Domination number of the non-commuting graph of finite groups

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Abstract

Let $G$ be a non-abelian group. The non-commuting graph of group $G$, shown by $\Gamma_G$, is a graph with the vertex set $G \setminus Z(G)$, where $Z(G)$ is the center of group $G$. Also two distinct vertices of $a$ and $b$ are adjacent whenever $ab \neq ba$. A set $S \subseteq V(\Gamma)$ of vertices in a graph $\Gamma$ is a dominating set if every vertex $v \in V(\Gamma)$ is an element of $S$ or adjacent to an element of $S$. The domination number of a graph $\Gamma$ denoted by $\gamma(\Gamma)$, is the minimum size of a dominating set of $\Gamma$. Here, we study some properties of the non-commuting graph of some finite groups. In this paper, we show that $\gamma(\Gamma_G) < \frac{|G| - |Z(G)|}{2}$. Also we characterize all of groups $G$ of order $n$ with $t = |Z(G)|$, in which $\gamma(\Gamma_G) + \gamma(\Gamma_G) \in \{n - t + 1, n - t, n - t - 1, n - t - 2\}$.

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1. Introduction

Let $G$ be a non-abelian group and $Z(G)$ be the center of $G$. Associate a graph $\Gamma_G$ with $G$ as follows: Take $G \setminus Z(G)$ as the vertices of $\Gamma_G$. Two vertices $a$ and $b$ are adjacent if $ab \neq ba$. This graph is called the non-commuting graph of $G$. Let $\Gamma$ be a simple graph. A subset $S \subseteq V(\Gamma)$ is called a dominating set if each vertex $v \in V(\Gamma) \setminus S$ has at least one neighbor in $S$. The size of a smallest dominating set of $\Gamma$ is called domination number of $\Gamma$ and is denoted by $\gamma(\Gamma)$.
Erdős considered the non-commuting graph in 1975 for the first time. In 2004, Abdollahi, Akbari and Maimani studied some properties of the non-commuting graph of a group. For more results, see [1],[2], [6], [5] and [12].

Before starting, let us introduce some necessary notation and definitions. For every graph $\Gamma$, we denote the set of the vertices of $\Gamma$ by $V(\Gamma)$. The minimum degree of a graph $\Gamma$ denoted by $\delta(\Gamma)$. The complete graph, path and cycle on $n$ vertices are denoted by $K_n$, $P_n$ and $C_n$, respectively. The open neighborhood of a vertex $v$ in $\Gamma$ is the set $N_{\Gamma}(v)$ of vertices joined to $v$ by an edge. The closed neighborhood of $v$ is the set $N_{\Gamma}[v] = N_{\Gamma}(v) \cup \{v\}$. The complement of $\Gamma$ denoted by $\overline{\Gamma}$. If $u$ and $v$ are vertices in $\Gamma$, then $d(u,v)$ denotes the length of the shortest path between $u$ and $v$. A graph $\Gamma$ is connected if there is a path between each pair of the vertices of $\Gamma$. The maximum value of $d(u,v)$ between all pair of the vertices of connected graph $\Gamma$ is called the diameter of $\Gamma$ and denoted by $\text{diam}(\Gamma)$. Let $\Gamma_1$ and $\Gamma_2$ be two graphs. The corona $\Gamma = \Gamma_1 \circ \Gamma_2$ is the graph formed from one copy of $\Gamma_1$ and $|V(\Gamma_1)|$ copies of $\Gamma_2$ such that the $i$th vertex of $\Gamma_1$ is adjacent to every vertex in the $i$th copy of $\Gamma_2$.

For each $x \in G$, $C_G(x) = \{g \in G \mid gx = xg\}$. We denote the symmetric group and the alternating group on $n$ letters by $S_n$ and $A_n$, respectively. Also $Q_8 = \langle A, B \mid A^4 = 1, A^2 = B^2, B^{-1}AB = A^{-1} \rangle$ and $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$ are the quaternion group with 8 elements and the dihedral group of order $2n$, respectively.

In this paper, we study the domination number of the non-commuting graphs. In particular, we characterize all groups $G$ of order $n$ with $|Z(G)| = t$, in which $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) \in \{n - t + 1, n - t, n - t - 1, n - t - 2\}$.

2. Preliminaries

In this section, we provide some useful results which will be applied in the next section.

