is $\mathscr{E}_H(G) = \mathscr{E}(G)$, then H is called a *generator subgraph* of G.

Clearly, $\mathscr{E}_H(G) \subseteq \mathscr{E}(G)$. To show that a subgraph H is a generator subgraph of G, it is sufficient to show that $\mathscr{E}(G) \subseteq \mathscr{E}_H(G)$. That is, the basis $\{\{e_1\}, \{e_2\}, \dots, \{e_m\}\} \subseteq \mathscr{E}_H(G)$. Equivalently, we have the following remark.

Remark 2.1. Let H be a subgraph of G. Then H is a generator subgraph of G if and only if for every $e \in E(G)$ the singleton $\{e\} \in \mathscr{E}_H(G)$.

For example, let $G = K_4$, a complete graph of order 4, where $E(K_4) = \{e_1, e_2, \dots, e_6\}$ as shown in Figure 1. Let $H = P_4$, a path of order 4. We show that P_4 is a generator subgraph of K_4.

First, we identify the elements of $E_{P_4}(K_4)$. Let $A_1 = \{e_2, e_4, e_5\}$. Then $A_1 \in E_{P_4}(K_4)$ since $G[A_1]$ is isomorphic to P_4, as shown in Figure 2.

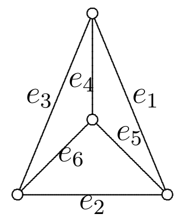

Figure 1. The labeling of K_4

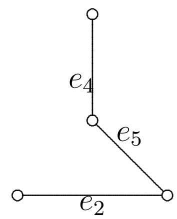

Figure 2. The graph $G[A_1]$

By enumerating all the elements of $E_{P_4}(K_4)$, we have the following:

$$
\begin{array}{llll}
A_1 = & \{e_2, e_4, e_5\}; & A_7 = & \{e_3, e_4, e_5\} \\
A_2 = & \{e_2, e_4, e_6\}; & A_8 = & \{e_1, e_3, e_6\} \\
A_3 = & \{e_1, e_2, e_6\}; & A_9 = & \{e_2, e_3, e_4\} \\
A_4 = & \{e_2, e_3, e_5\}; & A_{10} = & \{e_1, e_2, e_4\} \\
A_5 = & \{e_1, e_5, e_6\}; & A_{11} = & \{e_1, e_4, e_6\} \\
A_6 = & \{e_3, e_5, e_6\}; & A_{12} = & \{e_1, e_3, e_5\}
\end{array}
$$

Next, we show that each singleton is an element of $\mathscr{E}_{P_4}(K_4)$. By trial and error, we have

$$
\begin{aligned}
A_1 + A_2 + A_5 &= (A_1 + A_2) + A_5 \\
&= (A_1 \Delta A_2) \Delta A_5 \\
&= (\{e_2, e_4, e_5\} \Delta \{e_2, e_4, e_6\}) \Delta \{e_1, e_5, e_6\} \\
&= \{e_5, e_6\} \Delta \{e_1, e_5, e_6\} \\
&= \{e_1\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\{e_2\} &= A_2 + A_6 + A_7 \\
\{e_3\} &= A_5 + A_7 + A_{11} \\
\{e_4\} &= A_2 + A_4 + A_6 \\
\{e_5\} &= A_8 + A_{11} + A_{12} \\
\{e_6\} &= A_1 + A_5 + A_{10}
\end{aligned}
$$

This shows that P_4 is a generator subgraph of K_4 by Remark 2.1.

2.1. Some Known Results

Here, we give some useful results which will be applied in the succeeding sections.

Theorem 2.1 (Gervacio, [2]). *Let H be a subgraph of the graph G. If H is a generator subgraph of G, then $|E(H)|$ is odd.*

For a nonempty graph G and considering the path P_2, it can be observed that $E_{P_2}(G)$ is precisely the set of all singletons in $\mathscr{E}(G)$, which is a basis of $\mathscr{E}(G)$. Consequently, we have the following theorem.

Theorem 2.2. *Let G be a graph with $|E(G)| = m > 0$. Then the path P_2 is a generator subgraph of G.*

Let G be a graph and consider a subgraph H of G that contain an isolated vertex. It is obvious that $E_H(G) = \emptyset$. Thus, $\mathscr{E}_H(G) = \emptyset$. A useful remark is stated below.

