On topological integer additive set-labeling of star graphs

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Abstract

For integer \( k \geq 2 \), let \( X = \{0, 1, 2, \ldots, k\} \). In this paper, we determine the order of a star graph \( K_{1,n} \) of \( n + 1 \) vertices, such that \( K_{1,n} \) admits a topological integer additive set-labeling (TIASL) with respect to a set \( X \). We also give a condition for a star graph \( K_{1,n} \) such that \( K_{1,n} \) is not a TIASL-graph on set \( X \).

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1. Introduction

Research on graph labeling was started after Rosa introduced the concept of \( \beta \)-valuation of graphs \cite{2}. The concept of set-assignment \cite{1}, which is defined as follows, is analogous to the number valuations of graphs. Let \( G(V,E) \) be a graph, \( X \) be a non-empty set, and \( \mathcal{P}(X) \) be the power set of \( X \). Then the set-valued function \( f : V(G) \rightarrow \mathcal{P}(X) \) is called the set-assignment of \( G \). We can also define a set-assignment of edges or both elements (vertices and edges)
in a similar way. A set-assignment of a graph $G$ is called a set-labeling (or a set-valuation) of $G$ if it is injective.

In this paper, we combine the concept of the vertex set-labeling and the set topology. A topology on a non-empty set $X$ is a collection $\mathcal{T}$ of subsets of $X$ having the following properties:

1. The set $X$ and $\emptyset$ are in $\mathcal{T}$.
2. The union of the elements of any sub-collection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite sub-collection of $\mathcal{T}$ is in $\mathcal{T}$.

Let $G$ be a connected, simple, and finite graph. Let $X$ be a finite non-empty set of non-negative integers. A vertex set-labeling $f : V(G) \to P(X) - \{\emptyset\}$ is called a topological integer additive set-labeling (TIASL) of $G$ if $f$ is an injective function, $\{f(V(G)) \cup \{\emptyset\}\}$ is a topology of $X$, and there exists the corresponding function $f^+ : E(G) \to P(X) - \{\emptyset\}$ such that for every edge $uv \in E(G)$, $f^+(uv) = f(u) + f(v)$. We recall that the sumset (or Minkowski sum) of two non-empty sets $A$ and $B$, denoted by $A + B$, is defined by $A + B = \{a + b \mid a \in A; b \in B\}$. A graph $G$ which admits TIASL is called a topological integer additive set-labelled graph (in short, TIASL-graph).

The topological integer additive set-labeling was introduced by Sudev and Germina [3]. They give a tight condition for a TIASL-graph. They proved that $G$ is a TIASL-graph if and only if $G$ has at least one pendant vertex. They also characterized all TIASL-graphs with respect to either the indiscrete topology or Sierpenski’s topology.

Let $G$ be a graph having a pendant vertex. For integer $k \geq 2$, let $X = \{0, 1, 2, \ldots, k\}$. It seems that every graph $G$ admits a topological integer additive set-labeling on set $X$ if the cardinality of $X$ is big enough. In [3], Sudev and Germina proved that an $(n, m)$-tadpole graph is a TIASL-graph. An $(n, m)$-tadpole graph is a graph obtained from one copies of cycle $C_n$, $n \geq 3$, and path $P_m$, $m \geq 2$, by identifying an end point of the path $P_m$ to a vertex of cycle $C_n$. They have shown that an $(n, m)$-tadpole graph of $n + m - 1$ vertices admits a topological integer additive set-labeling on set $X = \{0, 1, 2, \ldots, k\}$ where $k = 2(m + n) - 5$.

In this paper, we consider a star graph $K_{1,n}$ of $n+1$ vertices and a given set $X = \{0, 1, 2, \ldots, k\}$ where $k \geq 2$. We obtain two main results. The first result is related to the order of a star graph $K_{1,n}$ such that $K_{1,n}$ is a TIASL-graph on the set $X$.