**Theorem 2.1.** (Ore) [10] Let $\Gamma$ be a graph with no isolated vertices. Then $\gamma(\Gamma) \leq \frac{n}{2}$.

**Theorem 2.2.** [7] For a graph $\Gamma$ with even order $n$ and no isolated vertices, $\gamma(\Gamma) = \frac{n}{2}$ if and only if the components of $\Gamma$ are the cycle $C_4$ or the Corona $H \circ K_1$ for any connected graph $H$.

**Theorem 2.3.** [1], 2.1. For any non-abelian group $G$, $\text{diam}(\Gamma_G) = 2$. In particular, $\Gamma_G$ is connected. Also the girth of $\Gamma_G$ equal 3.

**Theorem 2.4.** [11] Let $\Gamma$ be a graph of order $n$. Then the following holds.

i) $\gamma(\Gamma) + \gamma(\overline{\Gamma}) \leq n + 1$.

ii) $\gamma(\Gamma)\gamma(\overline{\Gamma}) \leq n$.

**Lemma 2.1.** Let $G$ be a finite non-abelian group. Then $\delta(\Gamma_G) \geq 3$.

**Proof.** Suppose that $\text{deg}_{\Gamma_G}(v) < 2$, for some $v \in V(\Gamma_G)$. Since $\text{deg}_{\Gamma_G}(v) \geq \frac{|G|}{2}$, then $|G| \leq 4$. So, $G$ is an abelian group, which is a contradiction. \qed

**Remark 2.1.** Here, we get figures of the non-commuting graph of some groups. These figures are useful in proving some theorems in the third section. (See Figures 1, 2, 3 and 4.)
Lemma 2.2. Let $G$ be a non-abelian group of odd order. Then, the graph $\Gamma_G$ contains no isolated vertex.

Proof. Assume to the contrary, $a$ is an isolated vertex of $\Gamma_G$. Then, for each $x \in G \setminus Z(G)$ we have $ax \neq xa$. Hence $o(a) = 2$, which is a contradiction. 

Lemma 2.3. Let $G$ be a non-abelian group and $|Z(G)| = 1$. Then, the vertices of degree one in $\Gamma_G$ occur only at the edges. Furthermore, if $\deg_{\Gamma_G} a = 1$ then $o(a) = 3$.

Proof. Let $a \in V(\Gamma_G)$, $\deg_{\Gamma_G} a = 1$ and $a$ be adjacent to $b$. Then, $C_G(a) = \{1, a, b\}$ and so $b = a^{-1}$. Hence, $C_G(b) = \{1, b, a\}$. Thus, $\deg_{\Gamma_G} b = 1$. Furthermore, $o(a) = o(b) = 3$. 

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Lemma 2.4. Let $G$ be a non-abelian group. Then $\Gamma_G$ does not have $C_n$, $(n > 3)$ as component.

Proof. Let $C_n$ be a component of $\Gamma_G$ and $\{a,b\} \subset V(C_n)$ such that $b \in N_{\Gamma_G}(a)$. Then $N_{\Gamma_G}[a] = \{a, b, f\}$ and $N_{\Gamma_G}[b] = \{a, b, c\}$. So $C_G(a) = \{a, b, f\} \cup Z(G)$ and $C_G(b) = \{a, b, c\} \cup Z(G)$. Since $|C_G(a) \cap C_G(b)|$ divides $|C_G(a)|$, then $2 + |Z(G)|$ divides $3 + |Z(G)|$, that is a contradiction. $\square$

Definition 2.1. Let $G$ be a group. Write $S$ and $T$ for the set of elements of $G$ of order two and three, respectively. Then $G$ is acceptable when neither $S$ nor $T$ is empty and $G = S^* \cup T$, where $S^* = S \cup \{e\}$.

Theorem 2.5. [3] If $G$ is acceptable, then either $S^* \leq G$ or $T^* \leq G$.

3. Main results

In this section, we prove our main results.

Theorem 3.1. Let $G$ be a non-abelian group. Then, the following holds.

i) $\delta(\Gamma_G) = 3$ if and only if $G$ is isomorphic to $S_3$.

ii) $\delta(\Gamma_G) = 4$ if and only if $G$ is isomorphic to $D_8$ or $Q_8$.

iii) $\delta(\Gamma_G) = 5$ if and only if $G$ is isomorphic to $D_{10}$.

