On cycle-irregularity strength of ladders and fan graphs

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Abstract

A simple graph $G = (V(G), E(G))$ admits an $H$-covering if every edge in $E(G)$ belongs to at least one subgraph of $G$ isomorphic to a given graph $H$. A total $k$-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ is called to be an $H$-irregular total $k$-labeling of the graph $G$ admitting an $H$-covering if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ there is $wt_\varphi(H') \neq wt_\varphi(H'')$, where $wt_\varphi(H) = \sum_{v \in V(H)} \varphi(v) + \sum_{e \in E(H)} \varphi(e)$. The total $H$-irregularity strength of a graph $G$, denoted by $\text{ths}(G, H)$, is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling. In this paper we determine the exact value of the cycle-irregularity strength of ladders and fan graphs.

Keywords: total $H$-irregular labeling, total cycle-irregularity strength, ladder, fan graph
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1. Introduction

Let $G$ be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is the vertex-set or the edge-set, the labelings are

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called respectively vertex labelings or edge labelings. If the domain is $V(G) \cup E(G)$ then we call the labeling total labeling. The most complete recent survey of graph labelings is [12].

Bača, Jendroľ, Miller and Ryan in [9] defined the total labeling $\varphi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$ to be an edge irregular total $k$-labeling of the graph $G$ if for every two different edges $xy$ and $x'y'$ of $G$ one has

$$wt(xy) = \varphi(x) + \varphi(xy) + \varphi(y) \neq wt(x'y') = \varphi(x') + \varphi(x'y') + \varphi(y').$$

The total edge irregularity strength, $\text{tes}(G)$, is defined as the minimum $k$ for which $G$ has an edge irregular total $k$-labeling.

Ivančo and Jendroľ [14] posed a conjecture that for arbitrary graph $G$ different from $K_5$ and maximum degree $\Delta(G)$,

$$\text{tes}(G) = \max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}.$$

This conjecture has been verified for complete graphs and complete bipartite graphs in [15] and [16], for the Cartesian, categorical and strong product of two paths in [17, 3, 2], for the categorical product of two cycles in [4], for generalized Petersen graphs in [13], for generalized prisms in [16], for the Cartesian, categorical and strong products of two paths in [17, 3, 2], for the categorical labeling introduced in [9] and the vertex labelings introduced in [1]. In [20] there is confirmed the conjecture proposed by Nurdin, Baskoro, Salman and Gaos [18] for all trees with maximum degree five. The edge irregularity strength of some chain graphs is determined in [5].

There are several modifications of irregularity strength, namely the total vertex irregularity strength introduced in [9] and the edge irregularity strength introduced in [1]. In [20] there is confirmed the conjecture proposed by Nurdin, Baskoro, Salman and Gaos [18] for all trees with maximum degree five. The edge irregularity strength of some chain graphs is determined in [5].

An edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. Then it is said that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. If every subgraph $H_i$ is isomorphic to a given graph $H$, then the graph $G$ admits an $H$-covering. Note, that in this case every subgraph isomorphic to $H$ must be in the $H$-covering.

Let $G$ be a graph admitting $H$-covering. For the subgraph $H \subseteq G$ under the total $k$-labeling $\varphi$, we define the associated $H$-weight as

$$wt_\varphi(H) = \sum_{v \in V(H)} \varphi(v) + \sum_{e \in E(H)} \varphi(e).$$

A total $k$-labeling $\varphi$ is called an $H$-irregular total $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ there is $wt_\varphi(H') \neq wt_\varphi(H'')$. The total $H$-irregularity strength of a graph $G$, denoted $\text{ths}(G, H)$, is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling. If $H$ is isomorphic to $K_2$, then the $K_2$-irregular total $k$-labeling is isomorphic to the edge irregular total $k$-labeling and thus the total $K_2$-irregularity strength of a graph $G$ is equivalent to the total edge irregularity strength, that is $\text{ths}(G, K_2) = \text{tes}(G)$.

Analogously we can define an $H$-irregular edge $k$-labeling and an $H$-irregular vertex $k$-labeling. For the subgraph $H \subseteq G$ under the vertex $k$-labeling $\alpha$, $\alpha : V(G) \rightarrow \{1, 2, \ldots, k\}$, the
associated $H$-weight is defined as
\[ \text{wt}_\alpha(H) = \sum_{v \in V(H)} \alpha(v) \]
and under the edge $k$-labeling $\beta, \beta : E(G) \to \{1, 2, \ldots, k\}$, we define the associated $H$-weight
\[ \text{wt}_\beta(H) = \sum_{e \in E(H)} \beta(e). \]

A vertex $k$-labeling $\alpha$ is called an $H$-irregular vertex $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ there is $\text{wt}_\alpha(H') \neq \text{wt}_\alpha(H'')$. The vertex $H$-irregularity strength of a graph $G$, denoted by $\text{vhs}(G, H)$, is the smallest integer $k$ such that $G$ has an $H$-irregular vertex $k$-labeling. Note, that $\text{vhs}(G, H) = \infty$ if there exist two subgraphs in $G$ isomorphic to $H$ that have the same vertex sets. An edge $k$-labeling $\beta$ is called an $H$-irregular edge $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ there is $\text{wt}_\beta(H') \neq \text{wt}_\beta(H'')$. The edge $H$-irregularity strength of a graph $G$, denoted by $\text{ehs}(G, H)$, is the smallest integer $k$ such that $G$ has an $H$-irregular edge $k$-labeling.

The notion of the vertex (edge) $H$-irregularity strength was introduced in [6]. The total $H$-irregularity strength was defined in [7] and its lower bound is given by the following theorem.