Remark 2.2. If H is a generator subgraph of G, then H contains no isolated vertex.

The next theorem is equivalent to the known theorem in linear algebra about dimension of a subspace of a vector space over a field.

Theorem 2.3. *Let G be a graph with $|E(G)| = m$. If H is a generator subgraph of G, then $|E_H(G)| \geq m$.*

The converse of the above theorem is not true. For instance, let $G = W_4$, a wheel of order 5 and $H = S_3$, a star graph of order 4. It can be shown that $|E_{S_3}(W_4)| = 8 = \dim \mathscr{E}(W_4)$. It can be verified that the subspace generated by $E_{S_3}(W_n)$ has dimension 7. Hence, $E_{S_3}(W_n)$ does not span $\mathscr{E}(W_4)$ so S_3 is not a generator subgraph of W_4.

3. Even Edge Space of a Graph

By $\mathscr{E}^*(G)$, we mean the set of all subsets of $E(G)$ with even cardinality. The first result gives a relation between $\mathscr{E}^*(G)$ and $\mathscr{E}(G)$.

Theorem 3.1. *Let G be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$. Then $\mathscr{E}^*(G)$ is a subspace of $\mathscr{E}(G)$. Moreover, $\dim \mathscr{E}^*(G) = m - 1$.*

Proof. Clearly, $\mathscr{E}^*(G)$ is a subset of $\mathscr{E}(G)$ and $\mathscr{E}^*(G)$ is not empty since $\emptyset \in \mathscr{E}^*(G)$. Let $X_1, X_2 \in \mathscr{E}^*(G)$, then $X_1 + X_2 \in \mathscr{E}^*(G)$ since $|X_1 + X_2| = |X_1 \Delta X_2| = |X_1| + |X_2| - 2|X_1 \cap X_2|$ is even. Further, let $c \in \mathbb{Z}_2$ and $X \in \mathscr{E}^*(G)$, then either $c \cdot X = \emptyset$ or $c \cdot X = X$. In both cases, $|c \cdot X|$ is even so $c \cdot X \in \mathscr{E}^*(G)$. Hence, $\mathscr{E}^*(G)$ is a subspace of $\mathscr{E}(G)$.

Now, we find the dimension of $\mathscr{E}^*(G)$. Let $\mathscr{E}'(G) = \{X \in \mathscr{E}(G) : |X| \text{ is odd}\}$. We know that $\mathscr{E}(G)$ is the power set of a non-empty set $E(G)$. Klasar [7] showed that if S is a non-empty set and $\mathscr{P}(S)$ is the power set of S then the number of elements of $\mathscr{P}(S)$ with even cardinality is equal to the number of elements of $\mathscr{P}(S)$ with odd cardinality. Thus, $|\mathscr{E}^*(G)| = |\mathscr{E}'(G)| = \frac{1}{2}|\mathscr{E}(G)| =$

2^{m-1}. Now, let $\dim \mathscr{E}^*(G) = k$ and let $\mathscr{B} = \{X_1, X_2, \ldots, X_k\}$ be a basis for $\mathscr{E}^*(G)$. Then any vector $X \in \mathscr{E}^*(G)$ is of the form

$$c_1X_1 + c_2X_2 + \ldots + c_kX_k$$

and, since $\mathscr{B}$ is linearly independent, the coefficients c_i are uniquely determined by X. Since c_i is either 0 or 1 for each i, the total number of vectors in $\mathscr{E}^*(G)$ must be 2^k. Since $|\mathscr{E}^*(G)| = 2^{m-1}$, it follows that $k = m - 1$. □

In this paper, we shall call the vector space $\mathscr{E}^*(G)$ the *even edge space* of a graph G.

The following remark is a known result in linear algebra.

Remark 3.1. If $A \subseteq \mathscr{E}^*(G)$, then the set of all linear combinations of the elements of A is a subspace of $\mathscr{E}^*(G)$.

Consequently, we have the next theorem.

