Exponent-critical primitive graphs and 
the Kronecker product

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Abstract

A directed graph is primitive of exponent \( t \) if it contains walks of length \( t \) between all pairs of vertices, and \( t \) is minimal with this property. Moreover, it is exponent-critical if the deletion of any arc results in an imprimitive graph or in a primitive graph with strictly greater exponent. We establish necessary and sufficient conditions for the Kronecker product of a pair of graphs to be exponent-critical of prescribed exponent, defining some refinements of the concept of exponent-criticality in the process.

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1. Introduction

A directed graph \( \Gamma \) is called primitive if there exists a positive integer \( t \) with the property that for all vertices \( u \) and \( v \) of \( \Gamma \), there exists a walk of length \( t \) from \( u \) to \( v \) in \( \Gamma \). The least such \( t \) is called the exponent of \( \Gamma \). This article is concerned with primitive graphs that are exponent-critical in the sense that deletion of any arc would result either in a primitive graph with increased exponent or in an imprimitive graph. In particular, we consider the behaviour of this critical property and some variants under the Kronecker product of graphs.

This article extends some work that is reported in the 2017 PhD thesis [12] of the first author, whose theme is the study of finite edge-minimal undirected graphs of exponent 2, referred to as

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me_2-graphs. In an undirected me_2-graph, every pair of distinct vertices has at least one neighbour, but this property does not survive deletion of an edge. A motivation for the study of this property is provided by the famous Erdős-Rényi-Sós Theorem or Friendship Theorem [3], which classifies finite undirected graphs with the friendship property, namely that every pair of distinct vertices has exactly one shared neighbour. The Friendship Theorem states that a finite graph with this property is a union of triangles that all share a single vertex and are otherwise disjoint, often referred to as a windmill. For undirected graphs, the friendship property may be expressed as the statement that there is a unique walk of length 2 from every vertex to every other. In a me_2-graph, it is not necessarily the case every pair of distinct vertices is connected by a unique walk of length 2, but such pairs are sufficiently abundant that every edge is involved in the unique walk of length 2 between some pair of vertices. Thus the undirected me_2-property may be regarded as a generalization of the friendship property. It is established in [12] that undirected me_2-graphs are plentiful, for example in the sense that every graph of order n is an induced subgraph of a me_2-graph of order at most 3n + 2. A detailed analysis of embeddings of trees as induced subgraphs of me_2-graphs is given in [13].

Considerable effort has been devoted to extending the Friendship Theorem in numerous directions. The formulation of the friendship property in terms of unique paths of length 2 has the advantages of admitting an obvious adaptation to the directed setting, and a natural extension to primitive graphs of exponent greater than 2. The problem of classifying directed graphs of order n in which there is a unique directed walk of length t from u to v for every pair of distinct vertices u and v was first investigated by Lam and Van Lint in [8], with the additional stipulation that the graph contains no closed walk of length t. A graph of this type has exponent t + 1 (provided that n ≥ 3) and need not be exponent-critical, as the examples in [8] demonstrate. Such a graph has adjacency matrix A satisfying A^t = J − I_n, where J is the n × n matrix whose entries are all equal to 1. Results on the identification and classification of (0, 1)-matrices satisfying this equation, and their corresponding graphs, can be found in [15] and [17] (for example).

The me_t-property, which like the friendship property was first formulated for undirected graphs of exponent 2, also admits straightforward adaptations to the directed context and to arbitrary exponent. We will abbreviate the property of being exponent-critical of exponent t as the me_t-property. A directed graph Γ of exponent t has the me_t-property if for every arc e of Γ there is a pair (u, v) of (not necessarily distinct) vertices such that every walk of length t from u to v in Γ includes the arc e. Thus the deletion of any arc from Γ results in a graph that does not have exponent t. This does not necessarily require that all or many pairs of vertices are connected by unique walks of length t, or that every arc belongs to a such a unique walk, but it requires a prevalence of pairs (u, v) without arc-disjoint t-walks from u to v. The adjacency matrix A of a me_t-graph is a (0, 1)-matrix for which A^t is positive, but B^t has a zero entry for every matrix B obtained from A by replacing a single 1 with a zero.

Kim, Song and Hwang determine the least possible number of edges in a primitive undirected graph of specified order and exponent in [6] and [7], and provide a corresponding analysis for directed graphs in the case of exponent 2. In many cases they identify all graphs in which these minima are attained, which obviously belong to the general class of exponent-critical (directed or undirected) primitive graphs. In the case of undirected graphs of exponent 2 and odd order n, they show that the minimum possible number of edges is \( \frac{3}{2}(n - 1) \) and that this minimum is uniquely
attained by the windmill on \( n \) vertices. Thus the work of Kim et al. connects to the Friendship Theorem in a natural way.

Another familiar theme to which the concept of exponent-criticality is related is the analogous property for diameter. The diameter of a (strongly) connected graph \( G \) is the minimum over all ordered pairs \( (u, v) \) of vertices in \( G \) of the length of the shortest path from \( u \) to \( v \) in \( G \). A connected graph is called diameter-critical if the deletion of any arc either disconnects the graph or leaves a graph of strictly higher diameter. The study of diameter-critical graphs originated in the 1960s (see for example [14], [11]) and has been a subject of the attention of numerous authors, mostly in the undirected setting. An up to date summary of the literature on this general topic can be found in a recent survey by Haynes et al [4].

The theme of this article is exponent-criticality for finite primitive graphs and its behaviour under the Kronecker product. The article is organised as follows. Section 2 provides relevant background information on primitivity and on the Kronecker product of graphs. In Section 3 we define a minimally primitive graph and characterize minimally primitive Kronecker products. The main technical content is in Section 4, which discusses conditions under which the Kronecker product of a pair of graphs is exponent-critical, and in the process introduces some refinements of the key property, of possible independent interest. In Section 5 we specialize some of the results of Section 4 to the relatively uncomplicated case of exponent 2.

We use the following terminology, notation and conventions. Graphs are assumed to be finite and are generally considered to be directed, except in Section 5 which includes a discussion of undirected graphs of exponent 2. In a directed graph, an arc is an ordered pair of vertices, respectively referred to as the initial and terminal vertices. We only consider graphs without loops, which means that the initial and terminal vertices of an arc are always distinct. When there is a need to specify the initial and terminal vertices, we will write the arc \( e = (u, v) \) as \( u \rightarrow v \) or as \( e : u \rightarrow v \).

An undirected graph may be considered to be a directed graph in which \( v \rightarrow u \) is an arc whenever \( u \rightarrow v \) is an arc and consider edges to be unordered pairs of vertices, writing \( e = uv \) where necessary to indicate that the edge \( e \) consists of the vertices \( u \) and \( v \). The degree of a vertex in an undirected graph is the number of edges incident with that vertex. In an directed graph, the outdegree and indegree of a vertex \( v \) are respectively the number of arcs having \( v \) as initial or terminal vertex. If \( e \) is an arc (or an edge) in a graph \( \Gamma \), then \( \Gamma \setminus e \) denotes the graph obtained from \( \Gamma \) by deleting \( e \).

