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On the dominating partition dimension of Möbius ladders and burger graphs

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Abstract

Let G be a connected graph and let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a partition of $V(G)$. The partition Π is called a *dominating resolving partition* of G if distinct vertices have distinct distance representations with respect to Π, and every vertex has a neighbor in a partition class different from its own. The minimum number of classes in such a partition is called the *dominating partition dimension* of G, denoted by $\eta_p(G)$. In this paper, we determine the dominating partition dimension of two families of cubic graphs: Möbius ladders and burger graphs with range one. We show that the dominating partition dimension is equal to 4 for both families within the specified ranges.

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1. Introduction

The problem of distinguishing vertices of a graph by means of distance information is a fundamental topic in metric graph theory. One of its classical formulations is the notion of a resolving set, introduced independently by Slater [21] and Harary and Melter [11]. A set $W \subseteq V(G)$ is *resolving* if every vertex of G is uniquely determined by its vector of distances to the vertices of W.

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The minimum cardinality of such a set is called the *metric dimension* of G. Since its introduction, the metric dimension and related parameters have been studied extensively, both from structural and algorithmic points of view [15, 6, 9].

A natural partition analogue of the metric dimension was introduced by Chartrand, Salehi, and Zhang [7, 8]. Given an ordered partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$, each vertex $v \in V(G)$ is represented by the vector

$$r(v \mid \Pi) = \big(d(v, S_1), d(v, S_2), \ldots, d(v, S_k)\big),$$

where

$$d(v, S_i) = \min\{d(v, x) : x \in S_i\}.$$

The partition Π is called a *resolving partition* if every two distinct vertices have distinct representations with respect to Π. The *partition dimension* of G, denoted by $\beta_p(G)$, is the minimum number of classes in a resolving partition of G. Chartrand, Salehi, and Zhang [8] showed that a connected bipartite graph G with partite sets of cardinalities r and s satisfies $\beta_p(G) \leq r + 1$ if $r = s$ and $\beta_p(G) \leq \max\{r, s\}$ if $r \neq s$, where equality holds if and only if G is a complete bipartite graph.

The partition dimension has subsequently been investigated for a variety of graph classes. In particular, exact values or sharp bounds have been obtained for graphs of diameter two [2], trees [10], generalized Möbius ladders [14], daisy graphs and related constructions [1], generalized hexagonal cellular networks [3], and several other structured graph families.

Combining vertex identification with domination leads to a more restrictive parameter. Motivated by the classical domination concept of Ore [16] and by the properties of a connecting resolving set and a dominating set [12, 4], Hernando, Mora, and Pelayo [13] introduced resolving dominating partitions. A resolving partition Π is *dominating* if every vertex has a neighbor belonging to a partition class different from its own. Equivalently, for every $v \in V(G)$, there exists a class S_i such that

$$d(v, S_i) = 1.$$

The *dominating partition dimension* of G, denoted by $\eta_p(G)$, is the minimum cardinality of a dominating resolving partition of G.

The additional domination requirement can substantially change the structure of an optimal resolving partition. Consequently, results for the ordinary partition dimension cannot, in general, be transferred directly to the dominating partition dimension. Hernando, Mora, and Pelayo [13] proved that

$$\beta_p(G) \leq \eta_p(G) \leq \beta_p(G) + 1$$

for every connected graph G. They also determined its value for some basic graph classes, namely $\eta_p(K_2) = 2$ and $\eta_p(P_n) = \eta_p(C_n) = 3$ for every integer $n \geq 3$, and proved that a connected graph G of order $n \geq 6$ satisfies $\eta_p(G) = n$ if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Ridwan and Baskoro [19] subsequently investigated graphs having large dominating partition dimension, while Ridwan, Assiyatun, and Baskoro [18] studied relationships among the partition dimension, dominating partition dimension, and locating-chromatic number in graphs. They showed that $\beta_p(G) \leq \eta_p(G) \leq \chi_L(G)$ for every connected graph G of order at least three, where $\chi_L(G)$ denotes the locating-chromatic number of G [5]. For trees, Ridwan and Abidin [20] proved

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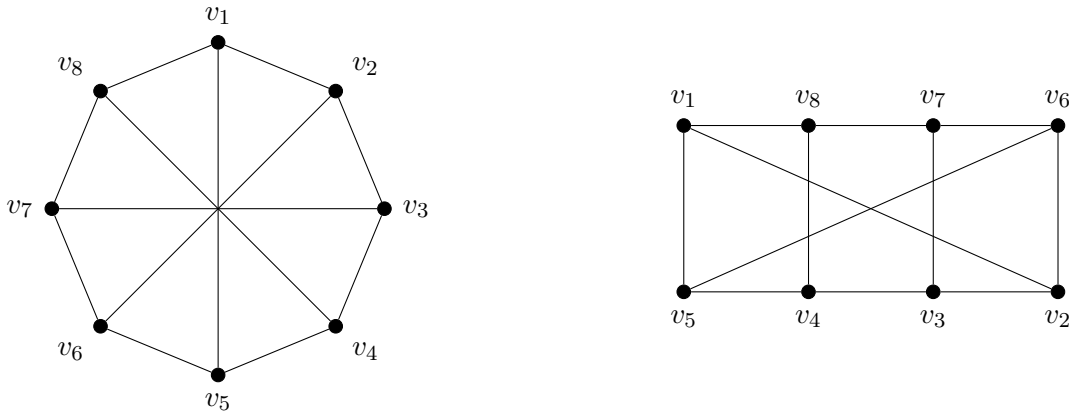


Figure 1. Two isomorphic illustrations of the Möbius ladder graph M_8.

that $\eta_p(T) = 3$ if and only if $\chi_L(T) = 3$, and that every tree T having two distinct strong support vertices satisfies $\eta_p(T) \geq 4$. Nevertheless, compared with the extensive literature on metric and partition dimensions, exact results on the dominating partition dimension remain relatively limited.

