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The second least eigenvalue of the signless Laplacian of the complements of trees

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Abstract

Suppose that $\mathfrak{T}_n^c$ is a set, such that the elements of $\mathfrak{T}_n^c$ are the complements of trees of order n. In 2012, Li and Wang gave the unique graph in the set $\mathfrak{T}_n^c \backslash \{K_{1,n-1}^c\}$ with minimum 1st 'least eigenvalue of the signless Laplacian' (abbreviated to a LESL). In the present work, we give the unique graph with 2nd LESL in $\mathfrak{T}_n^c \backslash \{K_{1,n-1}^c\}$, where $K_{1,n-1}^c$ represents the complement of star of order n.

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1. Introduction

All the graphs considered in this paper are finite, undirected and simple. Suppose $\Gamma = (V(\Gamma), E(\Gamma))$ is a graph, where $V(\Gamma)$ and $E(\Gamma)$ be the vertex set and the edge set respectively. The graph $\Gamma^c := (V(\Gamma), \bar{E}(\Gamma))$ be the complement of graph Γ and its edge set $\bar{E}(\Gamma) = \{xy \ : x, y \in V(\Gamma),\ xy \notin E(\Gamma)\}$. If a vertex v adjacent to a vertex u, then we simply write $v \sim u$, otherwise we write $v \nsim u$. Define $A(\Gamma) = [a_{ij}]$ be the *adjacency matrix* of a graph Γ with order n, where the entry $a_{ij} = 1$ if $i \sim j$, and $a_{ij} = 0$ if $i \nsim j$. The *degree matrix* of Γ is denoted by $D(\Gamma)$ and $D(\Gamma) = \text{diag}(d_\Gamma(v_1), \ldots, d_\Gamma(v_n))$, where $d_\Gamma(v)$ means degree of vertex v. The *Laplacian matrix* of a graph Γ, denoted by $L(\Gamma)$, is defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$. The Laplacian matrix of a graph has been extensively studied, see [2, 3, 14, 19, 20, 22, 26, 31]. Zero is the smallest eigenvalue of $L(\Gamma)$ and the 2nd smallest eigenvalue of $L(\Gamma)$ is known as the *algebraic connectivity* of Γ. We may refer to [4, 34], for undefined notations, the concepts of graph theory

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and for the study of distance matrix we refer to [29, 32]. The matrix $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ is the *signless Laplacian* matrix of Γ [28]. In particular, $Q(\Gamma)$ is positive semidefinite. It is easy to check that $Q(\Gamma)$ is real and symmetric, and so the eigenvalues of $Q(\Gamma)$ can be ordered as $q_1(Q(\Gamma)) \geq q_2(Q(\Gamma)) \geq \cdots \geq q_n(Q(\Gamma)) \geq 0$. In this case, the *signless Laplacian index* of Γ is $q_1(Q(\Gamma))$. If Γ is a connected graph of order n and m edges, then Γ is called k-*cyclic* if $m = n - 1 + k$. In particular, if $k = 0$, then Γ is called a *tree* [13, 39]. We denote the star graph of order n by $K_{1,n-1}$. Define $\mathfrak{T}_n = \{\Gamma \mid \Gamma \text{ is a tree of order } n\}$ and $\mathfrak{T}_n^c = \{\Gamma^c \mid \Gamma \in \mathfrak{T}_n\}$. In last few years, many researchers work on the eigenvalues of signless Laplacian matrix, especially they focus on signless Laplacian index and a brief survey on this work can be found in [9, 11]. Several bounds can be found in [6, 16, 24, 25, 33, 36, 37, 38] for the signless Laplacian eigenvalues. Furtheremore, for $Q(\Gamma)$-spread see [30]. Here, our focus is on the least eigenvalue of $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ which is denoted by $r(\Gamma)$.

Problem related to the signless Laplacian index was raised by Zhu in [38], he asked the following question: Let $\mathfrak{S}$ be a set of graphs, find an upper bound for the signless Laplacian index of graphs in $\mathfrak{S}$, and also determine the graphs which achieve the maximal index. Similar to the above problem, the following problem is also natural: Let $\mathfrak{S}$ be a set of graphs, for LESL, determine the lower bound. Also give the characterization of graphs which coincide the lower bound.