A walk of length \( k \) from \( u \) to \( v \) in a graph \( G \) is a sequence \( u_0, \ldots, u_k \) of vertices, where \( u_0 = u, u_k = v \) and \( u_i \rightarrow u_{i+1} \) is an arc of \( G \) for \( i = 0, \ldots, k - 1 \). A walk of length \( k \) involves \( k \) arcs and is referred to as a \( k \)-walk. The length of a walk \( P \) in a graph \( \Gamma \) is denoted \( l_{\Gamma}(P) \). A path is a walk in which no vertex appears more than once. A circuit is a walk in which the first and last vertices coincide. A cycle is a circuit in which the only repetition of vertices is that the first and last coincide. A directed graph is strongly connected if it possesses a walk from \( u \) to \( v \) for every pair \( (u, v) \) of vertices, and minimally strongly connected if this property does not survive the deletion of any arc. The distance from \( u \) to \( v \) in a graph \( \Gamma \) is the length of a shortest walk from \( u \) to \( v \) in \( \Gamma \), denoted \( d_{\Gamma}(u, v) \).
2. Primitivity and Kronecker Products

**Definition 2.1.** Let $\Gamma_1$ and $\Gamma_2$ be directed graphs. The **Kronecker product** $\Gamma_1 \otimes \Gamma_2$ is the directed graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$, in which $(u, x) \rightarrow (v, y)$ is an arc if and only if $u \rightarrow v$ is an arc in $\Gamma_1$ and $x \rightarrow y$ is an arc in $\Gamma_2$.

The Kronecker product of graphs is also known by other names, including the direct product and tensor product; it corresponds to the matrix Kronecker product when graph data are encoded in adjacency matrices. If $e : u \rightarrow v$ and $e : x \rightarrow y$ are arcs of $\Gamma_1$ and $\Gamma_2$, we will when convenient denote the arc $(u, x) \rightarrow (v, y)$ of $\Gamma_1 \otimes \Gamma_2$ by $(e, f)$.

**Definition 2.2.** A directed graph $\Gamma$ is **primitive** if there is a positive integer $t$ with the property that given any vertices $x$ and $y$ in $\Gamma$ (not necessarily distinct), there is a directed walk of length $t$ from $x$ to $y$ in $\Gamma$. The least $t$ for which this holds is called the **exponent** of $\Gamma$, denoted $\exp(\Gamma)$.

We mention here some elements of the theory of primitive graphs, and refer to [1] for a detailed discussion. Clearly a primitive graph must be strongly connected. The converse is not true, as demonstrated for any integer $n \geq 2$ by the graph consisting of a single directed cycle of length $n$. Let $A$ be the adjacency matrix of a directed graph $\Gamma$. For a positive integer $t$, the entries of $A^t$ count directed walks between pairs of vertices in $\Gamma$; thus $\Gamma$ is primitive if and only if $A^t$ is positive for some $t$, and the least $t$ for which this occurs is the exponent of $\Gamma$.

If $\Gamma$ is a primitive graph of exponent $t$ with vertices $u$ and $v$, then there is a walk of length $k$ from $u$ to $v$ for every integer $k$ with $k \geq t$; this is equivalent to the observation that if $A^t$ is positive for some non-negative matrix $A$, then all subsequent powers of $A$ are positive also.

A graph is primitive if and only if it is strongly connected and the greatest common divisor of the lengths of its directed circuits (or equivalently cycles) is 1. The greatest common divisor of the lengths of the directed circuits of a strongly connected graph $\Gamma$ is called the **index of imprimitivity** of $\Gamma$, denoted by $\mu(\Gamma)$. If $u$ and $v$ are vertices of $\Gamma$ and $P_1$ and $P_2$ are directed walks from $u$ to $v$ in $\Gamma$, then by concatenating both $P_1$ and $P_2$ with a walk from $v$ to $u$ we can observe that the lengths of $P_1$ and $P_2$ are congruent modulo $\mu(\Gamma)$. If $\mu(\Gamma) > 1$ it is then immediate that $\Gamma$ cannot be primitive, since there is no $t$ with the property that $\Gamma$ contains a walk of length $t$ from $u$ to $v$ for all $k \geq t$. Moreover, given vertices $u$ and $v$, there exists an integer $N_{uv}$ with the property that every integer that exceeds $N_{uv}$ and is congruent modulo $\mu(\Gamma)$ to $d_\Gamma(u, v)$ occurs as the length of some $\Gamma$-walk from $u$ to $v$.

The characterization of primitivity in terms of cycle lengths takes a particularly simple form for undirected graphs. Since every (non-null) undirected graph possesses 2-cycles, an undirected graph (on at least two vertices) is primitive if and only if it possesses a cycle of odd length. Equivalently, an undirected graph is imprimitive if and only if it is bipartite, in which case its index of imprimitivity is 2.

We note the following necessary and sufficient conditions on the factors $\Gamma_1$ and $\Gamma_2$ in order for the Kronecker product $\Gamma_1 \otimes \Gamma_2$ to be strongly regular or primitive.

**Theorem 2.1.** Let $\Gamma_1$ and $\Gamma_2$ be strongly connected graphs. Then

1. $\Gamma_1 \otimes \Gamma_2$ is strongly connected if and only if $\mu(\Gamma_1)$ and $\mu(\Gamma_2)$ are relatively prime.
2. \( \Gamma_1 \otimes \Gamma_2 \) is primitive if and only if both \( \Gamma_1 \) and \( \Gamma_2 \) are primitive. In this case the exponent of \( \Gamma_1 \otimes \Gamma_2 \) is the maximum of the exponents of \( \Gamma_1 \) and \( \Gamma_2 \).

**Proof.** The first item is Theorem 2 of [10]. We do not provide a detailed proof, but remark that the necessity of the condition is clear; let \( d = \gcd(\mu(\Gamma_1), \mu(\Gamma_2)) \) and suppose that \( d > 1 \). There exists a pair \((u, v)\) of vertices of \( \Gamma \) for which \( l_{\Gamma_1}(P_{uv}) \equiv 0 \mod d \) for every walk \( P_{uv} \) from \( u \) to \( v \) in \( \Gamma_1 \), and there exists a pair \((x, y)\) of vertices of \( \Gamma_2 \) for which \( l_{\Gamma_2}(P_{xy}) \equiv 1 \mod d \) for every walk \( P_{xy} \) from \( x \) to \( y \) in \( \Gamma_2 \). Then there is no walk from \((u, x)\) to \((v, y)\) in \( \Gamma_1 \otimes \Gamma_2 \).

The sufficiency of the condition follows from the Chinese Remainder Theorem, along with the fact that every sufficiently large integer in the congruence class of \( d \) is a multiple of either \( \mu(\Gamma_1) \) or \( \mu(\Gamma_2) \), and thus can be paired to produce a walk of length \( t \) from \((u, x)\) to \((v, y)\) in \( \Gamma_1 \otimes \Gamma_2 \).

On the other hand suppose that \( \Gamma_1 \) and \( \Gamma_2 \) are primitive, and let \( t \) be the greater of their exponents. Let \((u, x)\) and \((v, y)\) be vertices of \( \Gamma_1 \otimes \Gamma_2 \). There exist walks of length \( t \) from \( u \) to \( v \) in \( \Gamma_1 \) and from \( x \) to \( y \) in \( \Gamma_2 \), which can be paired to produce a walk of length \( t \) from \((u, x)\) to \((v, y)\) in \( \Gamma_1 \otimes \Gamma_2 \). Thus \( \Gamma_1 \otimes \Gamma_2 \) is primitive of exponent at most \( t \). \( \Box \)

Theorem 2.1 shows that the property of strong connectedness is not preserved by the Kronecker product of directed graphs, whereas the property of primitivity is. For undirected graphs, the first part of Theorem 2.1 amounts to the statement that the Kronecker product of two undirected graphs is connected if and only if both graphs are connected and at least one of them is primitive (or equivalently at most one is bipartite).