In this paper, we focus on two families of cubic graphs: Möbius ladders and burger graphs with range one. Cubic graphs provide a natural setting for this problem because their regular local structure, together with the symmetries occurring in these two families, can produce vertices having similar distance profiles. This makes the simultaneous requirements of resolving and domination particularly nontrivial.

We first present the definitions of these two families.

Definition 1.1. Let $n \geq 4$ be an even integer. The *Möbius ladder graph* M_n is a cubic graph obtained from the cycle graph C_n by adding edges joining each vertex to its antipodal vertex, that is, the vertex at distance $\frac{n}{2}$ on the cycle. Formally,

$$V(M_n) = \{v_i \mid 1 \leq i \leq n\}$$

and

$$E(M_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n\} \cup \left\{v_i v_{i+\frac{n}{2}} \mid 1 \leq i \leq \frac{n}{2}\right\},$$

where the indices are taken modulo n.

To illustrate the construction of a Möbius ladder graph, Figure 1 shows an example of M_n.

Definition 1.2. Let $n \geq 3$ be an integer and let σ be a permutation on $\{1, 2, \ldots, n\}$. For any integer $q \geq 1$, the *burger graph* with *range* q, denoted by $B_\sigma(n, q)$, is a simple undirected graph with a vertex set

$$V(B_\sigma(n, q)) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\},$$

where $|\sigma(i) - i| \leq q$ for every $i \in [1, n]$ and $\sigma(1) = q + 1$. The edge set is given by

$$E(B_\sigma(n, q)) = \{v_i v_{i+1}, u_i u_{i+1} \mid 1 \leq i \leq n\} \cup \{v_i u_{\sigma(i)} \mid 1 \leq i \leq n\},$$

where the indices in the cycles are taken modulo n.

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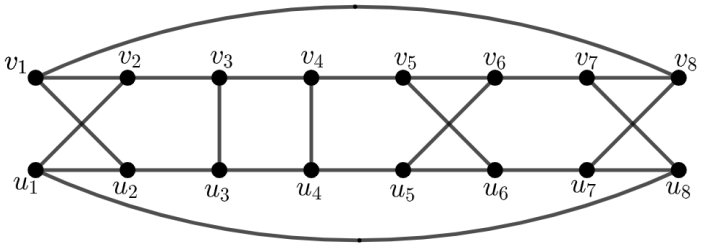


Figure 2. The burger graph $B_\sigma(8,1)$.

For example, the burger graph $B_\sigma(8,1)$ is depicted in Figure 2 with $\sigma = (12)(3)(4)(56)(78)$.

Our main results determine the dominating partition dimension of these two families, as follows.

Theorem 1.1. *Let M_n be a Möbius ladder with even $n \geq 4$. Then*

$$\eta_p(M_n) = 4.$$

Theorem 1.2. *Let $B_\sigma(n,1)$ be a burger graph with range one and $n \geq 4$. Then*

$$\eta_p\big(B_\sigma(n,1)\big) = 4.$$

For both families, the upper bounds are obtained by explicitly constructing dominating resolving partitions with four classes, while the corresponding lower bounds are established by showing that a dominating resolving partition with three classes cannot exist.

The remainder of the paper is organized as follows. In Section 2, we prove Theorem 1.1 on Möbius ladders. In Section 3, we prove Theorem 1.2 on burger graphs with range one.

2. Proof of Theorem 1.1: Möbius Ladders

Proof. We first consider even $n \geq 8$. To show the upper bound, namely $\eta_p(M_n) \leq 4$, consider a partition $\Pi = \{S_1, S_2, S_3, S_4\}$ of $V(M_n)$ defined by

$$\begin{aligned}
S_1 &= \{v_1\},\\
S_2 &= \left\{v_{\frac{n}{2}+1}\right\},\\
S_3 &= \left\{v_{\frac{n}{2}+3}, v_{\frac{n}{2}+4}, \ldots, v_{n-1}\right\},\\
S_4 &= \left\{v_2, v_3, \ldots, v_{\frac{n}{2}}, v_{\frac{n}{2}+2}, v_n\right\}.
\end{aligned}$$

An illustration of the partition is shown in Figure 3. If $x \in \{v_2, v_n\}$ then x is dominated by v_1, since $d(x, S_1) = 1$. If $x \in \{v_{\frac{n}{2}}, v_{\frac{n}{2}+2}\}$ then x is dominated by $v_{\frac{n}{2}+1}$, since $d(x, S_2) = 1$. Now, if $v_i \notin \{v_2, v_{\frac{n}{2}}, v_{\frac{n}{2}+2}, v_n\}$ then it is dominated by $v_{\frac{n}{2}+i \,(mod\, n)}$. Hence, Π is dominating.