**Theorem 1.1.** [7] Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Then
\[ \text{ths}(G, H) \geq \left[ 1 + \frac{t-1}{|V(H)|+|E(H)|} \right]. \]

The precise value of the total $H$-irregularity strength of $G$-amalgamation of graphs is given in [8] and it proves that the lower bound in Theorem 1.1 is tight.

Let $G$ be a graph admitting $H$-covering. By the symbol $\mathbb{H^S}_m = (H^S_1, H^S_2, \ldots, H^S_m)$, we denote the set of all subgraphs of $G$ isomorphic to $H$ such that the graph $S, S \neq H$, is their maximum common subgraph. Thus $V(S) \subseteq V(H^S_i)$ and $E(S) \subseteq E(H^S_i)$ for every $i = 1, 2, \ldots, m$. The next theorem presented in [7] gives another lower bound of the total $H$-irregularity strength.

**Theorem 1.2.** [7] Let $G$ be a graph admitting an $H$-covering. Let $S_i, i = 1, 2, \ldots, z$, be all subgraphs of $G$ such that $S_i$ is a maximum common subgraph of $m_i$, $m_i \geq 2$, subgraphs of $G$ isomorphic to $H$. Then
\[ \text{ths}(G, H) \geq \max \left\{ \left[ 1 + \frac{m_1-1}{|V(H/S_1)|+|E(H/S_1)|} \right], \ldots, \left[ 1 + \frac{m_z-1}{|V(H/S_z)|+|E(H/S_z)|} \right] \right\}. \]

In this paper we determine the exact value of the cycle-irregularity strength of ladders and fan graphs.

2. Total cycle-irregular labelings of ladders

Let $L_n \cong P_n \boxtimes P_2$, $n \geq 3$, be a ladder with the vertex set $V(L_n) = \{v_i, u_i : i = 1, 2, \ldots, n\}$ and the edge set $E(L_n) = \{v_i u_{i+1}, u_i u_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_i u_i : i = 1, 2, \ldots, n\}$.

In [7] is determined the exact value of the total cycle-irregularity strength of ladders when the cycle is either of length 4 or 6.
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Theorem 2.1. [7] Let \( L_n \cong P_n \square P_2, n \geq 3 \), be a ladder admitting a \( C_{2m} \)-covering, \( m = 2, 3 \). Then

\[
\text{ths}(L_n, C_{2m}) = \lceil \frac{3m+n}{4m} \rceil.
\]

In this section we extend the previous result for all feasible cycle-coverings.

Theorem 2.2. Let \( L_n \cong P_n \square P_2, n \geq 3 \), be a ladder admitting a \( C_{2m} \)-covering, \( 2 \leq m \leq \lceil (n+1)/2 \rceil \). Then

\[
\text{ths}(L_n, C_{2m}) = \lceil \frac{3m+n}{4m} \rceil.
\]

Proof. It is easy to see that the ladder \( L_n \cong P_n \square P_2, n \geq 3 \), admits a \( C_{2m} \)-covering for \( m = 2, 3, \ldots, \lceil (n+1)/2 \rceil \). Put \( k = \lceil \frac{3m+n}{4m} \rceil \). According to Theorem 1.1 \( k \) is the lower bound of \( \text{ths}(L_n, C_{2m}) \). In order to show the converse inequality, it only remains to describe a \( C_{2m} \)-irregular total \( k \)-labeling \( \varphi_m : V(L_n) \cup E(L_n) \to \{1, 2, \ldots, k\} \) as follows

\[
\begin{align*}
\varphi_m(v_i) &= \left\lceil \frac{i+3m}{4m} \right\rceil & \text{for } i = 1, 2, \ldots, n, \\
\varphi_m(u_i) &= \left\lceil \frac{i}{4m} \right\rceil & \text{for } i \equiv 0, 3m \pmod{4m}, i = 3m, 4m, 7m, 8m, \ldots, n, \\
\varphi_m(v_iv_{i+1}) &= \left\lceil \frac{i+m}{4m} \right\rceil & \text{for } i = 1, 2, \ldots, n-1, \\
\varphi_m(u_iu_{i+1}) &= \left\lceil \frac{i+1}{4m} \right\rceil & \text{for } i = 1, 2, \ldots, n-1, \\
\varphi_m(v_iu_i) &= \left\lceil \frac{i+2m}{4m} \right\rceil & \text{for } i = 1, 2, \ldots, n.
\end{align*}
\]

We can see that all edge labels and vertex labels are at most \( k \).

Every cycle \( C_{2m} \) in \( L_n \) is of the form

\[
C_{2m}^i = v_iv_{i+1} \cdots v_{i+m-1}u_{i+m-1}u_{i+m-2} \cdots u_iv_i,
\]

where \( i = 1, 2, \ldots, n-m+1 \). It is easy to see that every edge of \( L_n \) belongs to at least one cycle \( C_{2m}^i \) if \( m = 2, 3, \ldots, \lceil (n+1)/2 \rceil \).