3. Minimally primitive graphs

**Definition 3.1.** A directed graph \( \Gamma \) is minimally primitive if \( \Gamma \) is primitive, and the deletion of any arc from \( \Gamma \) leaves an imprimitive graph.

Every arc in a strongly connected directed graph belongs to a directed cycle, and a directed graph is primitive if and only if the greatest common divisor of its cycle lengths is 1. If a graph \( \Gamma \) is minimally primitive, then for every arc \( e : u \to v \) of \( \Gamma \), at least one of the following occurs:

- there is no walk from \( u \) to \( v \) in \( \Gamma \setminus e \);
- there is a prime \( p \) with the property that the length of every cycle of \( \Gamma \) that does not include \( e \) is a multiple of \( p \).

The class of minimally primitive graphs includes all primitive graphs with the property that deletion of an arc always leaves a graph that is not strongly connected. Such graphs are called **minimally strongly connected**, and it is shown in [1] that the maximum possible exponent of a minimally strongly connected primitive graph of order \( n \) is \( n^2 - 4n + 6 \). Examples of minimally primitive graphs that are not minimally strongly connected include the Wielandt graphs. For \( n \geq 3 \), the
maximum possible exponent of a primitive graph of order \( n \) is \( n^2 - 2n + 2 \) and the Wielandt graph \( W_n \) is the unique graph that attains this bound \([16], [5]\). It has \( n + 1 \) arcs and consists of a directed cycle of length \( n \) and an additional arc \( u \to v \), where the distance from \( u \) to \( v \) in the directed \( n \)-cycle is 2. Thus \( W_n \) consists of an \( n \)-cycle and a \((n - 1)\)-cycle that share \( n - 2 \) arcs; it is clearly primitive since \( n \) and \( n - 1 \) are relatively prime. It is minimally primitive, since deletion of the unique chord in the \( n \)-cycle leaves a single cycle of length \( n \), and deletion of any other arc leaves a graph that is not strongly connected.

From any finite primitive graph \( \Gamma \) we may obtain a minimally primitive graph of the same order, by repeating the step of deleting an arc. The outcome of this process depends on the choice of deletion at each step, and its exponent is not determined by \( \Gamma \) and typically exceeds that of \( \Gamma \). The following example shows a primitive graph of order 5 and its two minimally primitive subgraphs of order 5, which have different exponents but have the same number of arcs.

**Example 3.1.**
The graph \( G \) of Figure 1 is primitive of exponent 8. Deletion of the arc \( x_1 \to x_3 \) leaves a graph of exponent 14; deletion of the arc \( x_5 \to x_3 \) leaves the Wielandt graph \( W_5 \) which is primitive of exponent 17, and deletion of any other arc leaves an imprimitive graph.

![Figure 1. A minimally primitive graph.](image)

We have the following condition for minimal primitivity of a Kronecker product.

**Theorem 3.1.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be directed graphs. Then \( \Gamma_1 \otimes \Gamma_2 \) is minimally primitive if and only if both \( \Gamma_1 \) and \( \Gamma_2 \) are primitive and at least one of them is minimally primitive.

**Proof.** Suppose that \( \Gamma_1 \) and \( \Gamma_2 \) are primitive and that \( \Gamma_1 \) is minimally primitive. Then \( \Gamma_1 \otimes \Gamma_2 \) is primitive by Theorem 2.1. Let \( e : (u, x) \to (v, y) \) be an arc of \( \Gamma_1 \otimes \Gamma_2 \), and let \( e_1 \) denote the arc \( u \to v \) of \( \Gamma_1 \). Either \( \Gamma_1 \setminus e_1 \) has no walk from \( u \) to \( v \) or \( \Gamma_1 \setminus e_1 \) is strongly connected but imprimitive. In the first case, \( (\Gamma_1 \otimes \Gamma_2) \setminus e \) has no walk from \( (u, z) \) to \( (v, z) \) for any vertex \( z \) of \( \Gamma_2 \). In the second case, let \( d \) be the greatest common divisor of the cycle lengths in \( \Gamma_1 \setminus e_1 \); note \( d > 1 \) since \( \Gamma_1 \setminus e_1 \) is imprimitive. The projection on \( \Gamma_1 \) of any cycle in \( (\Gamma_1 \otimes \Gamma_2) \setminus e \) is a circuit in \( \Gamma_1 \setminus e_1 \). Thus the length of every cycle in \( (\Gamma_1 \otimes \Gamma_2) \setminus e \) is a multiple of \( d \). In neither case is \( (\Gamma_1 \otimes \Gamma_2) \setminus e \) primitive, and we conclude that \( \Gamma_1 \otimes \Gamma_2 \) is minimally primitive.

Now suppose that \( \Gamma_1 \otimes \Gamma_2 \) is minimally primitive. Then \( \Gamma_1 \) and \( \Gamma_2 \) are primitive by Theorem 2.1. Suppose that neither of them is minimally primitive, and let \( e_1 \) and \( e_2 \) be arcs of \( \Gamma_1 \) and \( \Gamma_2 \).
respectively for which $\Gamma_1 \setminus e_1$ and $\Gamma_2 \setminus e_2$ are primitive graphs. Then $(\Gamma_1 \setminus e_1) \otimes (\Gamma_2 \setminus e_2)$ is primitive by Theorem 2.1. Since $(\Gamma_1 \setminus e_1) \otimes (\Gamma_2 \setminus e_2)$ is a subgraph of $(\Gamma_1 \otimes \Gamma_2) \setminus (e_1, e_2)$ with the same vertex set, it follows that $(\Gamma_1 \otimes \Gamma_2) \setminus (e_1, e_2)$ is primitive also, whence $\Gamma_1 \otimes \Gamma_2$ is not minimally primitive. This contradiction completes the proof.

4. Exponent-critical graphs

If a directed graph $\Gamma$ is primitive but not minimally so, then it has an arc $e$ whose deletion leaves a primitive graph. In this case the exponent of $\Gamma \setminus e$ is at least equal to that of $\Gamma$ and may be higher, as Example 3.1 shows. We consider graphs which are primitive and arc-minimal with respect to their exponent, in the sense that upon deletion of any arc, the property of primitivity is either lost or it is retained but with increased exponent. We refer to such graphs as exponent-critical, and for ease of exposition we introduce some further terminology.

**Definition 4.1.** A primitive graph $\Gamma$ has the me$_t$ (minimal exponent $t$) -property, or is a me$_t$-graph, if $\Gamma$ is exponent-critical of exponent $t$, i.e. if $\Gamma$ has exponent $t$ and there is no arc $e$ of $\Gamma$ for which $\Gamma \setminus e$ is primitive of exponent $t$.

**Definition 4.2.** For a positive integer $k$, the directed graph $\Gamma$ is $k$-arc-essential if $\Gamma$ is primitive of exponent at most $k$ and there is no arc $e$ of $\Gamma$ for which $\Gamma \setminus e$ is primitive with exponent at most $k$.

The distinction between the properties described in Definitions 4.1 and 4.2 may not be immediately obvious. A me$_t$-graph is a graph of exponent $t$ that is $t$-arc-essential, and a $k$-arc-essential graph is one in which every arc is required for the existence of walks of length $k$ between all pairs of vertices. A $k$-arc-essential graph need not have exponent $k$, as the graph $G$ of Example 3.1 demonstrates. Deletion of the arc $x_1 \to x_3$ from $G$ leaves a graph of exponent 14; deletion of the arc $x_3 \to x_5$ leaves a graph of exponent 17, and deletion of any other arc breaks the primitivity (and the strong connectedness). Since $G$ has exponent 8 it possesses a walk of length 8 from every vertex to every vertex; however this property is not shared by $G \setminus e$ for any arc $e$ of $G$. Thus $G$ is an me$_8$-graph that is $k$-arc-essential for every $k$ in the range 8 to 13. It is not 14-arc-essential, since the arc $x_1 \to x_3$ is not required for the existence of walks of length 14 between all pairs of vertices.