Both problems are basically related to classical Brualdi-Solheid problem which base on signless Laplacian matrix, for adjacency matrix, we refer [5].

Recently Li and Wang [23] studied the unique graph which has first LESL over $\mathfrak{T}_n^c \backslash \{K_{1,n-1}^c\}$. In the present work, we give the unique graph which has 2nd LESL over the same class of trees.

2. Preliminaries

The eigenvectors correspoding to the eigenvalue $r(\Gamma)$ known as *least eigenvectors* of Γ. Assume $X \in \mathbb{R}^n$ be the vector defined on given graph Γ of order n. A one-one map φ from vertex set of Γ to entries of X, write as $X_u = \varphi(u)$ for each vertex u of $V(\Gamma)$. If $Q(\Gamma)$ has an eigenvector X, obviously this vector defined over $V(\Gamma)$. The entry in vector X with respect to the vertex u is X_u, it can be easily verified that for any $X \in \mathbb{R}^n$

$$X^T Q(\Gamma) X = \sum_{uv \in E_\Gamma} (X_u + X_v)^2 \tag{1}$$

and when X is the eigenvector corresponding to μ (signless Laplacian eigenvalue of Γ) $\Leftrightarrow X \neq 0$,

$$(\mu - d(v))X_v = \sum_{u \in N_\Gamma(v)} X_u. \tag{2}$$

Eq. (2) is called the *eigenvalue-equation* for the Γ. In Eq. (2), $d(v)$ and $N_\Gamma(v)$ denote the degree and the neighborhood of vertex $v \in V(\Gamma)$ respectively. Furthermore, for any arbitrary unit vector $X \in \mathbb{R}^n$,

$$r(\Gamma) = \min(X^T Q(\Gamma)X) \leq X^T Q(\Gamma)X. \quad (3)$$

Note that the equality sign in Eq. (3) holds if and only if X is a least eigenvector of Γ.

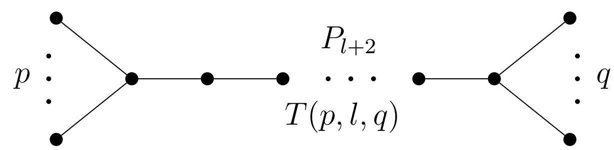

Figure 1. Graph $T(p, l, q)$ such that $p + l + q = n - 2$ where $l \geq 0$.

By Γ^c we denote the complement of Γ. It is trivial to see that $Q(\Gamma^c) = J - Q(\Gamma) + (n-2)I$, where J is the square matrix having all entries 1 and I is the identity matrix, with desired size. So, for each $X \in \mathbb{R}^n$,

$$X^T Q(\Gamma^c)X = X^T(J + (n-2)I)X - X^T Q(\Gamma)X. \quad (4)$$

A tree of order $n+1$ obtained by joining n isolated vertices to a specific vertex is called a *star*, we denote this by $K_{1,n}$. Let T be a tree and v, u be the two vertices in T, the distance between v and u is denoted by $d_T(v, u)$. Now, we define a special tree obtained by joining the center vertices of two disjoint stars $K_{1,p}$ and $K_{1,q}$ where $p, q \geq 0$ by a path having length $l + 1$, where $l \geq 0$, and it is denoted by $T(p, l, q)$. The tree $T(p, l, q)$ is shown in Figure 1 with some of vertices are labeled.

In the following results by $\lambda_{\min}(Q)$ we mean LESL of Γ.

Lemma 2.1 ([9]). *For a connected graph Γ, $\lambda_{\min}(Q) = 0 \Leftrightarrow \Gamma$ is bipartite.*

Lemma 2.2 ([9]). *Suppose Γ is a graph. Then $m(0) = \#\tau(\Gamma)$, where $m(0)$ is the multiplicity of signless Laplacian eigenvalue 0 and $\tau(\Gamma)$ is equal the bipartite components of Γ.*

Lemma 2.3 ([23]). *Given a graph Γ, $r(\Gamma) \leq \delta(\Gamma)$ where $\delta(\Gamma) = \min\{d_v, v \in V_\Gamma\}$.*

Lemma 2.4 ([23]). *For any $T \in \mathfrak{T}^c$ with $n \geq 5$, $\lambda_{\min}(T^c) = 0 \Leftrightarrow T \cong K_{1,n-1}$.*

3. Our Results

In the present section we are in the position to determine the unique graph with the 2nd LESL in the set $\mathfrak{T}_n^c \backslash K_{1,n-1}^c$. Before to do so 1st we give the following lemmas, which is crucial for the main result. Note, that from now p, q and n are positive integers, and of course the vector X is least eigenvector.