Proof. We prove as follows:

i) Let $\delta(\Gamma_G) = 3$ and $\deg_{\Gamma_G}(v) = 3$, for some $v \in V(\Gamma_G)$. Since $\deg_{\Gamma_G}(v) \geq \frac{|G|}{2}$, then $|G| \leq 6$. The only non-abelian group of order less than 7 is $S_3$.

Conversely, Suppose that $G$ is isomorphic to $S_3$. By considering the figure of the non-commuting graph associated to symmetric group $S_3$ (See Figure 1), we obtain $\delta(\Gamma_G) = 3$. 

Figure 4. The non-commuting graph of $D_{10}$. 

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ii) Let $\delta(\Gamma_G) = 4$. Then $|G| \leq 8$ and so $G \in \{S_3, D_8, Q_8\}$. By (i), $G \not\cong S_3$. So $G \cong D_8$ or $Q_8$. Conversely, Suppose that $G$ is isomorphic to $D_8$ or $Q_8$. By considering the figure of the non-commuting graph associated to $D_8$ and $Q_8$ (See Figures 2 and 3), we obtain $\delta(\Gamma_G) = 4$.

iii) Let $\delta(\Gamma_G) = 5$. Then $|G| \leq 10$. By (i) and (ii), $|G| = 10$. Since $G$ is not an abelian group, then $G$ is isomorphic to $D_{10}$. Conversely, Suppose that $G$ is isomorphic to $D_{10}$. By considering the figure of the non-commuting graph associated to the dihedral group $D_{10}$ (See Figure 4), we obtain $\delta(\Gamma_G) = 5$.

Corollary 3.1. All of 3-regular and 5-regular graphs cannot be non-commuting graphs.

Theorem 3.2. Let $\Gamma$ be a $(n - 2)$-regular graph of order $n$. Then $\Gamma$ is the non-commuting graph associated to a non-abelian group $G$ if and only if $n = 6$ and $G$ is isomorphic to $D_8$ or $Q_8$.

Proof. Let $\Gamma$ be a $(n - 2)$-regular graph of order $n$ and $G$ be a group such that $\Gamma_G = \Gamma$. Then $n$ is even and $\Gamma_G$ is a disjoint union of $\frac{n}{2}$ edges. If $a$ and $b$ are adjacent in $\Gamma_G$, then $C_G(a) = Z(G) \cup \{a, b\}$. It is clear that $|Z(G)| \leq 2$.

If $|Z(G)| = 1$, then $C_G(a) = \{1, a, a^{-1} = b\}$ and $o(a) = 3$. Since $\Gamma_G$ is a disjoint union of some edges, then for each $x \in G$ we have $o(x) = 3$. Thus, there is a positive integer $s$ such that $|G| = 3^s$. Hence $|Z(G)| > 1$, which is a contradiction.

If $|Z(G)| = 2$ and $Z(G) = \{1, x\}$, then for each $a \in G \setminus Z(G)$, $C_G(a) = \{1, x, a, b\}$ and so we have $a^2 = 1$ or $a^2 = x$. Therefore for each $a \in G \setminus Z(G)$ we have $a^2 \in Z(G)$. Hence $\frac{G}{Z(G)}$ is an elementary abelian 2-group. So $G' \leq Z(G)$, which implies $|G'| = 1$ or 2. Since $G$ is not an abelian group, then $G' \neq \{1\}$. Thus $G' = Z(G)$. Also we have $cl(a) = \{g^{-1}ag : g \in G\} \subseteq aG'$ and so $|cl(a)| \leq 2$. Since $|cl(a)| = \frac{|G|}{|C_G(a)|}$, then $|G| \leq 8$. Hence $G \cong S_3, D_8$ or $Q_8$. Since $Z(S_3) = 1$, then $G \cong D_8$ or $G \cong Q_8$.

Conversely, suppose that $G \cong D_8$ or $G \cong Q_8$. Then by considering the figures of the non-commuting graphs of these two groups (See Figures 2 and 3), we obtain $\Gamma_G$ is a 4-regular graph of order 6.

Theorem 3.3. Every $(n - 3)$-regular graph of order $n$ is not the non-commuting graph.

Proof. Let $\Gamma$ be a graph of order $n$ and $(n - 3)$-regular. Also, suppose that $G$ is a group and $\Gamma_G = \Gamma$. Thus, $\Gamma_G$ is a 2-regular graph. It means that $\Gamma_G$ is a disjoint union of cycles.