If the graph $\Gamma$ is $k$-arc-essential, then for every arc $e$ of $\Gamma$, there is pair $u$ and $v$ of vertices in $\Gamma$ for which every $k$-walk from $u$ to $v$ in $\Gamma$ includes the arc $e$. In a $k$-arc-essential graph, walks of length $k$ exist from every vertex to every vertex, but this property does not survive the deletion of an arc. Every minimally primitive graph of exponent $t$ is a me$_t$-graph, but a me$_t$-graph need not be minimally primitive. It is possible for a me$_t$-graph $\Gamma$ to have the property that $\Gamma \setminus e$ is primitive (of exponent exceeding $t$) for every arc $e$. For example the complete loopless directed graph on 3 vertices is a me$_3$-graph in which the deletion of any arc yields a me$_3$-graph.

**Lemma 4.1.** Let $\Gamma$ be a primitive graph of exponent $t$, that is $k$-arc-essential for some $k \geq t$. Then, $\Gamma$ is a me$_t$-graph.

**Proof.** We require to show that $\Gamma$ is $t$-arc-essential. Suppose not, and let $e$ be an arc of $\Gamma$ for which $\Gamma \setminus e$ is primitive of exponent $t$. Then $\Gamma \setminus e$ possesses $k$-walks from every vertex to every vertex, contrary to the hypothesis that $\Gamma$ is $k$-arc essential. \qed
It follows from the argument of Lemma 4.1 that a \( k \)-arc-essential graph of exponent \( t \) is \( k' \)-arc-essential for all \( k' \) in the range \( t \) to \( k \). If \( \Gamma \) is minimally primitive of exponent \( t \), then it is \( k \)-arc-essential all \( k \geq t \). If \( \Gamma \) is a me\( t \)-graph that is not minimally primitive, then \( \Gamma \) is \( k \)-arc-essential for all \( k \) in the range \( t \) to \( t' - 1 \) where \( t' \) is the minimum exponent of \( \Gamma \setminus e \), over all arcs \( e \) of \( \Gamma \) for which \( \Gamma \setminus e \) is primitive.

We refer to a walk of length \( k \) from a vertex \( u \) to a vertex \( v \) in a graph \( \Gamma \) as a *unique \( k \)-walk* if it is the only \( k \)-walk from \( u \) to \( v \) in \( \Gamma \). If every arc of a primitive graph \( \Gamma \) of exponent \( t \) belongs to some unique \( t \)-walk, then \( \Gamma \) is a me\( t \)-graph. The me\( t \)-property in general does not require that every arc belong to a unique \( t \)-walk. In fact the graph \( G \) of Example 4.1 below is a me\( 12 \)-graph in which there is no unique \( 12 \)-walk.

We now consider conditions under which the Kronecker product of a pair of graphs is exponent-critical, and hence \( k \)-arc-essential at least for some value of \( k \). Suppose that \( \Gamma_1 \) and \( \Gamma_2 \) are graphs for which \( \Gamma_1 \otimes \Gamma_2 \) is \( k \)-arc-essential for a positive integer \( k \). By Theorem 2.1, both \( \Gamma_1 \) and \( \Gamma_2 \) are primitive of exponent at most \( k \). Let \( u \to v \) and \( x \to y \) be arcs of \( \Gamma_1 \) and \( \Gamma_2 \) respectively, so that \((u, x) \to (v, y)\) is an arc of \( \Gamma_1 \otimes \Gamma_2 \). Then there exist vertices \((u', x')\) and \((v', y')\) in \( \Gamma_1 \otimes \Gamma_2 \) for which every walk of length \( k \) from \((u', x')\) to \((v', y')\) in \( \Gamma_1 \otimes \Gamma_2 \) includes the arc \((u, x) \to (v, y)\). This is equivalent to the statement that there is an integer \( i \) in the range 1 to \( k \) with the property that \( u \to v \) is the \( i \)th arc in every \( k \)-walk from \( u' \) to \( v' \) in \( \Gamma_1 \), and \( x \to y \) is the \( i \)th arc in every \( k \)-walk from \( x' \) to \( y' \) in \( \Gamma_2 \). Thus \( \Gamma_1 \) and \( \Gamma_2 \) are both \( k \)-arc-essential, and moreover all arcs of \( \Gamma_1 \) and all arcs of \( \Gamma_2 \) satisfy compatibility conditions on the positions in which they are required for \( k \)-walks in \( \Gamma_1 \) and \( \Gamma_2 \).

This observation motivates the following definitions.

**Definition 4.3.** Let \( \Gamma \) be a primitive graph of exponent at most \( k \) and let \( e \) be an arc of \( \Gamma \).

- The arc \( e \) is *\( k \)-required in position \( i \) in \( \Gamma \)* if there exist vertices \( u \) and \( v \) (not necessarily distinct) in \( \Gamma \) with the property that every \( k \)-walk from \( u \) to \( v \) in \( \Gamma \) has \( e \) as its \( i \)th arc.

- The arc \( e \) is *\( k \)-required in fixed position in \( \Gamma \)* if \( e \) is \( k \)-required in position \( i \) for some \( i \).

- We write

\[
C^k_\Gamma(e) = \{ i \in \{1, \ldots, k\} : e \text{ is } k \text{-required in position } i \}.
\]

and refer to \( C^k_\Gamma(e) \) as the *\( k \)-fixed position set of \( e \).*

- The graph \( \Gamma \) is *\( k \)-arc-static* if \( C^k_\Gamma(e) \) is non-empty for every arc \( e \) of \( \Gamma \), i.e. if every arc is \( k \)-required in fixed position.

Clearly a \( k \)-arc-static graph must be \( k \)-arc-essential. The following example shows that a \( k \)-arc-essential graph need not be \( k \)-arc-static.

**Example 4.1.** The graph \( G \) of Figure 2 is clearly primitive, since it is strongly connected and possesses directed cycles of lengths 2, 3 and 5.

A matrix computation confirms that the exponent of \( G \) is 12 (note that \( G \) has no 11-walk from \( x_5 \) to \( x_8 \)). Since \( G \) is minimally strongly connected, it is a me\( 12 \)-graph. We claim that the arc \( x_1 \to x_4 \) is not 12-required in fixed position in \( G \), whence \( C^1_{12}G(x_1 \to x_4) \) is empty and \( G \) is not
12-arc-static. To see this we partition the vertex set of $G$ as the union of $A = \{x_1, x_2, x_3\}$ and $B = \{x_4, x_5, x_6, x_7, x_8, x_9\}$.

First suppose that $x_i$ and $x_j$ are vertices of $A$. If $x_i = x_j$, then there is a 12-walk from $x_i$ to $x_i$ that involves only arcs of the triangle $T$ induced on $A$. If $x_i \rightarrow x_j$ is an arc, then there is a 10-walk from $x_i$ to $x_j$ that involves only arcs of $T$ and includes the vertex $x_1$ at least three times. This may be extended to a 12-walk by inserting the pair $x_4, x_1$ after any of these appearances of $x_1$. Similarly if the shortest path from $x_i$ to $x_j$ has length 2, then (for example) there is a 8-walk from $x_i$ to $x_j$ that involves only arcs of $T$ and includes $x_1$ three times. We may insert the segment $x_4, x_9, x_4, x_1$ after any appearance of $x_1$, to obtain different 12-paths from $x_i$ to $x_j$ not requiring the arc $x_1 \rightarrow x_4$ in the same position.