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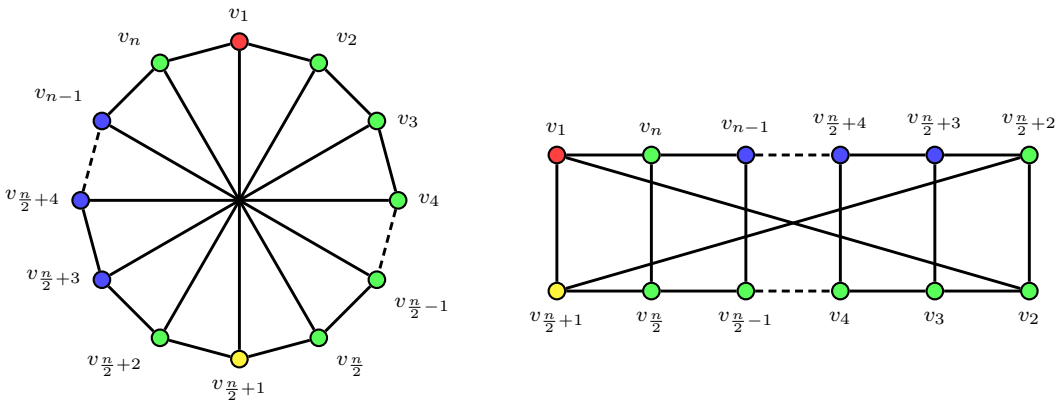


Figure 3. A partition pattern of the Möbius ladder graph M_n.

Now, we will show that Π is resolving. To do so, let x and y be two distinct vertices belonging to the same partition class S_q. Then, $q = 3$ or $q = 4$. If $q = 3$, then $x, y \in S_3$. Let $x = v_{n-i}$ and $y = v_{n-j}$ for some $1 \leq i < j \leq \frac{n}{2} - 3$. It is easy to verify that

$$d(v_{n-i}, S_1) = min\{i + 1, \frac{n}{2} - i\},$$

$$d(v_{n-i}, S_2) = min\{i + 2, \frac{n}{2} - i - 1\}.$$

If $d(x, S_1) = d(y, S_1)$ then $i + 1 = \frac{n}{2} - j$. Thus, $i + j = \frac{n}{2} - 1$, Now, calculate the distances of these two vertices to S_2. We obtain

$$\begin{aligned} d(x, S_2) &= min\{i + 2, \tfrac{n}{2} - i - 1\}, \\ d(y, S_2) &= min\{(\tfrac{n}{2} - 1 - i) + 2, \tfrac{n}{2} - (\tfrac{n}{2} - 1 - i) - 1\}, \\ &= min\{\tfrac{n}{2} - i + 1, i\}. \end{aligned}$$

It is obvious that $d(x, S_2) \neq d(y, S_2)$, since otherwise $min\{i+2, \frac{n}{2} - i - 1\} = min\{\frac{n}{2} - i + 1, i\}$. This holds if $i + 2 = \frac{n}{2} - i + 1$ or $\frac{n}{2} - i - 1 = i$ and both imply that $2i = \frac{n}{2} - 1$, and so $i = j$, a contradiction. Therefore, $r(x \mid \Pi) \neq r(y \mid \Pi)$.

Now, consider if $q = 4$, then $x, y \in S_4$. Let $x = v_i$ and $y = v_j$ for some $2 \leq i < j \leq \frac{n}{2}$.

$$d(v_i, S_1) = min\{i - 1, \frac{n}{2} - i + 2\},$$

$$d(v_i, S_2) = min\{i, \frac{n}{2} - i + 1\}.$$

If $d(x, S_1) = d(y, S_1)$ then $i - 1 = \frac{n}{2} - j + 2$. Thus, $i + j = \frac{n}{2} + 3$, Now, calculate the distances of these two vertices to S_2. We obtain

$$\begin{aligned} d(x, S_2) &= min\{i, \tfrac{n}{2} - i + 1\}, \\ d(y, S_2) &= min\{\tfrac{n}{2} + 3 - i, \tfrac{n}{2} - (\tfrac{n}{2} + 3 - i) + 1\}, \\ &= min\{\tfrac{n}{2} - i + 3, i - 2\}. \end{aligned}$$

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It is obvious that $d(x, S_2) \neq d(y, S_2)$, since otherwise $min\{i, \frac{n}{2} - i + 1\} = min\{\frac{n}{2} - i + 3, i - 2\}$. This holds if $i = \frac{n}{2} - i + 3$ or $\frac{n}{2} - i + 1 = i - 2$ and both imply that $2i = \frac{n}{2} + 3$, and so $i = j$, a contradiction. Therefore, $r(x \mid \Pi) \neq r(y \mid \Pi)$. Thus, Π is a dominating resolving partition of M_n, and therefore $\eta_p(M_n) \leq 4$.