Lemma 3.1. *$r(T(p, 2, q)^c)$ more than $r(T(p+1, 2, q-1)^c)$, for $n \geq 7$, $p \geq q \geq 2$ and $p+q = n-4$.*

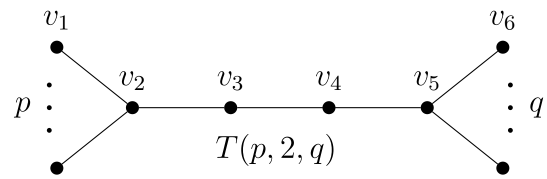

Figure 2. A tree $T(p,2,q)$.

Proof. Suppose that $T(p,2,q)$ is a graph with some vertices are labeled (see Figure 2). Assume that X is a vector of $T(p,2,q)^c$. By Eq. (2), as $r(T(p,2,q)^c)$ greater than 0, all pendent vertices adjacent to v_2 have the same values, write X_1. In the same way, all pendent vertices adjacent to v_5 have the same values, write X_6. Write $X_{v_i} =: X_i$, $2 \leq i \leq 5$ and $r(T(p,2,q)^c) := \mu_1$. By using Eq. (2) on vertices v_i where $1 \leq i \leq 6$, we obtains the following system of equations

$$\begin{cases} (\mu_1 - (p+q+2))X_1 = (p-1)X_1 + X_3 + X_4 + X_5 + qX_6 \\ (\mu_1 - (q+2))X_2 = X_4 + X_5 + qX_6 \\ (\mu_1 - (p+q+1))X_3 = pX_1 + X_5 + qX_6 \\ (\mu_1 - (p+q+1))X_4 = pX_1 + X_2 + qX_6 \\ (\mu_1 - (p+2))X_5 = pX_1 + X_2 + qX_6 \\ (\mu_1 - (p+q+2))X_6 = pX_1 + X_2 + X_3 + X_4 + (q-1)X_6 \end{cases}$$

transform the above system of equations into a matrix equation $(\mu_1 I - \mathbf{B}_1)X = 0$ where $X = (X_1, \ldots, X_6)$ and

$$\mathbf{B}_1 = \begin{bmatrix} \theta_1 & 0 & 1 & 1 & 1 & q \\ 0 & \theta_2 & 0 & 1 & 1 & q \\ p & 0 & \theta_3 & 0 & 1 & q \\ p & 1 & 0 & \theta_4 & 0 & q \\ p & 1 & 1 & 0 & \theta_5 & 0 \\ p & 1 & 1 & 1 & 0 & \theta_6 \end{bmatrix}$$

where $\theta_1 = 2p+q+1$, $\theta_2 = q+2$, $\theta_3 = \theta_4 = p+q-1$, $\theta_5 = p+2$ and $\theta_6 = p+2q+1$. Let

$f_1(\mu, p, q) := (\mu_1 I - \mathbf{B}_1)$, we get the following equation:

$$\begin{aligned} f_1(\mu,p,q) &= (1+p+q-\mu)(2p+2p^2+8p^3+4p^4+2q+12pq \\ &+ 22p^2q+16p^3q+2p^4q+2q^2+22pq^2+24p^2q^2 \\ &+ 6p^3q^2+8q^3+16pq^3+6p^2q^3+4q^4+2pq^4-4\mu \\ &- 15p\mu-30p^2\mu-20p^3\mu-2p^4\mu-15q\mu-60pq\mu \\ &- 59p^2q\mu-13p^3q\mu-30q^2\mu-59pq^2\mu-22p^2q^2\mu \\ &- 20q^3\mu-13pq^3\mu-2q^4\mu+14\mu^2+38p\mu^2+35p^2\mu^2 \\ &+ 7p^3\mu^2+38q\mu^2+69pq\mu^2+25p^2q\mu^2+35q^2\mu^2 \\ &+ 25pq^2\mu^2+7q^3\mu^2-16\mu^3-26p\mu^3-9p^2\mu^3 \\ &- 26q\mu^3-19pq\mu^3-9q^2\mu^3+7\mu^4+5p\mu^4+5q\mu^4-\mu^5), \end{aligned}$$