By Lemma 2.4, $\Gamma_G$ is a disjoint union of $\{a, b, c\}$ such that $bc = cb$. Thus, $|Z(G)| \leq 3$.

Case 1. If $|Z(G)| = 1$, then $|C_G(a)| = 4$. So there is a positive integer s such that $|G| = 2^4$. Thus, $|Z(G)| > 1$, which is a contradiction.

Case 2. If $|Z(G)| = 2$, then $|C_G(a)| = 5$. Since $|Z(G)|$ divides $|C_G(a)|$, then $2 | 5$, which is a contradiction.

Case 3. If $|Z(G)| = 3$, then $|C_G(a)| = 6$. Thus, $C_G(a) \cong Z_6$ and so for each $x \notin Z(G)$, $o(x) \in \{2, 6\}$. So for each $a \in G \setminus Z(G)$, $a^2 \in Z(G)$ which implies $G/Z(G)$ is an elementary abelian 2-group. Therefore, $G' \leq Z(G)$. Since $G$ is not an abelian group, then $G' = Z(G)$. We
know that $cl(a) \subseteq aG^\gamma$. So $|cl(a)| \leq 3$ Hence $6 < |G| \leq 18$. Since $|Z(G)| = 3$ and $|Z(G)|$ divides $|G|$, then $|G| \in \{9, 12, 15, 18\}$. Furthermore, since $G$ is not an abelian group, then we have $|G| \in \{12, 18\}$.

If $|G| = 12$, then $G \simeq A_4, D_{12}$ or $\langle a, b \mid a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle$. Since $G$ has only 2 elements of order three, then $G \not\simeq A_4$. Also since $|Z(D_{12})| = 2$, then $G \not\simeq D_{12}$. Hence $G \simeq \langle a, b \mid a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle$. So $G$ has an element of order 4, which is a contradiction.

If $|G| = 18$, then $\Gamma_G \simeq 5K_3$. So $G$ has exactly 5 elements of order 2. Let $n_p$ be the number of Sylow $p$-subgroup of $G$. By Sylow Theorem, $n_2(G) \in \{1, 3, 9\}$. Hence $G$ has 1, 3 or 9 elements of order 2, which is a contradiction. \qed

In [10], Ore proved that if $\Gamma$ is a graph with no isolated vertices, then $\gamma(\Gamma) \leq \frac{n}{2}$. In Theorems 3.4 and 3.5, we show that if $G$ is a non-abelian group with $|Z(G)| = t$, then $\gamma(\Gamma_G)$ and $\gamma(\tilde{\Gamma}_G)$ are less than $\frac{n-t}{2}$.

**Theorem 3.4.** Let $G$ be a non-abelian group of order $n$ and $|Z(G)| = t$. Then $\gamma(\Gamma_G) < \frac{n-t}{2}$.

*Proof.* We know $|V(\Gamma_G)| = n-t$. Since $\Gamma_G$ is a connected graph, it contains no isolated vertex. By Theorem 2.1, $\gamma(\Gamma_G) \leq \frac{n-t}{2}$. Now, we show $\gamma(\Gamma_G) < \frac{n-t}{2}$. Assume to the contrary, $\gamma(\Gamma_G) = \frac{n-t}{2}$. By Theorem 2.2, each component of the graph $\Gamma_G$ is the cycle $C_4$ or the corona product $K_1$ and a connected graph $H$, that is $H \circ K_1$. By Lemma 2.1, is a contradiction. Hence $\gamma(\Gamma_G) < \frac{n-t}{2}$. \qed

**Theorem 3.5.** Let $G$ be a non-abelian group of odd order $n$ and $|Z(G)| = t$. Then $\gamma(\tilde{\Gamma}_G) < \frac{n-t}{2}$.