For the \( C_{2m} \)-weight of the cycle \( C_{2m}^i, i = 1, 2, \ldots, n-m+1 \), under the total labeling \( \varphi_m \), we get

\[
wt_{\varphi_m}(C_{2m}^i) = \sum_{v \in V(C_{2m}^i)} \varphi_m(v) + \sum_{e \in E(C_{2m}^i)} \varphi_m(e)
\]

\[
= \sum_{j=0}^{m-1} \varphi_m(v_{i+j}) + \sum_{j=0}^{m-1} \varphi_m(u_{i+j}) + \sum_{j=0}^{m-2} \varphi_m(v_{i+j}v_{i+j+1}) + \sum_{j=0}^{m-2} \varphi_m(u_{i+j}u_{i+j+1})
\]

\[
+ \varphi_m(v_iu_i) + \varphi_m(v_{i+m-1}u_{i+m-1})
\]

(1)

and for the \( C_{2m} \)-weight of the cycle \( C_{2m}^{i+1}, i = 1, 2, \ldots, n-m \), we obtain

\[
wt_{\varphi_m}(C_{2m}^{i+1}) = \sum_{v \in V(C_{2m}^{i+1})} \varphi_m(v) + \sum_{e \in E(C_{2m}^{i+1})} \varphi_m(e)
\]

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Let us distinguish four cases.

**Case 1.** $i \equiv 0 \pmod{4m}$

For the difference of weights of cycles we get

$$\begin{align*}
\varphi_m(C_{2m}^{i+1}) - \varphi_m(C_{2m}^i) &= \varphi_m(v_{i+1}u_{i+1}) + \varphi_m(v_{i+m-1}u_{i+m}) + \varphi_m(v_{i+m}u_{i+m}) \\
&\quad + \varphi_m(u_{i+m}) + \varphi_m(u_{i+m-1}u_{i+m}) - \varphi_m(v_i) - \varphi_m(v_iu_{i+1}) \\
&\quad - \varphi_m(v_iu_i) - \varphi_m(u_i) - \varphi_m(u_iu_{i+1}) - \varphi_m(v_{i+m-1}u_{i+m-1}).
\end{align*}$$

Let us distinguish four cases.

**Case 1.** $i \equiv 0 \pmod{4m}$

For the difference of weights of cycles we get

$$\begin{align*}
\varphi_m(C_{2m}^{i+1}) - \varphi_m(C_{2m}^i) &= \left[\frac{i+1+2m}{4m}\right] + \left[\frac{i+2m-1}{4m}\right] + \left[\frac{i+4m}{4m}\right] + \left[\frac{i+3m}{4m}\right] + \left[\frac{i+3m-1}{4m}\right] + \left[\frac{i+m}{4m}\right] \\
&\quad - \left[\frac{i+3m}{4m}\right] - \left[\frac{i+m}{4m}\right] - \left[\frac{i+2m}{4m}\right] - \left[\frac{i+1}{4m}\right] - \left[\frac{i+3m-1}{4m}\right] \\
&= \left[\frac{i+2m+1}{4m}\right] + \left[\frac{i+2m}{4m}\right] + 1 - \left[\frac{i+2m}{4m}\right] - \left[\frac{i+1}{4m}\right] - \left[\frac{i+3m-1}{4m}\right].
\end{align*}$$

Since $i = 4mt$, $t = 1, 2, \ldots$, thus

$$\begin{align*}
\varphi_m(C_{2m}^{i+1}) - \varphi_m(C_{2m}^i) &= \left[\frac{4mt+2m+1}{4m}\right] + \left[\frac{4mt+2m-1}{4m}\right] + \left[\frac{4mt+2m}{4m}\right] + 1 - \left[\frac{4mt+2m}{4m}\right] - \left[\frac{4mt+1}{4m}\right] \\
&= t + \left[\frac{2m+1}{4m}\right] + t + \left[\frac{2m-1}{4m}\right] + 1 - t - \left[\frac{2m}{4m}\right] - t - \left[\frac{1}{4m}\right] = 1.
\end{align*}$$

**Case 2.** $i \equiv 2m \pmod{4m}$

For the difference of weights of cycles we get

$$\begin{align*}
\varphi_m(C_{2m}^{i+1}) - \varphi_m(C_{2m}^i) &= \left[\frac{i+1+2m}{4m}\right] + \left[\frac{i+2m-1}{4m}\right] + \left[\frac{i+4m}{4m}\right] + \left[\frac{i+3m}{4m}\right] + \left[\frac{i+m}{4m}\right] + \left[\frac{i+m}{4m}\right] \\
&\quad - \left[\frac{i+3m}{4m}\right] - \left[\frac{i+m}{4m}\right] - \left[\frac{i+2m}{4m}\right] - \left[\frac{i+1}{4m}\right] - \left[\frac{i+3m-1}{4m}\right] \\
&= \left[\frac{i+2m+1}{4m}\right] + 1 + \left[\frac{i+2m}{4m}\right] + \left[\frac{i+m}{4m}\right] - \left[\frac{i+2m}{4m}\right] - \left[\frac{i+1}{4m}\right] - \left[\frac{i+3m-1}{4m}\right].
\end{align*}$$

For $i = 4mt + 2m$, $t = 1, 2, \ldots$, we get

$$\begin{align*}
\varphi_m(C_{2m}^{i+1}) - \varphi_m(C_{2m}^i) &= \left[\frac{4mt+2m+2m+1}{4m}\right] + 1 + \left[\frac{4mt+2m}{4m}\right] + \left[\frac{4mt+2m+1}{4m}\right] \\
&\quad - \left[\frac{4mt+2m+2m}{4m}\right] - \left[\frac{4mt+2m+2}{4m}\right] - \left[\frac{4mt+2m+3m-1}{4m}\right] \\
&= t + 1 + \left[\frac{1}{4m}\right] + t + \left[\frac{2m+1}{4m}\right] + t + \left[\frac{3m}{4m}\right] - t - 1 - t - \left[\frac{2m+1}{4m}\right] \\
&= t - 1 - \left[\frac{m-1}{4m}\right] = 1.
\end{align*}$$

**Case 3.** $i \equiv 3m \pmod{4m}$

Now

$$\begin{align*}
\varphi_m(C_{2m}^{i+1}) - \varphi_m(C_{2m}^i) &= \left[\frac{i+1+2m}{4m}\right] + \left[\frac{i+2m-1}{4m}\right] + \left[\frac{i+4m}{4m}\right] + \left[\frac{i+3m}{4m}\right] + \left[\frac{i+m}{4m}\right] + \left[\frac{i+m}{4m}\right]
\end{align*}$$
This concludes the proof.