when $\mu = 0$, we have

$$\begin{aligned} f_1(0,p,q) &= (1+p+q)(2p+2p^2+8p^3+4p^4+2q+12pq+22p^2q \\ &+ 16p^3q+2p^4q+2q^2+22pq^2+24p^2q^2+6p^3q^2 \\ &+ 8q^3+16pq^3+6p^2q^3+4q^4+2pq^4) > 0, \end{aligned}$$

and

$$\begin{aligned} f_1(\mu;p,q)-f_1(\mu;p+1,q-1) &= (1+p-q)(1+p+q-\mu)(8-2p+2p^3-2q+6p^2q \\ &+ 6pq^2+2q^3+p\mu-5p^2\mu+q\mu-10pq\mu-5q^2\mu-\mu^2 \\ &+ 4p\mu^2+4q\mu^2-\mu^3). \end{aligned}$$

Lemma 2.3 and Lemma 2.4 $\Rightarrow \mu_1$ is a least zero of $f_1(\mu; p, q)$, for $0 \leq \mu_1 \leq q+2$. In addition, since $p \geq q$, we have $f_1(\mu; p, q) - f_1(\mu; p+1, q-1) > 0$. In particular, $f_1(\mu_1; p+1, q-1)$ less than 0, $\Rightarrow r(T(p, 2, q)^c)$ greater than $r(T(p+1, 2, q-1)^c)$. ☐

Remarks 1. Lemma 3.1 $\Rightarrow r(T(p,2,q)^c) > r(T(p+1,2,q-1)^c) > \cdots > r(T(n-5,2,1)^c) = r(T(n-5,3,0)^c)$, since $T(n-5,2,1) \cong T(n-5,3,0)$, this $\Rightarrow$ the last equality hold.

Lemma 3.2. *$r(T(p,3,q)^c)$ more than $r(T(p+1,3,q-1)^c) > \cdots > r(T(n-5,3,0)^c)$, for $n \geq 7$, $p \geq q \geq 1$ and $p+q = n-5$.*

Proof. Suppose that $T(p,3,q)$ is a graph with some vertices labeled (see Figure 3). Assume that X is a vector of $T(p,3,q)^c$. By the Eq. (2), as $r(T(p,3,q)^c)$ greater than 0, all the pendant vertices which are adjacent to v_2 have the same values given by X, write X_1. In the same way, all the pendant vertices adjacent to v_6 have the same values, write X_7. Write $X_{v_i} =: X_i$ where $2 \leq i \leq 6$ and $r(T(p,2,q)^c) := \mu_1$. Then, from Eq. (2) on v_i where $1 \leq i \leq 7$, we obtain the following system of equations

$$\begin{cases}(\mu_1-(p+q+3))X_1=(p-1)X_1+X_3+X_4+X_5+X_6+qX_7\\(\mu_1-(q+3))X_2=X_4+X_5+X_6+qX_7\\(\mu_1-(p+q+2))X_3=pX_1+X_5+X_6+qX_7\\(\mu_1-(p+q+2))X_4=pX_1+X_2+X_6+qX_7\\(\mu_1-(p+q+2))X_5=pX_1+X_2+X_3+qX_7\\(\mu_1-(p+3))X_6=pX_1+X_2+X_3+X_4\\(\mu_1-(p+q+3))X_7=pX_1+X_2+X_3+X_4+X_5+(q-1)X_7\end{cases}$$

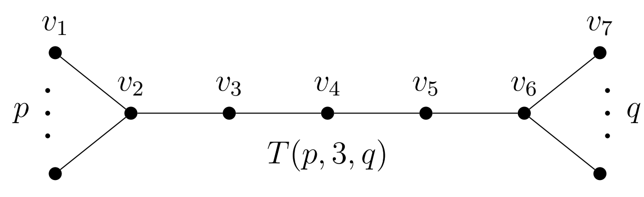

Figure 3. Graph $T(p,l,q)$ with $p+q=n-5$

transform the above system of equations into a matrix equation $(\mu_1 I-\mathbf{B}_2)X=0$ where $X=(X_1,\ldots,X_7)$ and