*Proof.* Since $n$ is odd, then $t$ is odd. By Lemma 2.2, the graph $\tilde{\Gamma}_G$ contains no isolated vertex. By Theorem 2.1, $\gamma(\tilde{\Gamma}_G) \leq \frac{n-t}{2}$. Now, suppose that $\gamma(\tilde{\Gamma}_G) = \frac{n-t}{2}$. Then, by Theorem 2.2, $\tilde{\Gamma}_G$ has connected components of kind of $C_4$ or $H \circ K_1$. By Lemma 2.4, all components of $\tilde{\Gamma}_G$ are the corona product $K_1$ and a connected graph $H$. Let $a$ be a vertex of degree 1 in $\Gamma_G$ and $b$ is adjacent to $a$. Then $C_4(a) = \{a, b\} \cup Z(G)$ and so $|Z(G)| \leq 2$. Since $|G|$ is odd, then $|Z(G)| = 1$. By Lemma 2.3, $|G| = 3^4$. Hence $|Z(G)| > 1$, which is a contradiction. \qed

In the following theorem, we characterize all groups $G$ of order $n$ with $|Z(G)| = t$, in which $\gamma(\Gamma_G) + \gamma(\tilde{\Gamma}_G) \in \{n - t + 1, n - t, n - t - 1, n - t - 2\}$.

**Theorem 3.6.** Let $G$ be a non-abelian group of order $n$ and $|Z(G)| = t$. Then the following holds.

i) $\gamma(\Gamma_G) + \gamma(\tilde{\Gamma}_G) < n - t + 1$.

ii) $\gamma(\Gamma_G) + \gamma(\tilde{\Gamma}_G) = n - t$ if and only if $G \simeq S_3$.

iii) $\gamma(\Gamma_G) + \gamma(\tilde{\Gamma}_G) = n - t - 1$ if and only if $G \simeq D_8$ or $G \simeq Q_8$.

iv) $\gamma(\Gamma_G) + \gamma(\tilde{\Gamma}_G) = n - t - 2$ if and only if $G \simeq D_{10}$.

*Proof.* Since $|G| = n$ and $|Z(G)| = t$, then $|V(\Gamma_G)| = n - t$. By Theorem 3.4, $\gamma(\Gamma_G) < \frac{n-t}{2}$.
i) By Theorem 2.4, \( \gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) \leq n - t + 1 \). Let \( \gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = n - t + 1 \). Then \( \gamma(\Gamma_G) > n - t + 1 - \frac{n-t}{2} = \frac{n-t+2}{2} \). By Theorem 2.1, \( \overline{\Gamma}_G \) contains at least one isolated vertex. So \( t = 1 \) and \( \gamma(\Gamma_G) = 1 \). Thus \( \gamma(\overline{\Gamma}_G) = n - 1 \). Therefore \( \Gamma_G = K_{n-1} \). By Theorem 2.3, it is impossible. Hence \( \gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) < n - t + 1 \).

ii) Let \( \gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = n - t \). Then \( \gamma(\overline{\Gamma}_G) > \frac{n-t}{2} \). By Theorem 2.1, \( \overline{\Gamma}_G \) contains at least one isolated vertex. So \( t = 1 \), \( \gamma(\Gamma_G) = 1 \) and \( \gamma(\overline{\Gamma}_G) = n - 2 \). Thus, \( \overline{\Gamma}_G \) is a disjoint union of \( P_2 \) and isolated vertices \( \{x_1, x_2, \ldots, x_{n-3}\} \). All isolated vertices \( x_i \) are of order 2. Let \( a \) and \( b \) be the vertices of the path \( P_2 \). By Lemma 2.3, \( b = a^{-1} \) and \( o(a) = o(b) = 3 \). We claim that if \( x_1 \) and \( x_2 \) are two elements of order 2, then \( o(x_1x_2) = 3 \). To see this, suppose \( o(x_1x_2) = 2 \). Then \( x_1x_2x_1x_2 = 1 \) and \( x_1x_2 = x_2x_1 \). Thus \( x_1 \) is adjacent to \( x_2 \) in \( \overline{\Gamma}_G \), which is a contradiction. It is easy to see that \( o(x_i b) = 2 \), for \( i = 1, 2, \ldots, n - 3 \).

If \( T = \{ x_1, b \mid x_1^2 = b^3 = 1, (x_1b)^2 = 1 \} \), then \( T \cong S_3 \). We prove \( G \cong T \). If \( x_2 \in G \setminus T \) and \( x_1x_2 = b \), then \( x_2 = x_1b \). Since \( x_1b \in T \), \( x_2 \notin T \). Also if \( x_1x_2 = b^{-1} \), then \( x_2 = x_1b^{-1} \in T \). However it is a contradiction. Hence \( G \cong T \cong S_3 \). Conversely, if \( G \cong S_3 \), then by Figure 1, \( \gamma(\Gamma_G) = 1 \) and \( \gamma(\overline{\Gamma}_G) = 4 \) and the proof is complete.