Case 4. $i \neq 0, 2m, 3m \pmod{4m}$

In this case for the difference of weights of cycles we obtain

$$\begin{align*}
wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^i) &= \left[\frac{i+1+2m}{4m}\right] + \left[\frac{i+2m-1}{4m}\right] + \left[\frac{i+4m}{4m}\right] + \left[\frac{i+3m}{4m}\right] + \left[\frac{i+3m-1}{4m}\right] + \left[\frac{i+m}{4m}\right] \\
&\quad - \left[\frac{i+3m}{4m}\right] - \left[\frac{i+m}{4m}\right] - \left[\frac{i+2m+1}{4m}\right] - \left[\frac{i+2m-1}{4m}\right] - \left[\frac{i+1}{4m}\right] - \left[\frac{i+3m-1}{4m}\right] \\
&= t + 1 + \left[\frac{s+1}{4m}\right] + t + 1 + \left[\frac{s+2m}{4m}\right] + t + 1 - t - 1 - \left[\frac{m}{4m}\right] - t \\
&\quad - \left[\frac{3m+1}{4m}\right] - t - 1 - \left[\frac{2m-1}{4m}\right] = 1.
\end{align*}$$

Let $i = 4mt + s$, $t = 0, 1, 2, \ldots$ and $1 \leq s \leq 4m - 1$, $s \neq 2m, 3m$. Then we have

$$\begin{align*}
\begin{align*}
wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^i) &= \left[\frac{4mt+s+2m+1}{4m}\right] + \left[\frac{4mt+s+4}{4m}\right] - \left[\frac{4mt+s+2m}{4m}\right] - \left[\frac{4mt+s+1}{4m}\right] \\
&= t + \left[\frac{s+2m+1}{4m}\right] + t + 1 + \left[\frac{s}{4m}\right] - t - \left[\frac{s+2m}{4m}\right] - t - \left[\frac{s+1}{4m}\right] \\
&= \left[\frac{s+2m+1}{4m}\right] + \left[\frac{s}{4m}\right] - \left[\frac{s+2m}{4m}\right] - \left[\frac{s+1}{4m}\right] + 1.
\end{align*}
\end{align*}$$

If $1 \leq s \leq 2m - 1$ then

$$\left[\frac{s+2m+1}{4m}\right] = 1, \quad \left[\frac{s}{4m}\right] = 1, \quad \left[\frac{s+2m}{4m}\right] = 1 \quad \text{and} \quad \left[\frac{s+1}{4m}\right] = 1.$$

If $2m + 1 \leq s \leq 3m - 1$ or $3m + 1 \leq s \leq 4m - 1$ then

$$\left[\frac{s+2m+1}{4m}\right] = 2, \quad \left[\frac{s}{4m}\right] = 1, \quad \left[\frac{s+2m}{4m}\right] = 2 \quad \text{and} \quad \left[\frac{s+1}{4m}\right] = 1.$$

We can see that for every value of parameter $s$

$$\begin{align*}
wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^i) &= 1.
\end{align*}$$

Previous cases prove that the labeling $\varphi_m$ is the desired $C_{2m}$-irregular total $k$-labeling of $L_n$. This concludes the proof. $\square$

3. Total cycle-irregular labelings of fan graphs

A fan graph $F_n$, $n \geq 2$, is a graph obtained by joining all vertices of a path $P_n$ to a further vertex. Thus $F_n$ contains $n+1$ vertices, say, $u, v_1, v_2, \ldots, v_n$ and $2n-1$ edges $uv_i$, $i = 1, 2, \ldots, n$, and $v_iv_{i+1}$, $i = 1, 2, \ldots, n-1$.

In [7] was given the exact value of the total $C_3$-irregularity strength of the fan graph $F_n$. 

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Theorem 3.1. [7] Let $F_n, n \geq 2$, be a fan graph on $n+1$ vertices. Then

\[ \text{ths}(F_n, C_m) = \left\lceil \frac{n+3}{5} \right\rceil. \]

The next theorem completes this result for arbitrary cycle-covering.

Theorem 3.2. Let $F_n$ be a fan graph on $n+1$ vertices, $n \geq 2$ and $3 \leq m \leq \lceil (n+3)/2 \rceil$. Then

\[ \text{ths}(F_n, C_m) = \left\lceil \frac{n+m}{2m-1} \right\rceil. \]