$$\mathbf{B}_2=\begin{bmatrix}\phi_1&0&1&1&1&1&q\\0&\phi_2&0&1&1&1&q\\p&0&\phi_3&0&1&1&q\\p&1&0&\phi_4&0&1&q\\p&1&1&0&\phi_5&0&q\\p&1&1&1&0&\phi_6&0\\p&1&1&1&1&0&\phi_7\end{bmatrix}$$

where $\phi_1=2p+q+2$, $\phi_2=q+3$, $\phi_3=\phi_4=\phi_5=p+q+2$, $\phi_6=p+3$ and $\phi_7=p+2q+2$

$$\begin{aligned}f_2(0,p,q)&=&&-2(32+140p+224p^2+195p^3+99p^4+27p^5+3p^6\\&+&&140q+472pq+626p^2q+422p^3q+149p^4q+24p^5q\\&+&&p^6q+224q^2+626pq^2+646p^2q^2+312p^3q^2\\&+&&69p^4q^2+5p^5q^2+195q^3+422pq^3+312p^2q^3\\&+&&96p^3q^3+10p^4q^3+99q^4+149pq^4+69p^2q^4\\&+&&10p^3q^4+27q^5+24pq^5+5p^2q^5+3q^6+pq^6)<0,\end{aligned}$$

and

$$\begin{aligned}f_2(\mu;p+1,q-1)-f_2(\mu;p,q)&=&&(1+p+q-\mu)(3+p+q-\mu)\\&&&(16+6p+4p^2+2p^3+6q+8pq+6p^2q\\&+&&4q^2+6pq^2+2q^3-7\mu-6p\mu-5p^2\mu-6q\mu\\&-&&10pq\mu-5q^2\mu+2\mu^2+4p\mu^2+4q\mu^2-\mu^3)(1+p-q).\end{aligned}$$

Lemma 2.3 and Lemma 2.4 $\Rightarrow 0 < \mu_1 \leq \delta(T^c) \leq q+1$ is a least zero of $f_2(\mu; p, q)$. And if $p \geq q$, then $f_2(\mu; p+1, q-1) - f_2(\mu; p, q)$. In particular, $f_2(\mu_1; p+1, q-1)$ greater than 0, we have $r(T(p, 3, q)^c)$ greater than $r(T(p+1, 3, q-1)^c)$. □

Lemma 3.3. *If the sequence $\{X_i : 1 \leq i \leq n\}$ is the decreasing one, with X_1 greater than 1 and X_n less than 0. Then for $i, j \in [1, n]$, $(X_i + X_j)^2 \leq \max\{(X_i + X_j)^2, (X_i + X_n)^2\}$ and $(X_i + X_j)^2 \leq \max\{(X_j + X_n)^2, (X_j + X_n)^2\}$ hold.*

Proof. If $X_i + X_j \geq 0$, where $1 \leq i, j \leq n$, then by monotone of $\{X_i, i = 1, 2, \ldots, n\}$, we have

$$0 \leq X_i + X_j \leq X_i + X_1,\ 0 \leq X_i + X_j \leq X_j + X_1, \tag{5}$$

Hence,

$$(X_i + X_j)^2 \leq (X_i + X_1)^2,\ (X_i + X_j)^2 \leq (X_j + X_1)^2. \tag{6}$$

Similarly if $X_i + X_j$ is at most 0, we have

$$0 \geq X_i + X_j \geq X_i + X_n, \tag{7}$$

then

$$(X_i + X_j)^2 \leq (X_i + X_n)^2. \tag{8}$$

□

Lemma 3.4. *For any tree $T \in \mathfrak{T}_n \backslash \{K_{1,n-1}\}$, $r(T^c) \geq r(T(p, l, q)^c)$ hold, where $n \geq 7$, $p, q \in [0, n-2]$, $p + q + l = n - 2$ and $l \in [2, 3]$.*

Proof. Suppose that X is a vector of T^c. Then X is not 0 and $X \perp 1$. Thus we can get a sequence $\{X_{v_i} : i = 1, 2, \ldots, n\}$ such that $X_{v_1} \geq X_{v_2} \geq \cdots \geq X_{v_n}$, $X_{v_1} > 0$, $X_{v_n} < 0$.