**Theorem 1.1.** Let $K_{1,n}$ be a star graph with $n + 1$ vertices. For $k \geq 2$, let $X = \{0, 1, 2, \ldots, k\}$. If $n$ is one of the positive integers below, then $K_{1,n}$ is a TIASL-graph on set $X$.

(a) $n \in \{1, 2, \ldots, 4k - 4\}$, or
(b) $n = 2^{r_1} + 2^{r_2} - 2$ for $r_1 \in \{2, 3, \ldots, k - 1\}$ and $r_2 \in \{1, 2\}$.

In the second result, we give a condition for a star graph $K_{1,n}$ such that $K_{1,n}$ is not a TIASL-graph on set $X$.

**Theorem 1.2.** Let $K_{1,n}$ be a star graph with $n + 1$ vertices. For $k \geq 2$, let $X = \{0, 1, 2, \ldots, k\}$. If $3 \cdot 2^{k-1} - 2 \leq n \leq 2^{k+1} - 2$, then $K_{1,n}$ is not a TIASL-graph on set $X$.

In order to prove both theorems above, we also consider the following useful proposition.

**Proposition 1.1.** Let $S$ be a finite non-empty set of non-negative integers with $s$ elements. Then $\mathcal{P}(S)$ is a topology of $S$ with $2^s$ elements.
2. Proof of Theorem 1.1

For an integer \( k \geq 2 \), let \( X = \{0, 1, 2, \ldots, k\} \). First we must consider the following proposition which has been proved by Sudev and Germina [3].

**Proposition 2.1.** Let \( f : V(G) \to X - \{\emptyset\} \) is a TIASL of a graph \( G \). Then, the vertices whose set-labels containing the maximal element of the ground set \( X \) are pendant vertices which are adjacent to the vertex having the set-label \( \{0\} \).

From Proposition 2.1, if \( f \) is a TIASL of a graph \( G \), then there exists a vertex \( v \) of \( G \) such that \( f(v) = \{0\} \). Therefore, we must construct a topology of \( X \) containing \( \{0\} \).

**Proposition 2.2.** There exists a topology \( T \) containing \( \{0\} \) on set \( X \) such that \(|T| = t\), where \( t \) is one of the positive integers as follows.

(a) \( 3 \leq t \leq 4k - 2 \), or 
(b) \( t = 2^{r_1} + r_2 \) for \( r_1 \in \{2, 3, \ldots, k - 1\} \) and \( r_2 \in \{1, 2\} \).

**Proof.** We distinguish two cases.

**Part 2.2.1.** \( 3 \leq t \leq 4k - 2 \)

Let \( I_0 = X \). For \( i \in \{1, 2, \ldots, k\} \), we define recursively

\[ I_i = I_{i-1} - \max(I_{i-1}) \]

and

\[ I_i = \{I_k\} \cup \{I_s \mid 0 \leq s \leq i - 1\} \].

Note that \(|I_i| = i+1\). We also define \( I_i^* = I_{k+1} - \{0\} \) and \( I_i^* = \{I_s^* \mid 1 \leq s \leq i\} \). In this case, \(|I_i^*| = i\). For \( j \in \{1, 2, \ldots, k-2\} \), we define

\[ \hat{I}_j = I_{j+2} \cup \{k-1\} \]

and

\[ \hat{I}_j^* = \hat{I}_j - \{0\} \].

We also define

\[ I_j^{**} = \hat{I}_j \cup \hat{I}_j^* \]

where \( \hat{I}_j = \{I_s \mid 1 \leq s \leq j\} \) and \( \hat{I}_j^* = \{I_s^* \mid 1 \leq s \leq j\} \). Note that \(|I_j^{**}| = 2j\).