Theorem 3.2. *Let H be a subgraph of G. If $|E(H)|$ is even, then $\mathscr{E}_H(G) \subseteq \mathscr{E}^*(G)$.*

Proof. Since $|E(H)|$ is even, each $A \in E_H(G)$ has even cardinality. Thus $E_H(G) \subseteq \mathscr{E}^*(G)$. By Remark 3.1, $\mathscr{E}_H(G) \subseteq \mathscr{E}^*(G)$. □

We now identify a basis for $\mathscr{E}^*(G)$. Let G be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$ and define $\mathscr{B} = \{X_1, X_2, \ldots, X_{m-1}\}$, where $X_1 = \{e_1, e_2\}, X_2 = \{e_1, e_3\}, \ldots, X_{m-1} = \{e_1, e_m\}$. Since $X \in \mathscr{E}^*(G)$ can be expressed as a union of disjoint sets $\{e_i, e_j\} = \{e_1, e_i\}\Delta\{e_1, e_j\}$, where $1 \le i, j \le m$, then $\mathscr{B}$ spans $\mathscr{E}^*(G)$. Since $|\mathscr{B}| = m - 1 = \dim \mathscr{E}^*(G)$, it follows that $\mathscr{B}$ forms a basis for $\mathscr{E}^*(G)$.

It is easily seen that $\mathscr{E}^*(G)$ is a maximal proper subspace of $\mathscr{E}(G)$.

4. Subspace generated by the set of all k - subset of the edge set of a graph

As we know from the previous discussions, given a subgraph H of a graph G, the uniform set $E_H(G)$ contains sets with the same cardinality- the size of H. Thus, in the study of generator subgraph, it is worth investigating the subspace of $\mathscr{E}(G)$ generated by the set of all subsets of $E(G)$ with exactly k elements, where k is a positive integer.

By a k-subset of $E(G)$, we mean a subset of $E(G)$ containing exactly k elements. Here, we determine the dimension of the subspace of $\mathscr{E}(G)$ generated by the set of all k-subsets of $E(G)$.

For convenience, we give the following definition.

Definition 1. *Let G be graph with $m > 0$ edges. For a positive integer k, denote by $E_k(G)$ the set of all k-subsets of $E(G)$ and let $\mathscr{E}_k(G)$ denote the subspace of $\mathscr{E}(G)$ generated by $E_k(G)$.*

For instance, let G be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$ for some positive integer m. Then $E_1(G) = \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$. Note that $E_1(G)$ is the natural basis for $\mathscr{E}(G)$ so $\mathscr{E}_1(G) = \mathscr{E}(G)$. Thus, $\dim \mathscr{E}_1(G) = m$. The set $E_m(G)$ contains exactly one element, the edge set of G. Since $E(G)$ is non-empty, $\dim \mathscr{E}_m(G) = 1$.

The following result shows the relation between $\mathscr{E}_k(G)$ and $\mathscr{E}^*(G)$.

Lemma 4.1. *Let G be a graph with size $m > 0$ and let k be a positive integer where $1 \leq k \leq m-1$. Then $\mathscr{E}^*(G) \subseteq \mathscr{E}_k(G)$.*

Proof. Let G be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let k be a positive integer where $1 \leq k \leq m-1$. Clearly, $\mathscr{E}^*(G) \subseteq \mathscr{E}_1(G)$ so we may assume that $k > 1$. Let e_i be an element of $E(G)$ for some i, $1 \leq i \leq m$. Let $A \in E_k(G)$ such that $e_i \in A$. Since $k < m$, there exists $e_j \in E(G)$ such that $e_j \notin A$ for some j, $1 \leq j \leq m$ and $j \neq i$. Define $B = \{e_j\} \cup A \backslash \{e_i\}$. Obviously, $B \in E_k(G)$. Thus, $\{e_i, e_j\} = A \Delta B \in \mathscr{E}_k(G)$. In particular, the set $\mathscr{B} = \{\{e_1, e_2\}, \{e_1, e_3\}, \ldots, \{e_1, e_m\}\}$ is a subset of $\mathscr{E}_k(G)$. Since $\mathscr{B}$ forms a basis for $\mathscr{E}^*(G)$, it follows that $\mathscr{E}^*(G) \subseteq \mathscr{E}_k(G)$. □

The next result gives the dimension of $\mathscr{E}_k(G)$ for all values of k.