It remains to show that $\eta_p(M_n) \geq 4$ for even $n \geq 8$. We first do this for even $n \geq 16$. For a contradiction, suppose that $\Pi = \{S_1, S_2, S_3\}$ is a dominating resolving partition of $V(M_n)$. It is easy to verify that

$$d = \text{diam}(M_n) = \left\lceil \frac{n}{4} \right\rceil.$$

Furthermore, let p be the maximum coordinate appearing among all vertex representations in M_n, namely

$$p = \max\{x_{i,1}, x_{i,2}, x_{i,3} \mid r(v_i|\Pi) = (x_{i,1}, x_{i,2}, x_{i,3}),\ v_i \in V(M_n)\}.$$

Without loss of generality, assume that this maximum coordinate is attained in the third coordinate by a vertex v_1 in S_1. Thus,

$$r(v_1|\Pi) = (0, 1, p).$$

We consider the following two cases.

Case 1. $p = d$. Then, let A be the set of all vertices at distance p from v_1, namely

$$A = \{x \in V(M_n) \mid d(v_1, x) = d\}.$$

By the distance structure of the Möbius ladder graph, we have

$$|A| = \begin{cases} 2, & n \equiv 2 \pmod 4, \\ 4, & n \equiv 0 \pmod 4. \end{cases}$$

Since $d(v_1, S_3) = p$ then $S_3 \subseteq A$. Now, we will count how many possible numbers of vertices in $S_1 \cup S_2$. These vertices could be at a distance of one from S_3 or at a distance greater than one from S_3. Since $S_3 \subseteq A$, every vertex at a distance of one from S_3 is at a distance $d - 1$ or d from v_1, and there are exactly four vertices at a distance $d-1$ from v_1 (see Figure 4). Hence, the number of vertices in S_3 together with the vertices at a distance of one from S_3 is at most $|A|+4$. Note that $N(S_3) \subset S_1 \cup S_2$. Now, all vertices at a distance greater than one from S_3 have representations of the form of either $(0, 1, t)$ or $(1, 0, t)$ with $t \in \{2, 3, \ldots, d\}$. Therefore, there are at most $2(d - 1)$ such vertices, and these vertices belong to $S_1 \cup S_2$. Thus, in total, the maximum number of vertices in $S_1 \cup S_2 \cup S_3$ is $|A| + 4 + 2(d - 1)$.

If $n \equiv 2 \pmod 4$, then $|A| = 2$ and $d = \frac{n+2}{4}$, and so the maximum number of vertices is

$$2 + 4 + 2(d - 1) = 2d + 4 = \frac{n + 10}{2} < n,$$

if $n \geq 14$, a contradiction.

If $n \equiv 0 \pmod 4$, then $|A| = 4$ and $d = \frac{n}{4}$, and so the maximum number of vertices is

$$4 + 4 + 2(d - 1) = 2d + 6 = \frac{n}{2} + 6 < n,$$

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if $n \geq 16$, a contradiction.

Case 2. $p < d$**.** Let $p = d - a$ for some integer a with $1 \leq a < d$. Again, without loss of generality, assume that the maximum coordinate is attained in the third coordinate by a vertex v_1 in S_1, namely $r(v_1|\Pi) = (0, 1, p)$. Define

$$A_p = \{x \in V(M_n) \mid d(v_1, x) = p\}.$$

Next, define

$$H_p = \{x \in V(M_n) \mid d(x, S_3) > 1 \text{ and } d(x, v_1) < p\}.$$

All vertices in H_p are in $S_1 \cup S_2$, and these vertices are not adjacent to any vertex of S_3. These vertices are easy to list. The vertex v_1 has exactly three neighbors, namely v_2, v_n, and $v_{\frac{n}{2}+1}$. For each j with $2 \leq j \leq d-1$, there are exactly four vertices at distance j from v_1: the two vertices v_{1+j} and v_{n+1-j}, reached along the cycle, and the two vertices $v_{\frac{n}{2}+2-j}$ and $v_{\frac{n}{2}+j}$, reached through the edge $v_1 v_{\frac{n}{2}+1}$. This is illustrated in Figure 4 for the Möbius ladder M_{20}, where $d = 5$. A vertex at a distance of $p-1$ from v_1 may be adjacent to S_3, so we consider only vertices at a distance of at most $p-2$ from v_1. Since $p < d$, the number of vertices at distance at most $p-2$ from v_1 is therefore

$$1 + 3 + 4(p-3) = 4p - 8,$$

and so $|H_p| \geq 4p - 8$.

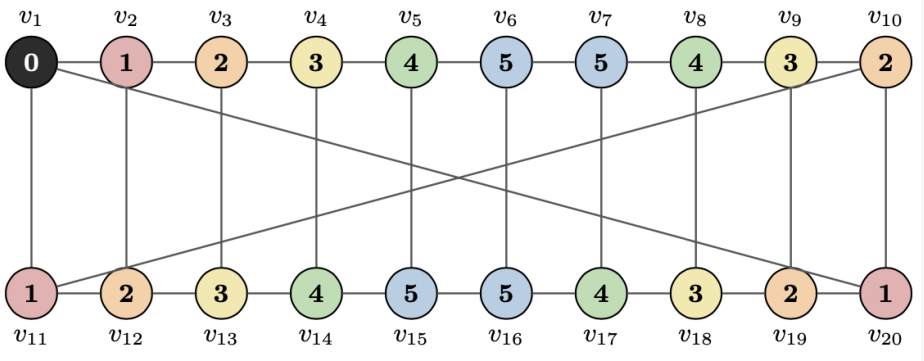


Figure 4. The Möbius ladder M_{20}, where each vertex is labelled by its distance from v_1.