By some definitions above, we define a collection-set \( T_1 \) with \( t \) elements as follows.

\[ T_1 = \{\emptyset\} \cup \begin{cases} 
I_{t-2}, & \text{if } 3 \leq t \leq k + 2, \\
I_k \cup I_{k-2}^*, & \text{if } k + 3 \leq t \leq 2k + 2, \\
I_k \cup I_{k-1} \cup I_{k+1}^{**}, & \text{if } 2k + 3 \leq t \leq 4k - 3 \text{ and } t \text{ is odd}, \\
I_k \cup I_{k-2} \cup I_{k+2}^{**}, & \text{if } 2k + 4 \leq t \leq 4k - 2 \text{ and } t \text{ is even}. 
\end{cases} \]

Note that \( I_k = \{0\} \in T_1 \). Now, we will show that \( T_1 \) is a topology of \( X \).

Let \( A \) and \( B \) be two distinct elements of \( T_1 \) where \(|A| \leq |B|\). If \( A \subset B \), then \( A \cap B = A \in T_1 \) and \( A \cup B = B \in T_1 \). Otherwise, we distinguish six cases.
Part 2.2.2.

1. \( A \in \mathcal{I}_k \) and \( B \in \mathcal{I}_k^* \) for \( i \in \{1, 2, \ldots, k\} \) (or \( B \in \mathcal{I}_k \) and \( A \in \mathcal{I}_k^* \))
   Then \( A \cap B \in \mathcal{I}_k^* \) and \( A \cup B \in \mathcal{I}_k \).

2. \( A \in \mathcal{I}_k \) and \( B \in \mathcal{I}_j \) for \( j \in \{1, 2, \ldots, k - 2\} \) (or \( B \in \mathcal{I}_k \) and \( A \in \mathcal{I}_j \))
   Then \( A \cap B \in \mathcal{I}_k \) and either \( A \cup B \in \mathcal{I}_k \) or \( A \cup B \in \mathcal{I}_j \).

3. \( A \in \mathcal{I}_k \) and \( B \in \mathcal{I}_j^* \) for \( j \in \{1, 2, \ldots, k - 2\} \) (or \( B \in \mathcal{I}_k \) and \( A \in \mathcal{I}_j^* \))
   Then \( A \cap B \in \mathcal{I}_k \) and either \( A \cup B \in \mathcal{I}_j^* \) or \( A \cup B \in \mathcal{I}_k \).

4. \( A \in \mathcal{I}_k^* \) and \( B \in \mathcal{I}_j \) for \( i \in \{k - 1, k\} \) and \( j \in \{1, 2, \ldots, k - 2\} \) (or \( B \in \mathcal{I}_k^* \) and \( A \in \mathcal{I}_j \))
   Then either \( A \cap B = \emptyset \) or \( A \cap B \in \mathcal{I}_k^* \) or \( A \cap B \in \mathcal{I}_j^* \). Also, we have either \( A \cup B \in \mathcal{I}_j^* \) or \( A \cup B \in \mathcal{I}_k \).

5. \( A \in \mathcal{I}_k^* \) and \( B \in \mathcal{I}_j^* \) for \( i \in \{k - 1, k\} \) and \( j \in \{1, 2, \ldots, k - 2\} \) (or \( B \in \mathcal{I}_k^* \) and \( A \in \mathcal{I}_j^* \))
   Then either \( A \cap B \in \mathcal{I}_k \) or \( A \cap B = \emptyset \). Also, we have either \( A \cup B \in \mathcal{I}_k^* \) or \( A \cup B \in \mathcal{I}_j^* \).

6. \( A \in \mathcal{I}_j \) and \( B \in \mathcal{I}_j^* \) for \( j \in \{1, 2, \ldots, k - 2\} \) (or \( B \in \mathcal{I}_j \) and \( A \in \mathcal{I}_j^* \))
   Then \( A \cap B \in \mathcal{I}_j^* \) and \( A \cup B \in \mathcal{I}_j \).