Theorem 4.1. *Let G be a graph with size $m > 0$ and let k be a positive integer where $1 \leq k \leq m$. Then*

$$\dim \mathscr{E}_k(G) = \begin{cases} 1 & \text{if } k = m, \\ m-1 & \text{if } k \text{ is even, and} \\ m & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let k be an integer where $1 \leq k \leq m$. We know earlier that $\dim \mathscr{E}_k(G) = 1$ if $k = m$ and $\dim \mathscr{E}_k(G) = m$ if $k = 1$. We now assume that $1 < k \leq m-1$. Consider the two cases: Case 1, k is even. Then $E_k(G)$ consists of sets with even cardinality. Thus, $\mathscr{E}_k(G) \subseteq \mathscr{E}^*(G)$ in view of Remark 3.1. By Lemma 4.1, $\mathscr{E}^*(G) \subseteq \mathscr{E}_k(G)$. Therefore $\mathscr{E}_k(G) = \mathscr{E}^*(G)$. It follows that $\dim \mathscr{E}_k(G) = m-1$. Case 2, k is odd. Let $e_i \in E(G)$, $1 \leq i \leq m$. Then there exists $A \in E_k(G)$ such that $e_i \in A$. Define $B = A \backslash \{e_i\}$. Since $|A| = k$ is odd, $|B|$ is even so $B \in \mathscr{E}^*(G)$. By Lemma 4.1, $B \in \mathscr{E}_k(G)$. Now, $\{e_i\} = A \Delta B \in \mathscr{E}_k(G)$. Meaning, $E_k(G)$ is a generating set for $\mathscr{E}(G)$. Hence, $\mathscr{E}(G) \subseteq \mathscr{E}_k(G)$. But we know that $\mathscr{E}_k(G) \subseteq \mathscr{E}(G)$. Therefore $\mathscr{E}_k(G) = \mathscr{E}(G)$. It follows that $\dim \mathscr{E}_k(G) = m$.

□

The next result determines another basis for the edge space of G.

Theorem 4.2. *Let G be a graph with size $m > 0$. If m is even, then the set $E_{m-1}(G)$ forms a basis for $\mathscr{E}(G)$.*

Proof. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let $A_i = E(G) \backslash \{e_i\}$ where $1 \leq i \leq m$. Then $E_{m-1}(G) = \{A_1, A_2, \ldots, A_m\}$. Since m is even, $m-1$ is odd. By Lemma 4.1, $\mathscr{E}_{m-1}(G) = \mathscr{E}(G)$. Thus, $E_{m-1}(G)$ spans $\mathscr{E}(G)$. Since $|E_{m-1}(G)| = m = \dim \mathscr{E}(G)$, it follows that $E_{m-1}(G)$ forms a basis for $\mathscr{E}(G)$. □

Corollary 4.1. *Let G be a graph with size $m > 0$. If m is odd, then the set $E_{m-1}(G)$ is a linearly dependent set.*

Proof. Since m is odd, $E_{m-1}(G)$ contains sets with even cardinality. By Theorem 3.2, $\mathscr{E}_{m-1}(G) \subseteq \mathscr{E}^*(G)$. Thus, $\dim \mathscr{E}_{m-1}(G) \leq \dim \mathscr{E}^*(G) = m-1$, in view of Theorem 3.1. We know from the proof of Theorem 4.2 that $|E_{m-1}(G)| = m$. Therefore, $E_{m-1}(G)$ is linearly dependent. □

5. Generator Subgraphs of Stars

There are several classes of graph where its generator subgraphs have not been explored yet. One of these is a star graph. Here, we determine the characterization of the generator subgraphs of star graph. Furthermore, direct applications of the results of the previous sections are shown in the discussion in this part.

By a *star* of order $n+1$, denoted by S_n, we mean a graph which consists of an independent set of n vertices each of which is adjacent to a common vertex called the *central vertex*. The size of S_n is n. Hence $\dim \mathscr{E}(S_n) = n$ and $\dim \mathscr{E}^*(S_n) = n-1$.

Let $E(S_n) = \{e_1, e_2, \ldots, e_n\}$. For a positive integer q, we can view $E_{S_q}(S_n)$ as $E_q(S_n)$, the set of all q-subsets of $E(S_n)$, since for each $A \in E_q(S_n)$, $S_n[A] \simeq S_q$. In fact, it is easy to verify that $E_{S_q}(S_n) = E_q(S_n)$. However, this equality holds only for some graphs.

First we establish a relation between $\mathscr{E}_{S_q}(S_n)$ and $\mathscr{E}^*(S_n)$.