If $x_i$ and $x_j$ are vertices of $B$, then it is easily confirmed that there is a 12-walk from $x_i$ to $x_j$ in $G$ that involves only arcs of the subgraph induced on $B$.

If $x_i \in B$ and $x_j \in A$, then there is a 12-walk from $x_i$ to $x_j$ in $G$ that does not involve the arc $x_1 \rightarrow x_4$. To see this note that there is a walk of even length at most 10 from $x_i$ to $x_j$, possibly involving a circuit of the triangle $T$, and including the arc $x_4 \rightarrow x_1$ once. Such a walk can be augmented to one of length 12 by inserting the necessary repetitions of the pair $x_9, x_4$ after the appearance of $x_1$.

Finally suppose that $x_i \in A$ and $x_j \in B$. We claim that the arc $x_1 \rightarrow x_4$ is not needed in fixed position for a 12-walk from $x_i$ to $x_j$. There exist walks of lengths $i, i + 3$ and $i + 6$ from $x_i$ to $x_4$, each involving the arc $x_1 \rightarrow x_4$ exactly once. It is straightforward to check that for each $x_j$ in $B$, at least two of these three walks from $x_i$ to $x_4$ may be extended to walks from $x_i$ to $x_j$, that have even length not exceeding 12, and that involve the arc $x_1 \rightarrow x_4$ exactly once each, in different positions. Since any walk of even length may be extended to one of length 12 using the 2-cycle at $x_9$, this ensures that the arc $x_1 \rightarrow x_4$ is not 12-required in fixed position for a walk from a vertex of $A$ to a vertex of $B$. We conclude that $C_{G}^{12}(x_1 \rightarrow x_4)$ is empty and $G$ is not 12-arc-static.

The exponent of a $k$-arc-static graph is at most $k$ but may be less, as demonstrated by the Wielandt graphs.

**Example 4.2.**
The Wielandt graph $W_4$, of exponent 10, is shown in Figure 3. We claim that each of the five arcs of $W_4$ is 11-required in fixed position in $W_4$. With the exception of $x_3 \rightarrow x_1$, each arc of $W_4$ involves...
a vertex with either an indegree or an outdegree of 1, and is therefore $k$-required either in position 1 or position $k$, for all $k \geq 10$. For the arc $x_3 \rightarrow x_1$, we note that the unique 11-walk in $W_4$ from $x_1$ to $x_3$ involves this arc in positions 3, 6 and 9. Thus, $W_4$ is 11-arc-static. The graph $W_4$ is also 12-arc-static, but not 13-arc-static. For $n \geq 3$, the Wielandt graph $W_n$ is $k$-arc-static for all $k$ in the range $n^2 - 2n + 2$ to $n^2 - n$.

![Figure 3. The Wielandt graph $W_4$.](image)

The following observation is an analogue of Lemma 4.1 for the property of $k$-arc-staticity.

**Lemma 4.2.** Let $\Gamma$ be a primitive graph of exponent $t$. If $\Gamma$ is $k$-arc-static for some $k \geq t$, then $\Gamma$ is $t$-arc-static.

**Proof.** Let $e$ be an arc of $\Gamma$. Then $e$ is $k$-required in position $i$ in $\Gamma$, for some $i \in \{1, \ldots, k\}$. Let $u$ and $v$ be vertices of $\Gamma$ with the property that every $k$-walk from $u$ to $v$ in $\Gamma$ has $e$ as its $i$th arc. Let $u, u_1, \ldots, u_{k-1}, v$ be such a $k$-walk. If $i \leq t$, then every $t$-walk from $u$ to $u_t$ has $e$ as its $i$th arc, so $e$ is $t$-required in position $i$ in $\Gamma$. If $i > t$, then every $t$-walk from $u_{i-t}$ to $u_i$ has $e$ as its final arc, so $e$ is $t$-required in position $t$ in $\Gamma$. □

In particular it follows from Lemma 4.2 (and its proof) that a $k$-arc-static graph of exponent $t$ is a me$_t$-graph and is $k'$-arc-static for all $k'$ in the range $t$ to $k$. It is possible for a graph of exponent $t$ to be $k$-arc-static for all $k \geq t$; for example the graph of order 5 that consists of a 3-cycle and a 4-cycle sharing a single arc has exponent 12 and has the property that every arc either originates at a vertex of outdegree 1 or terminates at a vertex of indegree 1. Thus, every arc is $k$-required either in position 1 or in position $k$, for all $k \geq 12$.

We now return to the task of articulating conditions under which the Kronecker product of a given pair of graphs is exponent-critical, beginning with a reformulation of our earlier observations in the language of Definition 4.3. Before stating necessary and sufficient conditions for the Kronecker product of a pair of graphs to be $k$-arc-essential, we recall that the definitions of the terms $k$-arc-essential and $k$-arc-static include the stipulation that the graph in question is primitive of exponent at most $k$.

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Theorem 4.1. Let $\Gamma_1$ and $\Gamma_2$ be directed graphs each having at least one arc. Then

1. $\Gamma_1 \otimes \Gamma_2$ is a $k$-arc-essential graph if and only if both $\Gamma_1$ and $\Gamma_2$ are $k$-arc-static and the $k$-fixed position sets of all arcs of $\Gamma_1$ and $\Gamma_2$ have pairwise non-empty intersection.

2. If $\Gamma_1 \otimes \Gamma_2$ is $k$-arc-essential then it is $k$-arc-static.

Proof. The “only if” direction of 1. is proved by the remarks preceding Definition 4.3. For the “if” direction, suppose that $\Gamma_1$ and $\Gamma_2$ satisfy the hypotheses. Then $\Gamma_1 \otimes \Gamma_2$ is primitive of exponent at most $k$ by Theorem 2.1. Let $(e, f)$ be an arc of $\Gamma_1 \otimes \Gamma_2$, where $e$ and $f$ are the corresponding arcs of $\Gamma_1$ and $\Gamma_2$, and let $i \in C^k_{\Gamma_1}(e) \cap C^k_{\Gamma_2}(f)$. Then there exist vertices $u, v$ of $\Gamma_1$ and $x, y$ of $\Gamma_2$ for which every $k$-walk from $u$ to $v$ in $\Gamma_1$ includes the arc $e$ in position $i$, and every $k$-walk from $x$ to $y$ in $\Gamma_2$ includes the arc $f$ in position $i$. It follows that every $k$-walk from $(u, x)$ to $(v, y)$ in $\Gamma_1 \otimes \Gamma_2$ includes the arc $(e, f)$ in position $i$. Thus $(e, f)$ is $k$-required in position $i$ in $\Gamma_1 \otimes \Gamma_2$, completing the proof of both 1. and 2.

A particular case in which the conditions of Theorem 4.1 are satisfied is where there is some $i \in \{1, \ldots, k\}$ for which every arc of $\Gamma_1$ and every arc of $\Gamma_2$ is $k$-required in position $i$ in $\Gamma_1$ or $\Gamma_2$ respectively. This is the case where the intersection over all arcs $e$ of $\Gamma_1$ and all arcs $f$ of $\Gamma_2$ of the sets $C^k_{\Gamma_1}(e)$ and $C^k_{\Gamma_2}(f)$ is non-empty. This can occur only if both $\Gamma_1$ and $\Gamma_2$ are $k$-uniformly static as defined below.