From the six cases above, we obtain that every two distinct elements \( A \) and \( B \) in \( \mathcal{T}_1 \) satisfy \( A \cap B \in \mathcal{T}_1 \) and \( A \cup B \in \mathcal{T}_1 \). Since \( \mathcal{T}_1 \) also contains \( \emptyset \) and \( X \), it implies that \( \mathcal{T}_1 \) is a topology of \( X \).

**Part 2.2.2.** \( t = 2^r_1 + r_2 \) for \( r_1 \in \{2, 3, \ldots, k - 1\} \) and \( r_2 \in \{1, 2\} \)

We define the sets \( J_{r_1} = \{0, 1, \ldots, r_1\} \). Now, we consider an element \( a \) of \( X \) such that \( a \neq \max(X) \). Let \( X^- = X - \{a\} \). By these definitions, we define a collection-set \( \mathcal{T}_2 \) with \( t \) elements as follows.

\[
\mathcal{T}_2 = \begin{cases} 
\mathcal{P}(J_{r_1}) \cup \{X\}, & \text{if } t = 2^r_1 + 1, \\
\mathcal{P}(J_{r_1}) \cup \{\{X\}, \{X^-\}\}, & \text{if } t = 2^r_1 + 2.
\end{cases}
\]

Now, we will show that \( \mathcal{T}_2 \) is a topology of \( X \).

Note that \( \emptyset, \{0\}, X \in \mathcal{T}_2 \). Let \( A \) and \( B \) be two distinct elements of \( \mathcal{T}_2 \). We distinguish three cases.

1. \( A, B \in \mathcal{P}(J_{r_1}) \)
   By Proposition 1.1, then \( A \cap B \in \mathcal{P}(J_{r_1}) \) and \( A \cup B \in \mathcal{P}(J_{r_1}) \).

2. \( A \in \mathcal{P}(J_{r_1}) \) or \( A = X^- \) and \( B = X \)
   Then \( A \cup B = B \) and \( A \cap B = A \).

3. \( A \in \mathcal{P}(J_{r_1}) \) and \( B = X^- \).
   Then \( A \cap B \in \mathcal{P}(J_{r_1}) \) and \( A \cup B \in \{X, X^-\} \).

From three cases above, we obtain that \( A \cap B, A \cup B \in \mathcal{T}_2 \).

Now, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( V(K_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\} \), where \( v_1 \) is the centre of \( K_{1,n} \). Let \( \mathcal{T}_i \) be a topology of \( X \) with \( t \) elements satisfying Proposition 2.2. Let \( \mathcal{T}_i' = \mathcal{T}_i - \{\emptyset\} \). Now, we define a vertex injective labeling \( f : V(S_n) \rightarrow \mathcal{T}_i' \) such that \( f(v_1) = \{0\} \). Since for \( 2 \leq i \leq n \), \( v_1 \) is adjacent to \( v_i \) and \( f(v_1) + f(v_i) = f(v_i) \in \mathcal{T}_i' \subseteq \mathcal{P}(X) \), we obtain that \( K_{1,n} \) is a TIASL-graph on the set \( X \).
3. Proof of Theorem 1.2

Let $S$ be a finite non-empty set of non-negative integers. From Proposition 1.1, it is clear that $\mathcal{P}(S)$ is a topology on the set $S$. Let $\mathcal{A} \subset \mathcal{P}(S)$. On some cases of $\mathcal{A}$, the collection $\mathcal{P}(S) - \mathcal{A}$ is not a topology on the set $S$. In proposition below, we prove that if $L \in \mathcal{P}(S)$ is not an element of a topology $\mathcal{T}$ on the set $S$, then there exists an element $l \in L$ such that $\{l\} \notin \mathcal{T}$.

Proposition 3.1. Let $S$ be a finite non-empty set of non-negative integers with $s$ elements, and $\mathcal{T}$ be a topology of $S$. Let $A \in \mathcal{P}(S)$ but $A \notin \mathcal{T}$. Then there exists an element $a$ of $A$ such that $\{a\} \notin \mathcal{T}$.