Lemma 5.1. *Let S_q be a subgraph of S_n for some positive integers q and n. If $q < n$, then $\mathscr{E}^*(S_n) \subseteq \mathscr{E}_{S_q}(S_n)$.*

Proof. Let S_q be a subgraph of S_n where $q < n$. We know earlier that $\mathscr{E}_{S_q}(S_n) = \mathscr{E}_q(S_n)$. Thus, by Lemma 4.1, $\mathscr{E}^*(S_n) \subseteq \mathscr{E}_{S_q}(S_n)$. □

The next theorem gives a family of generator subgraphs of S_n.

Theorem 5.1. *For positive integers q and n where $q < n$, the star S_q is a generator subgraph of S_n if and only if q is odd.*

Proof. The necessary condition of the theorem follows directly from Theorem 2.1. Conversely, assume that q is odd. We know that $\mathscr{E}_{S_q}(S_n) = \mathscr{E}_q(S_n)$. Thus, by Theorem 4.1, $\dim \mathscr{E}_{S_q}(S_n) = n = \dim \mathscr{E}(S_n)$. Hence, $\mathscr{E}_{S_q}(S_n) = \mathscr{E}(S_n)$. Therefore S_q is a generator subgraph of S_n. □

The following theorem is a special case of Theorem 4.1.

Theorem 5.2. *Let S_q be a subgraph of S_n for some positive integers q and n where $q < n$. If q is even, then* $\dim \mathscr{E}_{S_q}(S_n) = n-1$.

The next result determines the dimension of the subspace generated by the uniform sets of the subgraphs of star S_n.

Theorem 5.3. *Let H be a subgraph of S_n, $n > 0$. If H contains an isolated vertex then* $\dim \mathscr{E}_H(S_n) = 0$. *Moreover, if H does not contain an isolated vertex, then*

$$\dim \mathscr{E}_H(S_n) = \begin{cases} 1 & \text{if } |E(H)| = n, \\ n-1 & \text{if } |E(H)| \text{ is even, and} \\ n & \text{if } |E(H)| \text{ is odd.} \end{cases}$$

Proof. Let H be a subgraph of S_n. Then either H contains an isolated vertex or H does not contain an isolated vertex. Suppose H contains an isolated vertex, then $E_H(S_n) = \emptyset$ in view of Remark 2.2. It follows that $\dim \mathscr{E}_H(S_n) = 0$. If H does not contain an isolated vertex, then $H \simeq S_q$ for some positive integer q where $1 \le q \le n$. Consider the following three cases: Case 1, $1 \le q < n$ and q is odd. By Theorem 5.1, H is a generator subgraph of S_n so $\dim \mathscr{E}_H(S_n) = n$. Case 2, $1 \le q < n$ and q is even. By Theorem 5.2, $\dim \mathscr{E}_H(S_n) = n-1$. Case 3, $q = n$. Then $E_H(S_n)$ contains exactly one element, the edge set of S_n. Hence, $\dim \mathscr{E}_H(S_n) = 1$. □

6. Star as a Generator Subgraph of Some Graphs

This section determines some properties of graphs wherein star is one of its generator subgraphs.

Theorem 6.1. *Let $p > 0$ be an odd integer. If G is a graph such that for every edge $[a, b]$ in G either $\deg(a) > p$ or $\deg(b) > p$, then star S_p is a generator subgraph of G.*

Proof. Let $[a, b]$ be an edge of G. We show that $\{[a, b]\} \in \mathscr{E}_{S_p}(G)$. Without loss of generality, assume that $\deg(a) = r > p$ for some integer r. Let $A = \{e_1, e_2, \ldots, e_r\}$ be the set of all edges in G incident with a. Let $B \subseteq A$ with $|B| = p$. Then $G[A] \simeq S_r$ and $G[B] \simeq S_p$. Since p is odd, $G[B]$ is a generator subgraph of $G[A]$ in view of Theorem 5.1. Thus, $\{e_i\} \in \mathscr{E}_{S_p}(G[A]) \subseteq \mathscr{E}_{S_p}(G)$ for all i, $1 \leq i \leq r$. Since $[a, b]$ is one of the $e_i's$, it follows that $\{[a, b]\} \in \mathscr{E}_{S_p}(G)$. Therefore S_p is a generator subgraph of G. □

Below is an immediate consequence of Theorem 6.1.

Corollary 6.1. *Let $p > 0$ be odd. If G is k- regular and $k > p$ then star S_p is a generator subgraph of G.*

The converse of Theorem 6.1 is not true for $p = 1$ since a star $S_1 \simeq P_2$ is a generator subgraph of the graph $G = kP_2$, a graph consisting of k vertex-disjoint copies of P_2. If $p \neq 1$, we have the following result.