iii) Let \( \gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = n - t - 1 \). Then \( \gamma(\overline{\Gamma}_G) > \frac{n-t-2}{2} \). We consider two cases:

**Case 1.** Let \( \gamma(\overline{\Gamma}_G) = \frac{n-t}{2} \). Then \( \gamma(\Gamma_G) = \frac{n-t}{2} - 1 \). If \( \Gamma_G \) has an isolated vertex, then \( t = 1 \) and so \( n \) is odd. By Theorem 3.5, \( \gamma(\overline{\Gamma}_G) < \frac{n-t}{2} \), which is false. If \( \overline{\Gamma}_G \) does not have an isolated vertex, then by Theorem 2.2 and Lemma 2.4, \( \Gamma_G \) has a vertex of degree one. So \( t = 1 \) or \( t = 2 \). If \( t = 1 \), then \( n \) is odd. By Theorem 3.5, \( \gamma(\overline{\Gamma}_G) < \frac{n-t}{2} \), which is false.

If \( t = 2 \), then \( \gamma(\overline{\Gamma}_G) = \frac{n-2}{2} \) and \( \gamma(\Gamma_G) = \frac{n-4}{2} \). By Theorem 2.4, \( \frac{n-2}{2} \cdot \frac{n-4}{2} \leq n - 2 \). Since \( n \neq 2 \), then \( n \leq 8 \). So \( G \) is isomorphic to \( S_3, D_8 \) or \( Q_8 \). But \( Z(S_3) = 1 \). Hence \( G \) is isomorphic to \( D_8 \) or \( Q_8 \).

**Case 2.** If \( \gamma(\Gamma_G) > \frac{n-t}{2} \), then \( \overline{\Gamma}_G \) contains at least one isolated vertex. So \( t = 1 \) and so \( \gamma(\Gamma_G) = 1 \). Therefore \( \gamma(\overline{\Gamma}_G) = n - 3 \). By Lemmas 2.3, 2.4 and \( \gamma(\Gamma_G) = n - 3 \), we have the following subcases.

**Subcase 1.** Let \( \overline{\Gamma}_G \) be a union of the isolated vertices \( \{x_1, x_2, \ldots, x_{n-4}\} \) and \( K_3 \) with vertices \( a, b, c \). Then \( C_G(a) = C_G(b) = C_G(c) = \{1, a, b, c\} \). So orders of \( a, b \) and \( c \) are 2 or 4. If \( o(a) = o(b) = o(c) = 2 \), then order of each element of \( G \) is 2 and so \( G \) is an abelian group, which is a contradiction. If \( o(a) = 2 \) and \( o(b) = 4 \), then \( a = b^2 \) and for each \( i \) \((i = 1, 2, \ldots, n - 4)\), \( x_i bx_i = b \) or \( x_i bx_i = b^{-1} \). If \( x_i bx_i = b \), then \( bx_i = x_i b \). So \( b \) is adjacent to \( x_i \), which is a contradiction. If \( x_i bx_i = b^{-1} \), then \( \langle b, x_i \rangle \cong D_8 \). We claim that \( G \cong \langle b, x_1 \rangle \). Suppose that \( x \in G \setminus \langle b, x_1 \rangle \). Since \( xbx = b^{-1} \) and \( x_1bx_1 = b^{-1} \), then \( xbx = x_1bx_1 \) and so \( x_1x \in C_G(b) \). We know that \( C_G(b) = \langle b \rangle \). So \( x \in \{x_1b, x_1b^2, x_1b^3\} \). Hence \( x \in \langle b, x_1 \rangle \). Therefore \( G \cong D_8 \). Since in this case \( Z(G) = \{1\} \), then we have \( Z(D_8) = \{1\} \), which is a contradiction.
Subcase 2. Let $\Gamma_G$ be a union of the isolated vertices $\{x_1, x_2, \ldots, x_{n-5}\}$ and two edges with vertices $a \sim b$ and $c \sim d$. By Lemma 2.3, $o(a) = o(b) = o(c) = o(d) = 3$. Since $\{1, a, b, c, d\}$ is not a subgroup of $G$, by Theorem 2.5, $\{1, x_1, x_2, \ldots, x_{n-5}\}$ is a subgroup of $G$. So $o(x_1x_2) = 2$. Hence $x_1x_2 = x_2x_1$ and so $x_1$ is adjacent to $x_2$ in $\Gamma_G$, which is a contradiction. Conversely, by using the figures 2 and 3 we can obtain the proof.

iv) Let $\gamma(\Gamma_G) + \gamma(\Gamma_G) = n - t - 2$. By Theorem 3.4, $\gamma(\Gamma_G) < \frac{n-t}{2}$. We have the following cases.