Definition 4.4. Let $\Gamma$ be a primitive graph of exponent at most $k$, and let $r \in \{1, \ldots, k\}$. We say that $\Gamma$ is $(k, r)$-uniformly static if every arc $e$ of $\Gamma$ is $k$-required in position $r$. We say that $\Gamma$ is $k$-uniformly static if it is $(k, r)$-uniformly static for some $r$.

The following is an immediate consequence of Theorem 4.1. We note in particular that if there is some $r$ for which $\Gamma_1$ and $\Gamma_2$ are both $(k, r)$-uniformly static, then $\Gamma_1 \otimes \Gamma_2$ is $k$-arc-essential.

Corollary 4.1. Let $\Gamma_1$ and $\Gamma_2$ be primitive graphs of exponent at most $k$, and let $r \in \{1, \ldots, k\}$. Then $\Gamma_1 \otimes \Gamma_2$ is $(k, r)$-uniformly static if and only if both $\Gamma_1$ and $\Gamma_2$ are $(k, r)$-uniformly static.

Example 4.3. The graph $G$ shown in Figure 4 has exponent 5 and is $(5, 3)$-uniformly static. Thus $G \otimes G$ is also a $(5, 3)$-uniformly static graph, by Corollary 4.1.

![Figure 4. A $(5, 3)$-uniformly static graph.](image-url)
Example 4.4. The graph $G$ of Figure 5 has exponent 6 and is minimally strongly connected, so it has the me$_6$-property. By inspecting 6-walks between vertices of $G$, we may note for example that for each vertex $x$, there is a 6-walk from $x$ to $x_4$ in $G$ that has $x_1 \rightarrow x_4$ as its first arc, which means that the arc $x_1 \rightarrow x_2$ is not 6-required in position 1, and $1 \not\in C_6^d(x_1 \rightarrow x_2)$. The elements of $C_6^d(e)$ for each arc $e$ of $G$ are listed below.

\[
C_6^d(x_1 \rightarrow x_2) = \{2, 3, 5, 6\} \\
C_6^d(x_2 \rightarrow x_3) = \{1, 3, 4, 6\} \\
C_6^d(x_3 \rightarrow x_1) = \{1, 2, 4, 5\} \\
C_6^d(x_1 \rightarrow x_4) = \{1, 2, 3, 4, 5, 6\} \\
C_6^d(x_4 \rightarrow x_1) = \{1, 3, 5\}.
\]

Figure 5. A 6-arc-static graph that is not uniformly static.

This example shows that the $k$-arc-static Kronecker product of a pair of graphs need not be $k$-uniformly static; $G$ is a graph of exponent 6 that is 6-arc-static but not uniformly so. The pairwise intersections $C_6^d(e) \cap C_6^d(f)$ are non-empty for all arcs $e$ and $f$ of $G$, so the Kronecker product $G \otimes G$ is a me$_6$-graph that is 6-arc-static, by Theorems 4.1 and 2.1. However the intersection over all arcs $e$ of $G$ of the sets $C_6^d(e)$ is empty, so $G \otimes G$ is not 6-uniformly static.

We now consider the class of graphs that are uniformly static for all feasible parameters, which we will refer to as strong me$_t$-graphs. Suppose that $\Gamma$ is a $t$-arc-essential graph of order at least 3, with the additional property that $C_t^d(e) = \{1, \ldots, t\}$ for every arc $e$ of $\Gamma$, so that $\Gamma$ is $(t, r)$-uniformly static for every $r \in \{1, \ldots, t\}$. Then from Theorem 4.1 it follows that $\Gamma \otimes \Gamma'$ is $t$-arc-essential (and $t$-arc-static) for every $t$-arc-static graph $\Gamma'$. In this situation we note that the exponent of $\Gamma$ must be $t$. To see this let $x$ be a vertex of $\Gamma$ of outdegree at least 2 (such a vertex must exist since $x$ is primitive). Let $x \rightarrow y$ and $x \rightarrow y'$ be distinct arcs of $\Gamma$, and let $z$ be any vertex. If the exponent of $G$ is less than $t$, then there exists a $(t-1)$-walk from $y'$ to $z$ in $\Gamma$ and so there exists a $t$-walk from $x$ to $z$ that has $x \rightarrow y'$ as its first arc. Since this statement holds for every vertex $z$ of $\Gamma$, we reach the contradiction that the arc $x \rightarrow y$ is not $t$-required in position 1. The conclusion is that a $t$-arc-essential graph in which every edge is $t$-required in every position must have exponent $t$, and we make the following definition.

Definition 4.5. A primitive directed graph $\Gamma$ has the strong me$_t$-property, or is a strong me$_t$-graph, if it is $(t, r)$-uniformly static for every $r$ in the range $1, \ldots, k$.

By the above comments, a strong me$_t$-graph is necessarily a me$_t$-graph. We now present examples to demonstrate the existence of strong me$_t$-graphs for all $t \geq 2$. Our examples consist of separate families for even and odd exponent.
Example 4.5. *Strong me*$_{t}$-*graphs of even exponent.*

Let $t$ be an even positive integer. Let $C_{t+1}$ be the undirected cycle of length $t + 1$, interpreted as a directed graph consisting of two oppositely directed cycles on the same vertex set. We label the vertices of $C_{t+1}$ as $x_1, \ldots, x_{t+1}$, where $x_i \to x_j$ is an arc if and only if $|i - j| = 1$ or $|i - j| = t$. If $x_i$ and $x_j$ are distinct vertices, there are arc-disjoint paths from $x_i$ to $x_j$ that follow the two distinct directed $(t+1)$-cycles. One of these has even length at most $t$, and may be extended if necessary to a $t$-walk from $x_i$ to $x_j$ by adding repetitions of a 2-cycle at any vertex. There exists a $t$-walk in $C_{t+1}$ from each vertex to itself, since $t$ is even and every vertex belongs to a 2-cycle. Hence the exponent of $C_{t+1}$ is at most $t$. Finally, there is no walk of length $t - 1$ from any vertex to itself in $C_{t+1}$, so the exponent of $C_{t+1}$ is exactly $t$.

If $i$ and $j$ differ by 1 (or $t$), then there is a unique $t$-walk from $x_i$ to $x_j$ in $t$, that involves $t$ successive arcs of one of the two $t$-cycles. Inspection of these unique $t$-paths confirms that every arc is $t$-required in every fixed position, and so $C_{t+1}$ is a strong me$_{t}$-graph.

Example 4.6. *Strong me*$_{t}$-*graphs of odd exponent.*

For $t$ odd, $t \geq 3$, define the directed graph $G_{2t}$ of order $2t$ as follows.

- The vertex set of $G_{2t}$ is $\{x_1, \ldots, x_t, y_1, \ldots, y_t\}$.
- The arc set of $G_{2t}$ is defined as follows:
  - For $i = 1, \ldots, t - 1$, $x_i \to x_{i+1}$ and $y_i \to y_{i+1}$ are arcs; $x_t \to x_1$ and $y_t \to y_1$ are arcs;
  - For $i = 2, \ldots, t$, $x_i \to x_{i-1}$ and $y_i \to y_{i-1}$ are arcs;
  - The remaining arcs are $x_1 \to y_t$ and $y_1 \to x_t$.

The example $G_{10}$, for $t = 5$, is shown in Figure 6.