Theorem 6.2. *Let $p > 1$ be odd. Then S_p is a generator subgraph of G if and only if for every edge $[a, b]$ in G, either $\deg(a) > p$ or $\deg(b) > p$.*

Proof. Assume that S_p is a generator subgraph of G. Suppose, on the contrary, $\deg(a) \leq p$ and $\deg(b) \leq p$ for some $[a, b] \in E(G)$. Partition $E(G)$ into two sets A and B where $A = \{[a, b] \in E(G) : \deg(a) \leq p \ and \ \deg(b) \leq p\}$ and $B = \{[a, b] \in E(G) : \deg(a) > p \ or \ \deg(b) > p\}$. Clearly, $E_{S_p}(G[A]) \cap E_{S_p}(G[B]) = \emptyset$ and $E_{S_p}(G) = E_{S_p}(G[A]) \cup E_{S_p}(G[B])$. Now, let us consider the subgraph $G[A]$. Partition $V(G[A])$ into two sets X and Y where $X = \{x \in V(G[A]) : \deg(x) = p\}$ and $Y = \{y \in V(G[A]) : \deg(y) < p\}$. Observe that $|E_{S_p}(G[A])| = |X|$ and $|X|$ is maximum if $Y = \emptyset$. Let us assume that $Y = \emptyset$. Then $G[A]$ is p-regular. Thus, $\sum_{v \in V(G[X])} \deg(v) = p|X| = 2|E(G[A])|$. Since $p > 1$ is odd, $|X| = |E_{S_p}(G[A])| < |E(G[A])| = \dim \mathscr{E}(G[A])$. By Theorem 2.3, S_p is not a generator subgraph of $G[A]$. Meaning, there exists $e \in E(G[A]) \subseteq E(G)$ such that $\{e\} \notin \mathscr{E}_{S_p}(G[A])$. It follows that $\{e\} \notin \mathscr{E}_{S_p}(G)$. This is a contradiction to the fact that S_p is a generator subgraph of G. Therefore, for every edge $[a, b]$ in G, either $\deg(a) > p$ or $\deg(b) > p$. For the converse of the theorem, it follows by Theorem 6.1. □

The following result determines all graphs whose generator subgraph is the path P_2 only.

Theorem 6.3. *Let G be a graph with size $m > 0$. If $m \leq 3$, then the only generator subgraph of G is the path P_2.*

Proof. Let G be a graph with size m where $1 \leq m \leq 3$. We know by Theorem 2.2 that P_2 is a generator subgraph of G. Suppose there exists another generator subgraph of G, say H. Then $1 \leq |E(H)| \leq 3$. By Theorem 2.1, $|E(H)|$ is odd. Thus, either $|E(H)| = 1$ or $|E(H)| = 3$. Suppose $|E(H)| \neq 1$, then $|E(H)| = 3$. This implies that the size of G is 3. Hence, $E_H(G) = \{E(G)\}$. It follows that $\dim \mathscr{E}_H(G) = 1 < 3 = \dim \mathscr{E}(G)$. This is a contradiction to Theorem 2.3. Therefore $|E(H)| = 1$. But H does not contain isolated vertex by Remark 2.2. It follows that H is isomorphic to P_2. □

Equivalently, we have the following remark.

Remark 6.1. Let G be a graph with size m. If G has a generator subgraph which is not isomorphic to P_2, then $m \geq 4$.

7. Summary and Conclusions

The even edge space was introduced in this paper and found to be a maximal proper subspace of the edge space of a graph. This leads to a new method of finding the generator subgraph of a graph. Instead of expressing each singleton of the edge set of graph G as a linear combination of the elements of the uniform set of a subgraph H, one can show that the even edge space is a subset of the vector space generated by the uniform set of H. In addition, the dimension of the subspace generated by the set of all $k-$ subsets was identified. This result is useful in determining the bounds for the dimension of the subspace generated by the uniform set of a subgraph. Moreover, all generator subgraphs of star graphs were identified and a characterization for a graph G so that star graph is a generator subgraph of G was established. The characterization for the generator subgraph of a graph is still open. Interested researchers may focus on the generator subgraphs of some classes of graph. Moreover, it may be interesting to see if there is any upper bound on the order of generator subgraphs in random graphs.

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