Case 1. If $\gamma(\Gamma_G) = \frac{n-t}{2}$, then $\gamma(\Gamma_G) = \frac{n-t}{2}$. By Theorem 2.4, $\gamma(\Gamma_G) \leq n - t$. Hence, $1 \leq n - t \leq 7$. Since $n - t$ is even and $t \mid n$, then $n - t \in \{4, 6\}$. If $n - t = 4$, then $(n, t) = (6, 2)$ and so $G \cong S_3$. Which is a contradiction to the fact that $Z(S_3) = 1$. If $n - t = 6$, then $(n, t) \in \{(10, 4), (8, 2)\}$. Since $t \mid n$, we have $(n, t) = (8, 2)$. So $G \cong D_8$ or $G \cong Q_8$. By Figures 2 and 3, we have a contradiction.

Case 2. Let $\gamma(\Gamma_G) = \frac{n-t-4}{2}$ and $\gamma(\Gamma_G) = \frac{n-t}{2}$. If $\Gamma_G$ has an isolated vertex, then $t = 1$ and so $n$ is odd. By Theorem 3.5, $\gamma(\Gamma_G) < \frac{n-t}{2}$, which is false.

If $\Gamma_G$ does not have an isolated vertex, then by Theorem 2.2 and Lemma 2.4, $\Gamma_G$ has a vertex of degree 1. So $t = 1$ or $t = 2$. If $t = 1$, then $n$ is odd, which is false. If $t = 2$, then $\gamma(\Gamma_G) = \frac{n-2}{2}$ and $\gamma(\Gamma_G) = \frac{n-4}{2}$. Thus $n \leq 10$. By Figures 2, 3 and 4, we have a contradiction.

Case 3. Let $\gamma(\Gamma_G) < \frac{n-t-4}{2}$. Then $\gamma(\Gamma_G) > \frac{n-t}{2}$ and so $\Gamma_G$ has at least one isolated vertex. Thus $t = 1$, $\gamma(\Gamma_G) = 1$ and $\gamma(\Gamma_G) = n - 4$. Let $u \in V(\Gamma_G)$. If $deg_{\Gamma_G}(u) > 3$, then $\gamma(\Gamma_G) < n - 4$, which is not true. Hence for every $u \in V(\Gamma_G)$, $deg_{\Gamma_G}(u) \leq 3$ and $o(u) \leq 5$. We have the following subcases.

Subcase 1. Let $deg_{\Gamma_G}(u) \leq 1$, where $u \in V(\Gamma_G)$. Then $\Gamma_G$ is isomorphic to union of 3 copies of $P_2$ and $n - 7$ isolated vertices. It is clear that isolated vertices are of order 2. By Lemma 2.3, $G$ is an abelian acceptable group. By Theorem 2.5, $T \cup \{1\} \neq G$ or $S \cup \{1\} \neq G$. Since 3 does not divide $|T \cup \{1\}|$, then $T \cup \{1\}$ is not a subgroup of $G$. Hence $S^* = S \cup \{1\}$ is a subgroup of $G$. Since $|S^*| \leq \frac{|G|}{2}$, then $|G| \leq 12$. We know that there is no group of order less than 12 with exactly 6 elements of order 3, which implies that $|G| = 12$. Hence $G \cong A_4, D_{12}$ or $L = \langle a, b \mid a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle$. Since $G$ has exactly 6 elements of order 3, we have $G \notin \{A_4, D_{12}\}$. Also in $L$ we have $o(b) = 4$. So $G \not\cong L$.