Note that every vertex in H_p has a representation of the form either $(0, 1, t)$ or $(1, 0, t)$, where $t \in \{2, \ldots, p\}$. Consequently, $|H_p| \leq 2(p-1)$. This implies that $4p - 8 \leq 2(p-1)$, and hence $p \leq 3$.

Since p is the maximum coordinate appearing in any representation with respect to Π, every representation has each coordinate at most p. The vertices in S_3 can only have representations of the form $(t, 1, 0)$ or $(1, t, 0)$, where $1 \leq t \leq p$; since these two forms coincide at $t = 1$, there are at most $2p - 1$ such representations. The vertices in $N(S_3)$ can only have representations of the form $(0, t, 1)$ or $(t, 0, 1)$, where $1 \leq t \leq p$, giving at most $2p$ such representations. Every remaining vertex has a distance greater than 1 from S_3, so its representation is of the form $(0, 1, t)$ or $(1, 0, t)$,

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where $t \in \{2, \ldots, p\}$, giving at most $2(p-1)$ such representations. Therefore, the total number of available representations is at most

$$(2p-1) + 2p + 2(p-1) = 6p - 3 \leq 6 \cdot 3 - 3 = 15.$$

This is a contradiction for even $n \geq 16$. Therefore, we have shown that $\eta_p(M_n) = 4$ for every even $n \geq 16$.

Now, it remains to show that $\eta_p(M_n) = 4$ for even $4 \leq n \leq 14$.

First, consider the case $n = 4$ or $n = 6$. The graphs M_4 and M_6 are isomorphic to K_4 and $K_{3,3}$, respectively. In K_4, any two vertices have the same distance to every other vertex, so all four vertices must belong to different classes, and hence $\eta_p(M_4) = 4$, since four singletons clearly form a dominating resolving partition. In M_6, it holds for any two vertices in the same partite set $\{v_1, v_3, v_5\}$ or $\{v_2, v_4, v_6\}$. Hence, with only three classes, every class would consist of one vertex from each partite set, and these two vertices would have the same representation. Thus, $\eta_p(M_6) \geq 4$, and since $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5\}, \{v_6\}\}$ is a dominating resolving partition, $\eta_p(M_6) = 4$.

Now, let $n \in \{8, 10, 12, 14\}$, and suppose that $\Pi = \{S_1, S_2, S_3\}$ is a dominating resolving partition of M_n.

First, let $n = 8$. Since M_8 has diameter 2, each class has at most three possible representations. Since M_8 has eight vertices and each class has at most three vertices, two classes have exactly three vertices, and the remaining class has exactly two vertices. Renaming the classes if necessary, we may assume that $|S_1| = |S_2| = 3$ and $|S_3| = 2$.

Let $b_1, c_1 \in S_1$ be the vertices with representations $(0, 2, 1)$ and $(0, 1, 1)$, and let $b_2, c_2 \in S_2$ be the vertices with representations $(2, 0, 1)$ and $(1, 0, 1)$. Write $S_3 = \{s, s'\}$. The representations of s and s' are distinct, and both are among $(1, 1, 0)$, $(2, 1, 0)$ and $(1, 2, 0)$, so one of them, say s, has no neighbor in S_1 or no neighbor in S_2. By the symmetry between S_1 and S_2, we may assume that s has no neighbor in S_1. Since the third coordinates of b_1 and c_1 are equal to 1, both b_1 and c_1 are adjacent to s'. Moreover, b_1 has no neighbor in S_2, so its three neighbors lie in $\{a_1, c_1, s'\}$, where a_1 is the third vertex of S_1. In particular, b_1 is adjacent to c_1, and so $b_1 c_1 s' b_1$ is a cycle C_3 in M_8. This is impossible, since M_8 contains no C_3.

Next, let $n \in \{10, 12\}$. Then $d = 3$. If $p = d$, then $S_3 \subseteq A$ with $A = \{x \in V(M_n) | d(v_1, x) = d\}$. For $n = 10$, we have $|A| = 2$. If $|S_3| = 1$, then the number of possible representations is at most $1 + 3 + 2(d-1) = 8$. If $|S_3| = 2$, then $S_3 = A$ and v_1 is the only vertex at distance 3 from S_3, so the form $(1, 0, 3)$ cannot occur, and the number of possible representations is at most $2 + 4 + \big((d-1) + (d-2)\big) = 9$. In both cases, the number of possible representations is less than the number of vertices $|V(M_{10})| = 10$, a contradiction. For $n = 12$, we have $|A| = 4$. If $|S_3| = 1$, the number of possible representations is again at most $1 + 3 + 2(d-1) = 8$. If $2 \leq |S_3| \leq 4$, the same computation gives at most 11 representations: if v_1 is the only vertex at distance 3 from S_3, then the remaining vertices can use at most $(d-1) + (d-2)$ representations, and otherwise $|S_3| + |N(S_3)| = 6$, which gives at most $6 + 2(d-1) = 10$. In every case, the number of possible representations is less than the number of vertices $|V(M_{12})| = 12$, a contradiction.