Proof. Clearly, for every $m$, $3 \leq m \leq \lceil (n+3)/2 \rceil$, the fan graph $F_n$ admits a $C_m$-covering with exactly $n-m+2$ cycles $C_m$. In view of the lower bound from Theorem 1.2 it suffices to prove the existence of a $C_m$-irregular total labeling $\varphi : V(F_n) \cup E(F_n) \rightarrow \{1, 2, \ldots, \lceil (n+m)/(2m-1) \rceil\}$ such that $wt_\varphi(C_m) \neq wt_\varphi(C_m)$ for every $i, j = 1, 2, \ldots, n-m+2$, $j \neq i$. We describe the irregular total labeling $\varphi_m$ in the following way

\[
\begin{align*}
\varphi_m(u) &= 1, \\
\varphi_m(v_i) &= \begin{cases} 
\left\lceil \frac{i+2}{2m-1} \right\rceil & \text{for } i \equiv m+1 \pmod{2m-1}, i = 1, 2, \ldots, n, \\
\left\lceil \frac{i+2}{2m-1} \right\rceil + 1 & \text{for } i \equiv m+1 \pmod{2m-1}, i = m+1, 3m, \ldots, n,
\end{cases} \\
\varphi_m(v_i v_{i+1}) &= \begin{cases} 
\left\lceil \frac{i+m}{2m-1} \right\rceil & \text{for } i \equiv m+1, 2m-3, 2m-1 \pmod{2m-1}, \\
i = 1, 2, \ldots, n, \\
\left\lceil \frac{i+m}{2m-1} \right\rceil - 1 & \text{for } i \equiv m+1, 2m-3, 2m-1 \pmod{2m-1}, \\
i = m+1, 2m-3, 2m-1, 3m, 4m-4, 4m-2, \ldots, n,
\end{cases} \\
\varphi_m(v_i u) &= \begin{cases} 
\left\lceil \frac{i+m}{2m-1} \right\rceil - 1 & \text{for } i \equiv m+1, 2m-2 \pmod{2m-1}, \\
i = m+1, 2m-2, 3m, 4m-3, \ldots, n.
\end{cases}
\end{align*}
\]

Evidently, every edge label and vertex label is not greater than $\lceil (n+m)/(2m-1) \rceil$. Every cycle $C_m$ in $F_n$ is of the form $C_m^i = v_i v_{i+1} \ldots v_{i+m-2} u v_i$, where $i = 1, 2, \ldots, n-m+2$.

For the $C_m$-weight of the cycle $C_m^i$, $i = 1, 2, \ldots, n-m+2$, under the total labeling $\varphi_m$, we get

\[
\text{wt}_\varphi(C_m^i) = \sum_{v \in V(C_m^i)} \varphi_m(v) + \sum_{e \in E(C_m^i)} \varphi_m(e) = \sum_{j=0}^{m-2} \varphi_m(v_{i+j}) + \varphi_m(u) + \sum_{j=0}^{m-3} \varphi_m(v_{i+j} v_{i+j+1}) + \varphi_m(v_i u) + \varphi_m(v_{i+m-2} u) \tag{3}
\]
and for the $C_m$-weight of the cycle $C_m^{i+1}$, $i = 1, 2, \ldots, n - m + 1$, we obtain

$$wt_{\varphi_m}(C_m^{i+1}) = \sum_{v \in V(C_m^{i+1})} \varphi_m(v) + \sum_{e \in E(C_m^{i+1})} \varphi_m(e)$$

$$= \sum_{j=1}^{m-1} \varphi_m(v_{i+j}) + \varphi_m(u) + \sum_{j=1}^{m-2} \varphi_m(v_{i+j}v_{i+j+1}) + \varphi_m(v_{i+1}u) + \varphi_m(v_{i+m-1}u). \quad (4)$$

Now we count the difference between the $C_m$-weights of the cycle $C_m^{i+1}$ and $C_m^i$ for $i = 1, 2, \ldots, n - m + 1$. According to (3) and (4) we get

$$wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) = \varphi_m(v_{i+m-1}) + \varphi_m(v_{i+m-2}v_{i+m-1}) + \varphi_m(v_{i+m-1}u) + \varphi_m(v_{i+1}u)$$

$$- \varphi_m(v_i) - \varphi_m(v_{i+1}) - \varphi_m(v_i) - \varphi_m(v_{i+m-2}u).$$

Let us distinguish nine cases.

**Case 1.** $i \equiv 2 \pmod{(2m-1)}$, i.e., $i = 2 + (2m-1)t$, $t = 0, 1, \ldots, \text{then}$

$$wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) = \varphi_m(v_{m+1}(2m-1)t) + \varphi_m(v_{m+1}(2m-1)t)$$

$$+ \varphi_m(v_{3+(2m-1)t}u) + \varphi_m(v_{3+(2m-1)t}u) - \varphi_m(v_{2+(2m-1)t})$$

$$- \varphi_m(v_{2+(2m-1)t}v_{3+(2m-1)t}) - \varphi_m(v_{2+(2m-1)t}u)$$

$$- \varphi_m(v_{2+(2m-1)t}u)$$

$$= \left[ \frac{m+3+(2m-1)t}{2m-1} \right] + 1 + \left[ \frac{2m+1+(2m-1)t}{2m-1} \right] - 1$$

$$+ \left[ \frac{m+3+(2m-1)t}{2m-1} \right] - \left[ \frac{4+(2m-1)t}{2m-1} \right] - \left[ \frac{m+2+(2m-1)t}{2m-1} \right]$$

$$= (1 + t) + (2 + t) + (2 + t) - 1 + (1 + t) - (1 + t)$$

$$- (1 + t) - (1 + t) - (2 + t) = 1.$$
Case 3. \( i \equiv m - 1 \pmod{(2m - 1)} \), i.e., \( i = m - 1 + (2m - 1)t \), \( t = 0, 1, \ldots \), then we get