In our discussion of the graph $G_{2t}$ we consider $x_i$ and $y_i$ to be defined for all integers $i$ and we identify $x_i$ with $x_{i'}$, and $y_i$ with $y_{i'}$, whenever $i$ and $i'$ are congruent modulo $t$. This notational device is convenient for our discussion of properties of $G_{2t}$. 

Figure 6. The strong me$_5$-graph $G_{10}$. 

---
Since $G_{2t}$ is strongly connected and has a 4-cycle on $\{x_1, y_t, y_1, x_t\}$ and also has a cycle of odd length $t$, it is clearly primitive. Its exponent cannot be less than $t$, since for example the shortest walk from $x_{\frac{t+1}{2}}$ to $y_{\frac{t+1}{2}}$ has length $t$. On the other hand it is straightforward to confirm that $G_{2t}$ contains $t$-walks from every vertex to every vertex.

To verify that the strong $me_t$-property holds in $G_{2t}$, we consider arcs of three different types.

1. **The arcs $x_i \rightarrow x_{i+1}$ and $y_i \rightarrow y_{i+1}$**
   For $k \in \{1, \ldots, t\}$ the arc $x_i \rightarrow x_{i+1}$ is required in position $k$ for the unique $t$-walk in $G_{2t}$ from the vertex $x_{i-t+1}$ to itself. A similar observation applies to $y_i \rightarrow y_{i+1}$.

2. **The arcs $x_i \rightarrow x_{i-1}$ and $y_i \rightarrow y_{i-1}$, for $i \in \{2, \ldots, t\}$**
   Let $k \in \{1, \ldots, t\}$. If $i + k \leq t + 1$, the arc $x_i \rightarrow x_{i-1}$ is required in position $k$ for the unique $t$-walk in $G$ from $x_{i+k-1}$ to $y_{i+k-1}$. If $i + k > t + 1$, then $x_i \rightarrow x_{i-1}$ is required in position $k$ for the unique $t$-walk in $G_{2t}$ from $y_{i-k-1}$ to $x_{i+k-1}$. Similar analysis applies to $y_i \rightarrow y_{i-1}$.

3. **For $k \in \{1, \ldots, t\}$ the arc $x_1 \rightarrow y_k$ is $t$-required in position $k$ for the unique $t$-walk in $G_{2t}$ from $x_k$ to $y_k$, and $y_1 \rightarrow x_t$ is similarly required for $y_k$ to $x_k$**.

Thus $G_{2t}$ is a strong $me_t$-graph.

It follows from Corollary 4.1 that the Kronecker product of two graphs is a strong $me_t$-graph if and only if both factors are strong $me_t$-graphs. We conclude this section with a summary of criteria for a Kronecker product to be exponent-critical with specified additional properties. We recall from Theorem 2.1 that if the maximum of the exponents of two primitive graphs is $t$, then their Kronecker product is primitive of exponent $t$.

**Theorem 4.2.** Let $\Gamma_1$ and $\Gamma_2$ be primitive graphs each of order at least 2, and let $t$ be a positive integer.

1. $\Gamma_1 \otimes \Gamma_2$ is a $me_t$-graph if and only if $t = \max\{\exp(\Gamma_1), \exp(\Gamma_2)\}$ and $C^t_{\Gamma_1}(e) \cap C^t_{\Gamma_2}(f)$ is non-empty for all arcs $e$ of $\Gamma_1$ and $f$ of $\Gamma_2$.

2. $\Gamma_1 \otimes \Gamma_2$ is $(t, r)$-uniformly static if and only if $\Gamma_1$ and $\Gamma_2$ are both $(t, r)$-uniformly static.

3. $\Gamma_1 \otimes \Gamma_2$ is a $me_t$-graph if either $\Gamma_1$ or $\Gamma_2$ is a strong $me_t$-graph and the other is $t$-arc-static.

4. $\Gamma_1 \otimes \Gamma_2$ is a strong $me_t$-graph if and only if both $\Gamma_1$ and $\Gamma_2$ are strong $me_t$-graphs.

# 5. Graphs of exponent 2

In graphs of exponent 2, many of the special properties discussed in Section 4 turn out to be equivalent, which allows for a much more concise version of Theorem 4.2 in this special case, particularly in the undirected setting. Let $\Gamma$ be a (directed) graph with the $me_2$-property, and let $e : u \rightarrow v$ be an arc of $\Gamma$. Deletion of $e$ would either leave a graph with no 2-walk from $u$ to some vertex $x$, or a graph with no 2-walk from some vertex $y$ to $v$. Thus $e$ is either 2-required in position 1 for a walk originating at $u$, or 2-required in position 2 for a walk terminating at $v$. In particular every $me_2$-graph is 2-arc-static.

Now suppose that $\Gamma_1$ and $\Gamma_2$ are graphs for which $\Gamma_1 \otimes \Gamma_2$ has the $me_2$-property. Since a graph without loops cannot have exponent 1, it follows from Theorem 4.2 that $\Gamma_1$ and $\Gamma_2$ are $me_2$-graphs.
Suppose that neither $\Gamma_1$ nor $\Gamma_2$ has the strong me$_2$-property, and let $e$ be an arc of $\Gamma_1$ which is 2-required only in one position, say in position 1. Then every arc of $\Gamma_2$ is 2-required in position 1 by Theorem 4.2, so $\Gamma_2$ is $(2, 1)$-uniformly static. Since $\Gamma_2$ does not have the strong me$_2$-property, it has an arc $f$ for which $C_{\Gamma_2}^2(f) = \{1\}$, whence $\Gamma_1$ is also $(2, 1)$-uniformly static by the same reasoning.

Theorem 4.2 has the following formulation for $t = 2$.

Theorem 5.1. Let $\Gamma_1$ and $\Gamma_2$ be directed graphs. Then $\Gamma_1 \otimes \Gamma_2$ has the me$_2$-property if and only if $\Gamma_1$ and $\Gamma_2$ are me$_2$-graphs and either

- at least one of $\Gamma_1$ and $\Gamma_2$ has the strong me$_2$-property, or
- $\Gamma_1$ and $\Gamma_2$ are both $(2, r)$-uniformly static for the same $r \in \{1, 2\}$.

We remark that the inclusion of the second case is necessary in Theorem 5.2; it is possible for a directed graph of exponent 2 to be $(2, 1)$-uniformly static and not $(2, 2)$-uniformly static. The least possible order of such a graph is 6 and the unique example (up to isomorphism) of order 6 is shown below.

Example 5.1.
The graph $G$ of Figure 7 is a me$_2$-graph in which every arc is the first arc in some unique 2-path. However, the arc $x_3 \rightarrow x_1$ (for example) is not required in position 2 for a 2-walk from any vertex to $x_1$. By Theorem 5.2, $G \otimes G$ is a me$_2$-graph.

Moreover, $G \otimes G$ is $(2, 1)$-uniformly static but not $(2, 2)$-uniformly static.

![Figure 7. A me$_2$-graph that is $(2, 1)$-uniformly static but not $(2, 2)$-uniformly static.](image)

In a directed graph $\Gamma$ of exponent 2, the me$_2$-property is equivalent to the statement that every arc belongs to a unique 2-walk in $\Gamma$.

We now turn attention to undirected graphs of exponent 2. In general, care is needed in adapting the definition of the me$_t$-property to the context of undirected graphs, since the two interpretations below may need to be distinguished.
Definition 5.1. Let $\Gamma$ be a primitive undirected graph of exponent $t$.