If $p < d$, then $p \leq 2$. Hence, the number of available representations is at most

$$(2p-1) + 2p + 2(p-1) = 6p - 3 \leq 6 \cdot 2 - 3 = 9,$$

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which is less than the number of vertices $|V(M_n)| \in \{10, 12\}$, a contradiction.

Finally, let $n = 14$. Then $d = 4$ and $|A| = 2$. If $p = d$, then $S_3 \subseteq A$, and the same computation as above gives at most $1 + 3 + 2(d-1) = 10$ representations if $|S_3| = 1$, and at most $2 + 4 + 2(d-1) = 12$ representations if $|S_3| = 2$. Both numbers are less than the number of vertices $|V(M_{14})| = 14$, a contradiction.

If $p < d$, then $p \leq 3$. If $p \leq 2$, then the number of available representations is at most $6p - 3 \leq 9 < 14$, a contradiction. If $p = 3$, then $|H_3| = 4$, so H_3 consists of v_1 and its three neighbors. These four vertices can only have the four representations $(0, 1, 2)$, $(1, 0, 2)$, $(0, 1, 3)$ and $(1, 0, 3)$, so all of them are used, and some neighbor u of v_1 satisfies $d(u, S_3) = 3$. However, u has a neighbor w at a distance of 2 from v_1. Since $d(w, v_1) < p$ but $w \notin H_3$, the vertex w is adjacent to S_3. Hence, $d(u, S_3) \leq 2$ is a contradiction.

Hence $\eta_p(M_n) \geq 4$ for every even $n \geq 4$, which completes the proof. □

3. Proof of Theorem 1.2: Burger Graphs with Range One

To prove Theorem 1.2, we first determine a bound on the diameter of a burger graph with range one.

Lemma 3.1. *Let $B_\sigma(n, 1)$ be a burger graph with range one and $n \geq 5$. If*

$$d = \operatorname{diam}\big(B_\sigma(n, 1)\big),$$

then

$$d = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \text{ is odd and } \sigma \text{ contains at least } \lfloor \frac{n}{2} \rfloor \text{ consecutive fixed points}, \\ \lfloor \frac{n}{2} \rfloor, & \text{otherwise.} \end{cases}$$

Proof. Let x and y be any two distinct vertices of $B_\sigma(n, 1)$. First, let x and y be in the same cycle. Without loss of generality, let $x = v_i$ and $y = v_j$ for some $1 \leq i < j \leq n$. Then, $d(x, y) = \min\{j - i, n - j + i\} \leq \lfloor \frac{n}{2} \rfloor$. In particular, if $j = i + \lfloor \frac{n}{2} \rfloor \pmod n$, then $d(x, y) = \lfloor \frac{n}{2} \rfloor$. If x and y are in different cycles, without loss of generality, let $x = v_i$ and $y = u_j$ for $1 \leq i, j \leq n$. Let $t = \min\{|i - j|, n - |i - j|\}$ and $s = \min\{i, j\}$. Note that $t \leq \lfloor \frac{n}{2} \rfloor$. If there is an integer r with $s \leq r \leq s + t - 1$ such that $\sigma(r) = r + 1$, then $d(x, y) = t$; otherwise, $d(x, y) = t + 1$. In particular, if n is odd and $t = \lfloor \frac{n}{2} \rfloor$ and $\sigma(z) \neq z$ for $s \leq z \leq s + t - 1$ then $d(x, y) = t + 1 = \lceil \frac{n}{2} \rceil$. This concludes the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. First, we show that $\eta_p(B_\sigma(n, 1)) \leq 4$ for $n \geq 5$. Define a partition $\Pi = \{S_1, S_2, S_3, S_4\}$ of $V(B_\sigma(n, 1))$ as follows:

$$\begin{aligned} S_1 &= \{v_1\}, \\ S_2 &= \{v_2, v_3, \ldots, v_{n-1}\}, \\ S_3 &= \{u_1, u_2, \ldots, u_{n-1}\}, \\ S_4 &= \{v_n, u_n\}. \end{aligned}$$

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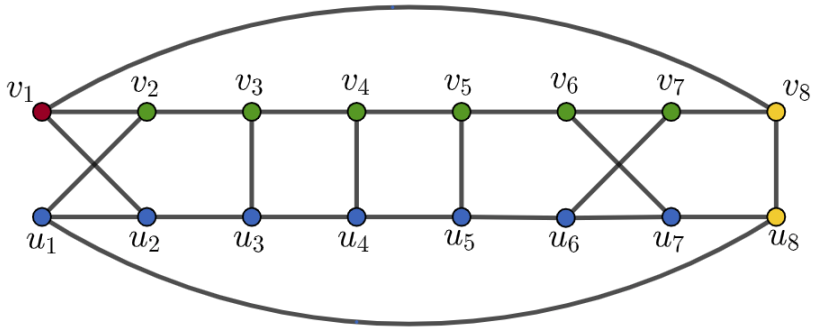


Figure 5. A partition pattern of the burger graph for $n \geq 5$.