Subcase 2. For each $u \in V(\Gamma_G)$, $deg_{\Gamma_G}(u) \leq 2$. Let $u, v \in V(\Gamma_G)$ and $deg_{\Gamma_G}(u) = deg_{\Gamma_G}(v) = 2$. If $u$ and $v$ are not adjacent in $\Gamma_G$, then by Lemmas 2.3 and 2.4, $\gamma(\Gamma_G) < n - 4$, which is not true. If $u$ and $v$ are adjacent in $\Gamma_G$, then by Lemmas 2.3 and 2.4, $u$ and $v$ are vertices of a $K_3$. Since $\gamma(\Gamma_G) = n - 4$, then $\Gamma_G$ is isomorphic to union of exactly one copy of $K_3$, one copy of $P_2$ and some isolated vertices. Suppose $V(K_3) = \{u_1, u_2, u_3\}$, $V(P_2) = \{v_1, v_2\}$ and isolated vertices are denoted by $x_i$, where $1 \leq i \leq n - 6$. We have
\[ C_G(u_i) = \{1, u_1, u_2, u_3\}. \] So \( o(u_i) \in \{2, 4\} \). By Lemma 2.3, \( o(v_1) = o(v_2) = 3 \). Also we have \( o(x_i) = 2 \). So \( |G| = 2^3 \ell \), where \( \ell \) is natural. By Sylow Theorem, there are two subgroup \( H \) and \( K \) of \( G \) such that \( |H| = 2^t \) and \( K = \{e, v_1, v_1^{-1} = v_2\} \).

If \( h \in H \) and \( o(h) = 2 \), then \( o(hv_1h) = o(v_1) = 3 \). So \( hv_1h = v_1 \) or \( v_1^{-1} \). If \( hv_1h = v_1 \), then \( v_1h = hv_1 \) and so \( o(v_1h) = 6 \), which is false. If \( hv_1h = v_1^{-1} \), then \( v_1h = hv_1^{-1} \) or \( hv_1 = v_1^{-1}h \). It is well known that \( |Z(H)| > 1 \).

Let \( z \in Z(H) \) and \( o(z) = 2 \). Then for every \( h \in H \setminus \{z\} \), we have
\[
(zh)v_1 = (hz)v_1 = h(zv_1) = h(v_1^{-1}z) = (hv_1)z = v_1(hz) = v_1(zh)
\]
Hence \( zh \in C_G(v_1) \). So \( zh \in \{v_1, v_1^{-1}\} \).

If \( |H| > 4 \), then \( |H| \geq 8 \). Thus there are \( h_1, h_2 \in H, h_1 \neq h_2 \) such that \( zh_1 = zh_2 \). Hence \( h_1 = h_2 \), which is a contradiction. Thus \( |H| = 4 \) and so \( |G| = 12 \). Since \( G \) has exactly two elements of order 3, then \( |cl(v_1)| = 1 \) or 2. Since \( |C_G(v_1)| = 3 \) and \( |G : C_G(v_1)| = |cl(v_1)| \), we have \( 4 = 1 \) or \( 4 = 2 \), which is not true.

**Subcase 3.** Let \( u \in V(\overline{\Gamma}_G) \), \( deg_{\overline{\Gamma}_G}(u) = 3 \) and \( N_{\overline{\Gamma}_G}(u) = \{x, y, z\} \). Then \( C_G(u) = \{1, u, x, y, z\} \) and so \( o(u) = 5 \). Hence induced subgraph on \( N_{\overline{\Gamma}_G}[u] \) is isomorphic to \( K_4 \).

Since \( \gamma(\overline{\Gamma}_G) = n - 4 \), then \( \overline{\Gamma}_G \cong K_4 \cup (n - 5)K_1 \). On the other hand if \( x \) is a isolated vertex in \( \overline{\Gamma}_G \), then \( o(x) = 2 \). Since \( xu \notin \{1, u, u^2, u^3, u^4\} \), we have \( o(xu) = 2 \). Thus \( xuux = u^{-1} \) and so \( \langle x, u \rangle \cong D_{10} \). Now let \( y \in G \setminus \langle x, u \rangle \) and \( y \) be an isolated vertex in \( \overline{\Gamma}_G \).

Then \( o(yu) = 2 \). Hence \( yuy = xux \). This implies that \( xy \in C_G(u) = \{1, u, u^2, u^3, u^4\} \).

Therefore \( y \in \{x, xu, xu^2, xu^3, xu^4\} \) and so \( y \in \langle x, u \rangle \), which is a contradiction. Hence \( G \cong \langle x, u \rangle \cong D_{10} \).

Conversely, if \( G \cong D_{10} \), then \( t = 1 \) and by Figure 4, \( \gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = 7 \) and the proof is complete.

\[ \Box \]

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**References**