First we consider $l = d_T(v_1, v_n) - 1 > 3$. Let the path $v_1 T v_n := v_1 \ldots u_1 v u_2 \ldots v_n$. For any $u \in V_T$, by Lemma 3.3, we have $(X_v + X_u)^2) \leq max\{(X_v + X_{v_1})^2,\ (X_v + X_{v_n})^2\}$ if $(X_v + X_{v_1})^2 \geq (X_v + X_{v_n})^2$, then remove the edge vu_1 and plus the edge vv_1; if not, then remove the edge vu_2 and plus the edge vv_n.

Now we get a T^* such that $l^* := d_{T^*}(v_1, v_n) - 1$ less than l. In this situation, we get the following:

$$\sum_{v_i v_j \in E_T} (X_{v_i} + X_{v_j})^2 \leq \sum_{v_i v_j \in E_{T^*}} (X_{v_i} + X_{v_j})^2.$$

This procedure repeated until $l = d_T(v_1, v_n) - 1 \leq 3$. If the pendant vertex v, exists in the new graph whose neighbor u is neither v_1 nor v_n satisfying $(X_v + X_{v_1})^2 \geq (X_v + X_{v_n})^2$, then remove uv and plus vv_1; if not, then remove vu and plus vv_n. Repeat this re-arranging until T' isomorphic to $T(p, l, q),$ where $2 \leq l \leq 3$. Lemma 3.3 $\Rightarrow$

$$\sum_{v_i v_j \in E_T} (X_{v_i} + X_{v_j})^2 \leq \sum_{v_i v_j \in E_{T'}} (X_{v_i} + X_{v_j})^2.$$

Now, consider $l = d_T(v_1, v_n) - 1 = 4$; see Figure 4, if $(X_{v_1} + X_{v_j})^2 \geq (X_{v_i} + X_{v_n})^2$, remove the edge v_iv_j and plus the edge v_jv_3, if not, then remove the edge v_iv_j and plus the edge v_iv_n. By Lemma 3.3, we have

$$\sum_{v_iv_j \in E_{T(p,4,q)}} (X_{v_i} + X_{v_j})^2 \leq \sum_{v_iv_j \in E_{T(p+1,3,q)}} (X_{v_i} + X_{v_j})^2,$$

or

$$\sum_{v_iv_j \in E_{T(p,4,q)}} (X_{v_i} + X_{v_j})^2 \leq \sum_{v_iv_j \in E_{T(p,3,q+1)}} (X_{v_i} + X_{v_j})^2.$$

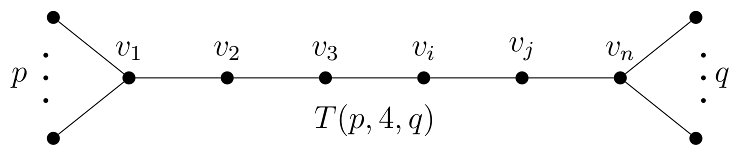

Figure 4. Graph $T(p, 4, q)$ with $p + q = n - 6$

Hence, for any $T \in \mathfrak{T}_n \backslash \{K_{1,n-1}\}$, there are some p, q, l with $p+q+l = n-2$, $p, q \in [0, n-2]$ and $l \in [2, 3]$, such that

$$\begin{aligned} r(T^c) &= X^TQ(T^c)X \\ &= X^T(J + (n-2)I)X - X^TQ(T)X \\ &\geq X^T(J + (n-2)I)X - X^TQ(T(p, l, q))X \\ &\geq X^TQ(T(p, l, q)^c)X \\ &\geq r(T(p, l, q)^c). \end{aligned}$$

□

By Lemmas 3.1 and 3.2, we get

$$r(T(p, 2, q)^c) > r(T(p+1, 2, q-1)^c) > \cdots > r(T(n-5, 2, 1)^c) = r(T(n-5, 3, 0))$$

also

$$r(T(p, 3, q)^c) > r(T(p+1, 3, q-1)^c) > \cdots > r(T(n-5, 3, 0)^c).$$

As consequence of Lemmas 3.1, 3.2 and 3.3. Now, we obtain the following:

Theorem 3.1. *For each $T \in \mathfrak{T}_n \backslash \{K_{1,n-1}\}$, $r(T^c) \geq r(T(n-5, 3, 0)^c)$ hold (where $n \geq 7$), with equality $\Leftrightarrow T \cong r(T(n-5, 3, 0))$.*

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