- If the undirected graph $\Gamma \setminus e$ is imprimitive or primitive of exponent exceeding $t$ for every edge $e$ of $\Gamma$, we say that $\Gamma$ has the undirected $me_t$-property.

- If $\Gamma$ has the $me_t$-property when interpreted as a directed graph (in which $v \to u$ is an arc whenever $u \to v$ is), we say that $\Gamma$ has the directed $me_t$-property.

It is clear that any undirected graph with the directed $me_t$-property also possesses the undirected $me_t$-property. That the two do not coincide in general is confirmed by the following example.

Example 5.2. The undirected graph shown in Figure 8 is primitive of exponent 3, and has the undirected $me_3$-property but not the directed $me_3$-property. Deletion of the arc $u \to v$ leaves a directed graph that is primitive of exponent 3. Deletion of the edge $uv$ leaves a primitive undirected graph of exponent 4. The key point here is that the edge $uv$ is 3-required only for a walk from $w$ to $u$, and there are two such walks that traverse this edge in opposite directions. Thus one of the two arcs between $u$ and $v$ can be deleted with no effect on the primitivity status of the graph or on the exponent.

![Figure 8. The undirected $me_3$-property is weaker than the directed $me_3$-property.](image)

We now show that in the special case of undirected graphs of exponent 2, we need not distinguish between the directed and undirected versions of the $me_2$-property.

Lemma 5.1. Let $\Gamma$ be a primitive undirected graph of exponent 2 that has the undirected $me_2$-property. Then $\Gamma$ has the directed $me_2$-property.

Proof. Let $u \to v$ be an arc of $\Gamma$ and let $e$ denote the edge $uv$ of $\Gamma$. We need to show that the directed graph obtained from $\Gamma$ by deleting the arc $u \to v$ is not primitive of exponent 2.

Since $\Gamma$ has the undirected $me_2$-property, deletion of the edge $e$ from $\Gamma$ would leave a graph with a vertex $x$ sharing no neighbour with $v$, or a vertex $y$ sharing no neighbour with $u$. In the first case, $x, u, v$ is the unique 2-walk from $x$ to $v$ in $\Gamma$, and in the second case $u, v, y$ is the unique 2-walk from $u$ to $y$ in $\Gamma$. In either case the arc $u \to v$ is required for the existence of 2-walks between all pairs of vertices in $\Gamma$. We conclude that $\Gamma$ has the directed $me_2$-property. \qed
Since every pair of vertices in a primitive undirected graph of exponent 2 has a common neighbour, every edge belongs to a triangle and every vertex has degree at least 2. It follows that no edge can be required for a 2-walk from any vertex to itself, and every edge of an undirected me$_2$-graph is included in the unique 2-walk between two distinct vertices.

For clarity we reformulate the me$_2$-property for undirected graphs as follows.

**Definition 5.2.** The undirected graph $\Gamma$ has the me$_2$-property if $\Gamma$ is primitive of exponent 2, and for every edge $uv$ of $\Gamma$ there exists a vertex $x$ whose only common neighbour with $v$ is $u$, or a vertex $y$ whose only common neighbour with $u$ is $v$.

This characterization recalls the original friendship property, which is the condition that every 2-path in an undirected graph is the unique 2-path between its endpoints. The me$_2$-property for undirected graphs does not require that all 2-paths are unique, but that unique 2-paths are sufficiently plentiful that every edge belongs to one.

Theorem 4.2 admits further refinement for undirected graphs of exponent 2, after the following observation.

**Lemma 5.2.** Let $\Gamma$ be an undirected me$_2$-graph. Then $\Gamma$ is $(2, 1)$-uniformly static if and only if $\Gamma$ is $(2, 2)$-uniformly static.

**Proof.** Suppose that $\Gamma$ is $(2, 1)$-uniformly static and let $u \to v$ be an arc of $\Gamma$. We require to show that $u \to v$ is the latter arc in some unique 2-walk in $\Gamma$. Since $v \to u$ is an arc of $\Gamma$, and $\Gamma$ is $(2, 1)$-arc static, $v \to u$ is the initial arc of the unique 2-walk in $\Gamma$ from $v$ to some vertex $w$. Since $\Gamma$ is undirected, it follows that $w, u, v$ is the unique 2-walk from $w$ to $v$ in $\Gamma$, hence that $u \to v$ is 2-required in position 2 in $\Gamma$ and $\Gamma$ is $(2, 2)$-uniformly static. \[\square\]

An immediate consequence of Lemma 5.2 is that an undirected me$_2$-graph $\Gamma$ has the strong me$_2$-property if it is 2-uniformly static. This means that every edge $uv$ is included in a unique 2-path originating at $u$ and one originating at $v$. Example 5.1 shows that the hypothesis that $\Gamma$ is undirected is necessary in Lemma 5.2. We may reformulate Definition 4.5 as follows, for undirected graphs of exponent 2.

**Definition 5.3.** An undirected graph $\Gamma$ has the strong me$_2$-property if $\Gamma$ has exponent 2 and for every edge $uv$ of $\Gamma$, there exist vertices $x$ and $y$ (not necessarily distinct) such that $u$ is the unique common neighbour of $x$ and $v$ in $\Gamma$, and $v$ is the unique common neighbour of $y$ and $u$ in $\Gamma$.

Examples of infinite families of undirected strong me$_2$-graphs include the windmills and the Kneser graphs with parameters $(3r, r)$ for $r \geq 1$. The Kneser graph $K_n(3r, r)$ is the graph of order $\binom{3r}{r}$ whose vertices represent the $r$-element subsets of a set of $3r$ elements, and in which two vertices are adjacent if the subsets that they represent are disjoint. More generally any undirected graph of exponent 2 in which every edge belongs to a unique 3-cycle is a strong me$_2$-graph, although strong me$_2$-graphs need not have this property. It is shown in [12] that every complete graph arises as an induced subgraph of a strong me$_2$-graph.

An infinite family of undirected graphs that have the me$_2$-property but not the strong me$_2$-property is the collection $\{G_n : n \geq 4\}$ defined as follows. The vertex set of $G_n$ is $\{x_1, x_2, \ldots, x_n\}$,
and the vertices $x_1$ and $x_2$ are both adjacent to all others. For $i \geq 3$, $x_i$ is adjacent only to $x_1$ and $x_2$. It is easily checked that $G_n$ is a me$_2$-graph. For $i \geq 3$, inspection of the edge $x_1x_i$ shows that $G_n$ is not a strong me$_2$-graph. The vertex $x_1$ is not the unique common neighbour of $x_1$ and any other vertex of $G_n$, whereas $x_1$ is the unique common neighbour of $x_i$ and $x_2$.

Theorem 4.2 takes the following form for undirected me$_2$-graphs. We remark that this theorem bears a structural resemblance to the undirected version of Theorem 2.1, which states that the Kronecker product of undirected graphs is connected if and only if both factors are connected and at least one is primitive, and that the product is primitive if and only if both factors are primitive.

**Theorem 5.2.** Let $\Gamma_1$ and $\Gamma_2$ be undirected graphs each of order at least 2. Then

1. $\Gamma_1 \otimes \Gamma_2$ is a me$_2$-graph if and only if both $\Gamma_1$ and $\Gamma_2$ are me$_2$-graphs and at least one of them is a strong me$_2$-graph.

2. $\Gamma_1 \otimes \Gamma_2$ is a strong me$_2$-graph if and only if both $\Gamma_1$ and $\Gamma_2$ are strong me$_2$-graphs.

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**References**