An illustration of the partition is shown in Figure 5. If $x = v_i$ with $1 \leq i \leq n-1$, then x is dominated by $u_{\sigma(i)}$, since $d(x, u_{\sigma(i)}) = 1$. If $x = u_i$ with $1 \leq i \leq n-1$, then x is dominated by v_j with $\sigma(j) = i$, since $d(x, v_j) = 1$. Finally, v_n is dominated by v_{n-1}, since $d(v_n, S_2) = 1$, and u_n is dominated by u_{n-1}, since $d(u_n, S_3) = 1$. Hence, Π satisfies the domination condition.

Now, we will show that Π is resolving. To do so, let x and y be two distinct vertices belonging to the same partition class S_q. Then, $q = 2$ or $q = 3$. If $q = 2$, then $x, y \in S_2$. Let $x = v_i$ and $y = v_j$ for some $2 \leq i < j \leq n-1$. It is easy to verify that

$$d(v_i, S_1) = \min\{i-1,\, n-i+1\},$$

$$d(v_i, S_4) = \min\{i,\, n-i\}.$$

If $d(x, S_1) = d(y, S_1)$ then $i - 1 = n - j + 1$. Thus, $i + j = n + 2$. Now, calculate the distances of these two vertices to S_4. We obtain

$$\begin{aligned} d(x, S_4) &= \min\{i, n-i\}, \\ d(y, S_4) &= \min\{n+2-i,\, n-(n+2-i)\}, \\ &= \min\{n+2-i,\, i-2\}. \end{aligned}$$

It is obvious that $d(x, S_4) \neq d(y, S_4)$, since otherwise $\min\{i, n-i\} = \min\{n+2-i, i-2\}$. This holds if $i = n+2-i$ or $n-i = i-2$, and both imply that $2i = n+2$, and so $i = j$, a contradiction. Therefore, $r(x \mid \Pi) \neq r(y \mid \Pi)$.

By the same argument, if $q = 3$, then $x, y \in S_3$, and writing $x = u_i$ and $y = u_j$ for some $1 \leq i < j \leq n-1$, it follows that $r(x \mid \Pi) \neq r(y \mid \Pi)$. Thus, Π is a dominating resolving partition of $B_\sigma(n, 1)$, and therefore $\eta_p(B_\sigma(n, 1)) \leq 4$ for $n \geq 5$.

It remains to show that $\eta_p(B_\sigma(n, 1)) \geq 4$ for $n \geq 10$. For a contradiction, suppose that $\Pi = \{S_1, S_2, S_3\}$ is a dominating resolving partition.

Let $d = \operatorname{diam}(B_\sigma(n, 1))$, and let p be the maximum coordinate appearing among all vertex representations in $B_\sigma(n, 1)$, namely

$$p = \max\{x_i \mid r(x|\Pi) = (x_1, x_2, x_3),\ x \in V(B_\sigma(n, 1)),\ i \in \{1, 2, 3\}\}$$

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Without loss of generality, assume that this maximum coordinate is attained in the third coordinate by a vertex $z \in S_1$. Thus

$$r(z|\Pi) = (0, 1, p).$$

We consider the following two cases.

Case 1: $p = d$**.** Let A be the set of all vertices at distance p from z, namely

$$A = \{x \in V(B_\sigma(n, 1)) \mid d(x, z) = d\}.$$

By the distance structure of a burger graph with range one, we have

$$|A| \leq \begin{cases} 4, & \text{if } n \text{ is even,} \\ 5, & \text{if } n \text{ is odd.} \end{cases}$$

and hence $|A| \leq 5$ for every n. Since $d(z, S_3) = p$ then $S_3 \subseteq A$, and so $|S_3| \leq 5$.

Now, we will count how many possible vertices are in S_1 or S_2. These vertices could be at distance one from S_3 or at distance greater than one from S_3. Since $S_3 \subseteq A$, every vertex in S_3 or adjacent to S_3 is at distance $d - 1$ or d from z, and so $|S_3| + |N(S_3)| \leq 10$, and these vertices belong to $S_1 \cup S_2$. Now, all vertices at distance greater than one from S_3 have representations of the form either $(0, 1, t)$ or $(1, 0, t)$ with $t \in \{2, 3, \ldots, d\}$. Therefore, there are at most $2(d - 1)$ such vertices, and these vertices belong to $S_1 \cup S_2$. Thus, in total, the maximum number of vertices in $S_1 \cup S_2 \cup S_3$ is

$$5 + 5 + 2(d - 1) = 2d + 8.$$

Since $d \leq \frac{n+1}{2}$ (Lemma 3.1), we have $2d + 8 \leq n + 9 < 2n$ if $n \geq 10$, a contradiction.

Case 2: $p < d$**.** Let $p = d - a$ for some integer a with $1 \leq a < d$. Without loss of generality, assume that the maximum coordinate is attained in the third coordinate by a vertex z_0 in S_1 such that $r(z_0 \mid \Pi) = (0, 1, p)$. Define

$$A_p = \{x \in V(B_\sigma(n, 1)) \mid d(x, z_0) = p\}, \text{and}$$

$$H_p = \{x \in V(B_\sigma(n, 1)) \mid d(x, S_3) > 1 \text{ and } d(x, z_0) < p\}.$$

All vertices in H_p are in $S_1 \cup S_2$ and these vertices not adjacent to any vertex of S_3.