\[
\begin{align*}
wt_{\varphi_m}(C_{m+1}^{i+1}) - wt_{\varphi_m}(C_m^i) &= \varphi_m(v_{2m-2+(2m-1)t}) + \varphi_m(v_{2m-3+(2m-1)t}v_{2m-2+(2m-1)t}) \\
&+ \varphi_m(v_{2m-2+(2m-1)t}u) + \varphi_m(v_{m+(2m-1)t}u) \\
&- \varphi_m(v_{m-1+(2m-1)t}) - \varphi_m(v_{m-1+(2m-1)t}v_{m+(2m-1)t}) \\
&- \varphi_m(v_{m-1+(2m-1)t}u) - \varphi_m(v_{2m-3+(2m-1)t}u) \\
&= \left[ \frac{2m+(2m-1)t}{2m-1} \right] + \left[ \frac{3m-3+(2m-1)t}{2m-1} \right] - 1 + \left[ \frac{3m-2+(2m-1)t}{2m-1} \right] \\
&- 1 + \left[ \frac{2m+(2m-1)t}{2m-1} \right] - \left[ \frac{m+1+(2m-1)t}{2m-1} \right] - \left[ \frac{2m-1+(2m-1)t}{2m-1} \right] \\
&= (2 + t) + (2 + t) - 1 + (2 + t) - 1 + (2 + t) - (1 + t) \\
&- (1 + t) - (1 + t) - (2 + t) = 1.
\end{align*}
\]

Case 4. \( i \equiv m \pmod{(2m - 1)} \), i.e., \( i = m + (2m - 1)t \), \( t = 0, 1, \ldots \) In this case holds

\[
\begin{align*}
wt_{\varphi_m}(C_{m+1}^{i+1}) - wt_{\varphi_m}(C_m^i) &= \varphi_m(v_{2m-2+(2m-1)t}) + \varphi_m(v_{2m-3+(2m-1)t}v_{2m-2+(2m-1)t}) \\
&+ \varphi_m(v_{2m-2+(2m-1)t}u) + \varphi_m(v_{m+1+(2m-1)t}u) - \varphi_m(v_{m+(2m-1)t}) \\
&- \varphi_m(v_{m+1+(2m-1)t}v_{m+1+(2m-1)t}) - \varphi_m(v_{m+1+(2m-1)t}u) \\
&- \varphi_m(v_{2m-2+(2m-1)t}u) \\
&= \left[ \frac{2m+1+(2m-1)t}{2m-1} \right] + \left[ \frac{3m-2+(2m-1)t}{2m-1} \right] + \left[ \frac{3m-1+(2m-1)t}{2m-1} \right] \\
&+ \left[ \frac{2m+1+(2m-1)t}{2m-1} \right] - 1 - \left[ \frac{m+2+(2m-1)t}{2m-1} \right] - \left[ \frac{2m+2+(2m-1)t}{2m-1} \right] \\
&- \left[ \frac{2m+2+(2m-1)t}{2m-1} \right] - \left[ \frac{3m-2+(2m-1)t}{2m-1} \right] + 1 \\
&= (2 + t) + (2 + t) + (2 + t) + (2 + t) - 1 - (1 + t) - (2 + t) \\
&- (2 + t) - (2 + t) + 1 = 1.
\end{align*}
\]

Case 5. \( i \equiv m + 1 \pmod{(2m - 1)} \), i.e., \( i = m + 1 + (2m - 1)t \), \( t = 0, 1, \ldots \), thus

\[
\begin{align*}
wt_{\varphi_m}(C_{m+1}^{i+1}) - wt_{\varphi_m}(C_m^i) &= \varphi_m(v_{2m+2+(2m-1)t}) + \varphi_m(v_{2m+1+(2m-1)t}v_{2m+2+(2m-1)t}) \\
&+ \varphi_m(v_{2m+2+(2m-1)t}u) + \varphi_m(v_{m+2+(2m-1)t}u) - \varphi_m(v_{m+1+(2m-1)t}) \\
&- \varphi_m(v_{m+1+(2m-1)t}v_{m+2+(2m-1)t}) - \varphi_m(v_{m+1+(2m-1)t}u) \\
&- \varphi_m(v_{2m+1+(2m-1)t}u) \\
&= \left[ \frac{2m+2+(2m-1)t}{2m-1} \right] + \left[ \frac{3m-1+(2m-1)t}{2m-1} \right] - 1 + \left[ \frac{3m+(2m-1)t}{2m-1} \right] \\
&+ \left[ \frac{2m+2+(2m-1)t}{2m-1} \right] - \left[ \frac{m+3+(2m-1)t}{2m-1} \right] - 1 - \left[ \frac{2m+1+(2m-1)t}{2m-1} \right] + 1 \\
&- \left[ \frac{2m+1+(2m-1)t}{2m-1} \right] + 1 - \left[ \frac{3m-1+(2m-1)t}{2m-1} \right] \\
&= (2 + t) + (2 + t) + (2 + t) + (2 + t) - 1 - (1 + t) - (2 + t) \\
&- (2 + t) - (2 + t) + 1 = 1.
\end{align*}
\]
\[
(2 + t) + (2 + t) - 1 + (2 + t) + (2 + t) - (1 + t) - 1 \\
- (2 + t) + 1 - (2 + t) + 1 - (2 + t) = 1.
\]