Proof. By the definition of a topology, we have $A \neq \emptyset$. Let $A = \{a_1, a_2, \ldots, a_r\}$. If $r = 1$, then we are done. Now, we assume that $r \geq 2$. Suppose that $\{a_i\} \in \mathcal{T}$ for $1 \leq i \leq r$. Note that $\bigcup_{i=1}^{r} \{a_i\} = A \notin \mathcal{T}$, a contradiction.

Let the collection $\mathcal{T}$ be a topology on the set $S$ which is satisfying Proposition 3.1 above and the set $L \in \mathcal{P}(S)$ but $L \notin \mathcal{T}$. Let $l \in L$ and $\{l\} \notin \mathcal{T}$. So, there are no two distinct sets $A_1$ and $A_2$ in $\mathcal{T}$ such that $A_1 \cap A_2 = \{l\}$. Therefore, we need to determine how many elements of $\mathcal{T}$ such that $\mathcal{T}$ may be a topology on the set $S$.

Proposition 3.2. Let $S$ be a finite non-empty set of non-negative integers with $s \geq 2$ elements. Let $\mathcal{A}$ be a non-empty collection-set, where every element of $\mathcal{A}$ is an element of $\mathcal{P}(S)$. If $\mathcal{P}(S) - \mathcal{A}$ is a topology of $S$, then $|\mathcal{P}(S) - \mathcal{A}| \leq 3 \cdot 2^{s-2}$.

Proof. Let $S = \{v_1, v_2, \ldots, v_s\}$. By Proposition 1.1, $\mathcal{P}(S)$ is a topology of $S$ with $2^s$ elements. Let $\mathcal{A}$ be a non-empty collection-set, where every element of $\mathcal{A}$ is element of $\mathcal{P}(S)$. Let $\mathcal{T} = \mathcal{P}(S) - \mathcal{A}$ be a topology of $S$.

Let $E \in \mathcal{A}$. Since $\mathcal{T}$ is a topology of $S$, it is clear that $E \neq \emptyset$ and $E \neq S$. By considering Proposition 3.1, without lost of generality, let $v_s \in E$ and $\{v_s\} \notin \mathcal{T}$. We can say that $\{v_s\} \in \mathcal{A}$.

Let $\mathcal{B} = \{\{v_s, v_i\} | 1 \leq i \leq s-1\}$. Note that $|\mathcal{B}| = s - 1$. Since $\mathcal{T}$ is a topology of $S$, then at least $s - 2$ elements of $\mathcal{B}$ are in $\mathcal{A}$. Without lost of generality, let $\mathcal{B} = \{\{v_s, v_i\} | 1 \leq i \leq s-2\} \subset \mathcal{A}$. Now, we define $\hat{\mathcal{B}} = \{v \in \mathcal{B} | \{v, v_s\} \notin \mathcal{B}\}$. We also define $\mathcal{C} = \{\{v_s\} \cup C | C \in \mathcal{P}(\mathcal{B})\}$. Note that $|\mathcal{C}| = 2^{s-2}$, $\{v_s\} \in \mathcal{C}$, and $\mathcal{B} \subset \mathcal{C}$. Note that for any distinct elements $C_1, C_2 \in \mathcal{C}$, we have $C_1 \cup C_2$ and $C_1 \cap C_2$ are also in $\mathcal{C}$. However, every $C \in \mathcal{C}$ satisfy $C \cap \{v_s, v_{s-1}\} = \{v_s\} \in \mathcal{A}$. So, it must be $\mathcal{C} \subset \mathcal{A}$. Therefore, we obtain

$$|\mathcal{P}(S) - \mathcal{A}| \leq 2^s - 2^{s-2} = 3 \cdot 2^{s-2}.$$  

Proof of Theorem 2. Theorem 1.2 is a direct consequence of Propositions 1.1 and 3.2.

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