**Case 6.** \(i \equiv 2m - 3 \pmod{2m - 1}\), i.e., \(i = 2m - 3 + (2m - 1)t, t = 0, 1, \ldots,\) thus

\[
\begin{align*}
wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) &= \varphi_m(v_{3m-4(2m-1)t}) + \varphi_m(v_{3m-5(2m-1)t})v_{3m-4(2m-1)t}) \\
&+ \varphi_m(v_{3m-4(2m-1)t}) + \varphi_m(v_{3m-4(2m-1)t})v_{3m-4(2m-1)t}) \\
&- \varphi_m(v_{2m-3(2m-1)t}) - \varphi_m(v_{2m-4(2m-1)t})v_{2m-2(2m-1)t}) \\
&- \varphi_m(v_{2m-3(2m-1)t}) - \varphi_m(v_{2m-4(2m-1)t})v_{2m-2(2m-1)t}) \\
&= \left[ \frac{3m-2+(2m-1)t}{2m-1} \right] + \left[ \frac{4m-5+(2m-1)t}{2m-1} \right] + \left[ \frac{4m-3+(2m-1)t}{2m-1} \right] \\
&+ \left[ \frac{3m-3+(2m-1)t}{2m-1} \right] - \left[ \frac{2m-1+(2m-1)t}{2m-1} \right] - \left[ \frac{3m-3+(2m-1)t}{2m-1} \right] \\
&+ 1 - \left[ \frac{3m-3+(2m-1)t}{2m-1} \right] - \left[ \frac{4m-5+(2m-1)t}{2m-1} \right] \\
&= (2 + t) + (2 + t) + (2 + t) + (2 + t) - (1 + t) \\
&- (2 + t) + 1 - (2 + t) - (2 + t) = 1.
\end{align*}
\]

**Case 7.** \(i \equiv 2m - 2 \pmod{2m - 1}\), i.e., \(i = 2m - 2 + (2m - 1)t, t = 0, 1, \ldots,\) thus

\[
\begin{align*}
wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) &= \varphi_m(v_{3m-3(2m-1)t}) + \varphi_m(v_{3m-4(2m-1)t})v_{3m-3(2m-1)t}) \\
&+ \varphi_m(v_{3m-3(2m-1)t}) + \varphi_m(v_{3m-4(2m-1)t})v_{3m-3(2m-1)t}) \\
&- \varphi_m(v_{2m-2(2m-1)t}) - \varphi_m(v_{2m-3(2m-1)t})v_{2m-1(2m-1)t}) \\
&- \varphi_m(v_{2m-2(2m-1)t}) - \varphi_m(v_{2m-3(2m-1)t})v_{2m-1(2m-1)t}) \\
&= \left[ \frac{3m-1+(2m-1)t}{2m-1} \right] + \left[ \frac{4m-4+(2m-1)t}{2m-1} \right] + \left[ \frac{4m-3+(2m-1)t}{2m-1} \right] \\
&+ \left[ \frac{3m-1+(2m-1)t}{2m-1} \right] - \left[ \frac{2m-1+(2m-1)t}{2m-1} \right] - \left[ \frac{3m-2+(2m-1)t}{2m-1} \right] \\
&+ 1 - \left[ \frac{3m-2+(2m-1)t}{2m-1} \right] - \left[ \frac{4m-4+(2m-1)t}{2m-1} \right] \\
&= (2 + t) + (2 + t) + (2 + t) + (2 + t) - (2 + t) \\
&- (2 + t) - (2 + t) + 1 - (2 + t) = 1.
\end{align*}
\]

**Case 8.** \(i \equiv 2m - 1 \pmod{2m - 1}\), i.e., \(i = 2m - 1 + (2m - 1)t, t = 0, 1, \ldots,\) thus

\[
\begin{align*}
wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) &= \varphi_m(v_{3m-2(2m-1)t}) + \varphi_m(v_{3m-3(2m-1)t})v_{3m-2(2m-1)t}) \\
&+ \varphi_m(v_{3m-2(2m-1)t}) + \varphi_m(v_{3m-3(2m-1)t})v_{3m-2(2m-1)t}) \\
&- \varphi_m(v_{2m-1(2m-1)t}) - \varphi_m(v_{2m-2(2m-1)t})v_{2m-1(2m-1)t}) \\
&- \varphi_m(v_{2m-1(2m-1)t}) - \varphi_m(v_{2m-2(2m-1)t})v_{2m-1(2m-1)t}) \\
&= \left[ \frac{3m+2(2m-1)t}{2m-1} \right] + \left[ \frac{4m-3+(2m-1)t}{2m-1} \right] + \left[ \frac{4m-2+(2m-1)t}{2m-1} \right] \\
&+ \left[ \frac{3m+2(2m-1)t}{2m-1} \right] - \left[ \frac{2m+1+(2m-1)t}{2m-1} \right] - \left[ \frac{3m-1+(2m-1)t}{2m-1} \right] + 1
\end{align*}
\]
For every $\phi$.

Now we distinguish three subcases.

Thus, according to all these cases we get that $wt_{\phi_m}(C_{m}^{i+1}) - wt_{\phi_m}(C_{m}^{i}) = 1$.

Case 9. $i \neq 2, 3, m - 1, m + 1, 2m - 3, 2m - 2, 2m - 1 \pmod{(2m - 1)}$. Then

$$wt_{\phi_m}(C_{m}^{i+1}) - wt_{\phi_m}(C_{m}^{i}) = \phi_m(v_{i+m-1}) + \phi_m(v_{i+m-2}v_{i+m-1}) + \phi_m(v_{i+m-1}u) + \phi_m(v_{i+1}u)$$

$$- \phi_m(v_i) - \phi_m(v_{i+1}) - \phi_m(v_{i+m-2})$$

$$= \left\lceil \frac{(i+m-1)+2}{2m-1} \right\rceil + \left\lceil \frac{(i+m-2)+m}{2m-1} \right\rceil + \left\lceil \frac{(i+m-1)+m}{2m-1} \right\rceil + \left\lceil \frac{(i+1)+m}{2m-1} \right\rceil$$

$$- \left\lceil \frac{i+2}{2m-1} \right\rceil - \left\lceil \frac{i+m}{2m-1} \right\rceil - \left\lceil \frac{i+m}{2m-1} \right\rceil + \left\lceil \frac{(i+m-2)+m}{2m-1} \right\rceil$$

$$= 2 \left\lceil \frac{i+m+1}{2m-1} \right\rceil - 2 \left\lceil \frac{i+m}{2m-1} \right\rceil + \left\lceil \frac{i}{2m-1} \right\rceil - \left\lceil \frac{i+2}{2m-1} \right\rceil + 1.$$

Now we distinguish three subcases.

If $i = 1 + (2m - 1)t$, $t = 0, 1, \ldots$, then

$$wt_{\phi_m}(C_{m}^{i+1}) - wt_{\phi_m}(C_{m}^{i}) = 2 \left\lceil \frac{1+(2m-1)t+m+1}{2m-1} \right\rceil - 2 \left\lceil \frac{1+(2m-1)t+m}{2m-1} \right\rceil + \left\lceil \frac{1+(2m-1)t}{2m-1} \right\rceil$$

$$= 2(1 + t) - 2(1 + t) + (1 + t) - (1 + t) + 1 = 1.$$

If $i = s + (2m - 1)t$, $t = 0, 1, \ldots$ and $4 \leq s \leq m - 2$, then

$$wt_{\phi_m}(C_{m}^{i+1}) - wt_{\phi_m}(C_{m}^{i}) = 2 \left\lceil \frac{s+(2m-1)t+m+1}{2m-1} \right\rceil - 2 \left\lceil \frac{s+(2m-1)t+m}{2m-1} \right\rceil + \left\lceil \frac{s+(2m-1)t}{2m-1} \right\rceil$$

$$= 2(1 + t) - 2(1 + t) + (1 + t) - (1 + t) + 1 = 1.$$

If $i = s + (2m - 1)t$, $t = 0, 1, \ldots$ and $m + 2 \leq s \leq 2m - 4$, in this case we get

$$wt_{\phi_m}(C_{m}^{i+1}) - wt_{\phi_m}(C_{m}^{i}) = 2 \left\lceil \frac{s+(2m-1)t+m+1}{2m-1} \right\rceil - 2 \left\lceil \frac{s+(2m-1)t+m}{2m-1} \right\rceil + \left\lceil \frac{s+(2m-1)t}{2m-1} \right\rceil$$

$$= 2(2 + t) - 2(2 + t) + (1 + t) - (1 + t) + 1 = 1.$$

Thus, according to all these cases we get that

$$(wt_{\phi_m}(C_{m}^{i+1}) - wt_{\phi_m}(C_{m}^{i})) = 1$$

for every $i$, $i = 1, 2, \ldots, n - m + 1$. This concludes the proof.
4. Conclusion

In this paper we determined the exact value of the cycle-irregularity strength of ladders and fan graphs. We proved that for the ladder \( L_n \cong P_n \Box P_2, n \geq 3, \) admitting a \( C_{2m} \)-covering, \( 2 \leq m \leq \lceil (n+1)/2 \rceil, \) \( \text{ths}(L_n, C_{2m}) = \lceil \frac{3m+n}{4m} \rceil. \) Moreover, for the fan graph \( F_n \) on \( n+1 \) vertices, \( n \geq 2 \) and \( 3 \leq m \leq \lceil (n+3)/2 \rceil, \) \( \text{ths}(F_n, C_m) = \lceil \frac{n+m}{2m-1} \rceil. \)

For the edge (vertex) cycle-irregularity strength of ladders was proved the following.

**Theorem 4.1.** [6] Let \( L_n \cong P_n \Box P_2, n \geq 2, \) be a ladder. Then

\[
\text{ehs}(L_n, C_4) = \left\lceil \frac{n+2}{4} \right\rceil.
\]

**Theorem 4.2.** [6] Let \( L_n \cong P_n \Box P_2, n \geq 3, \) be a ladder. Let \( m \) be a positive integer, \( m \leq \lceil (n+1)/2 \rceil. \) Then

\[
\text{vhs}(L_n, C_{2m}) = \left\lceil \frac{m+n}{2m} \right\rceil.
\]

In [6] is also given the exact value for the vertex cycle-irregularity strength for fan graphs.

**Theorem 4.3.** [6] Let \( F_n \) be a fan graph on \( n+1 \) vertices, \( n \geq 2 \) and \( 3 \leq m \leq \lceil (n+3)/2 \rceil. \) Then

\[
\text{vhs}(F_n, C_m) = \left\lceil \frac{n}{m-1} \right\rceil.
\]

According to results proved in [6] it is needed to find the edge cycle-irregularity strength for ladders and fans for every feasible length of cycles. We suppose that these parameters equal to the lower bounds. We conclude the paper with the following conjectures.

**Conjecture 1.** Let \( L_n \cong P_n \Box P_2, n \geq 2, \) be a ladder admitting a \( C_{2m} \)-covering, \( 3 \leq m \leq \lceil (n+1)/2 \rceil. \) Then

\[
\text{ehs}(L_n, C_{2m}) = \left\lceil \frac{m+n}{2m} \right\rceil.
\]

**Conjecture 2.** Let \( F_n \) be a fan graph on \( n+1 \) vertices, \( n \geq 2 \) and \( 3 \leq m \leq \lceil (n+3)/2 \rceil. \) Then

\[
\text{ehs}(F_n, C_m) = \left\lceil \frac{n+1}{m} \right\rceil.
\]

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References