As in the proof of Theorem 1.1, every vertex at distance at most $p - 2$ from z_0 belongs to H_p. Suppose $z_0 = v_i$. The $2(p - 2) + 1$ vertices $v_{i-(p-2)}, \ldots, v_{i+(p-2)}$ are at distance at most $p - 2$ from v_i along the cycle $v_1 v_2 \cdots v_n$. The vertex $u_{\sigma(i)}$ is a neighbor of v_i, so the $2(p-3)+1$ vertices $u_{\sigma(i)-(p-3)}, \ldots, u_{\sigma(i)+(p-3)}$ are also at distance at most $p - 2$ from v_i. This is illustrated in Figure 6 for the burger graph $B_\sigma(16, 1)$ with $\sigma = (12)(56)(78)(11\,12)(15\,16)$. Hence

$$|H_p| \geq \big(2(p - 2) + 1\big) + \big(2(p - 3) + 1\big) = 4p - 8.$$

Note that every vertex in H_p has a representation of the form either $(0, 1, t)$ or $(1, 0, t)$, where $t \in \{2, \ldots, p\}$. Consequently, $|H_p| \leq 2(p-1)$. This implies that, $4p-8 \leq 2(p-1)$, and so $p \leq 3$.

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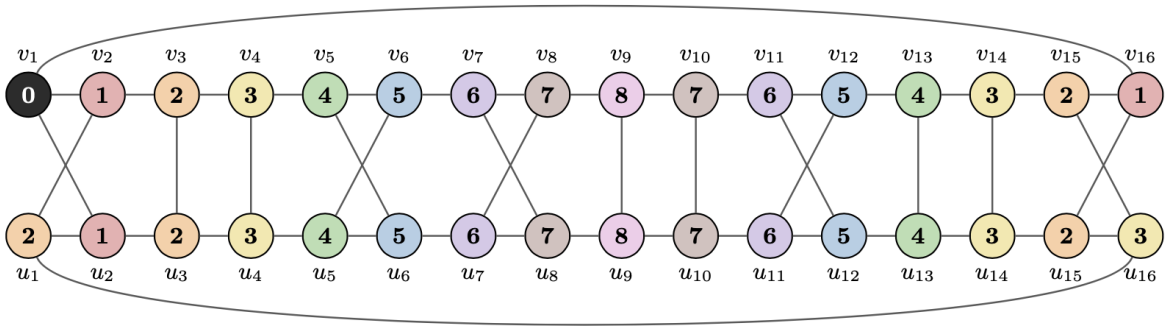


Figure 6. The burger graph $B_\sigma(16,1)$ with $\sigma = (12)(56)(78)(11\,12)(15\,16)$, where each vertex is labelled by its distance from v_1.

By the same counting argument as in the proof of Theorem 1.1 (Case 2), the total number of representations available for all vertices of $B_\sigma(n,1)$ is at most

$$(2p-1)+2p+2(p-1) = 6p-3 \le 6\cdot 3-3 = 15.$$

This is less than the number of vertices in $B_\sigma(n,1)$ if $n \ge 10$, a contradiction. Therefore, we conclude that $\eta_p(B_\sigma(n,1)) = 4$ if $n \ge 10$.

Now, it remains to show that $\eta_p(B_\sigma(n,1)) = 4$ for $4 \le n \le 9$. First, let $5 \le n \le 9$, and suppose that $\Pi = \{S_1, S_2, S_3\}$ is a dominating resolving partition, with p and z_0 as above, so that $r(z_0 \mid \Pi) = (0,1,p)$. The number of available representations is at most $(2p-1)+2p+2(p-1) = 6p-3$. If $p \le 2$, this number is at most 9, which is less than the number of vertices $2n \ge 10$, a contradiction. Hence, $p \ge 3$. On the other hand, every vertex at distance at most $p-2$ from z_0 belongs to H_p, and so $4p-8 \le |H_p| \le 2(p-1)$, that is, $p \le 3$. Therefore, $p = 3$. Then $|H_p| = 4$, so H_p consists of z_0 and its three neighbors. These four vertices can only have the four representations $(0,1,2)$, $(1,0,2)$, $(0,1,3)$ and $(1,0,3)$, so all of them are used, and some neighbor u of z_0 satisfies $d(u,S_3) = 3$. However, u has a neighbor w at distance 2 from z_0. Since $d(w,z_0) < p$ but $w \notin H_p$, the vertex w is adjacent to S_3. Hence, $d(u,S_3) \le 2$, a contradiction. Therefore, $\eta_p(B_\sigma(n,1)) = 4$ for $5 \le n \le 9$.

Now, let $n = 4$. By the definition of a burger graph, either $\sigma = (12)(3)(4)$ or $\sigma = (12)(34)$. In both cases, consider the partition $\Pi = \{S_1, S_2, S_3, S_4\}$ with

$$S_1 = \{v_1\}, \quad S_2 = \{v_2, v_3, v_4\}, \quad S_3 = \{u_1\}, \quad S_4 = \{u_2, u_3, u_4\}.$$

The vertices v_1, v_2, v_3 and u_1, u_2, u_3 have the representations $(0,1,2,1)$, $(1,0,1,2)$, $(2,0,2,1)$ and $(2,1,0,1)$, $(1,2,1,0)$, $(2,1,2,0)$ for both permutations. If $\sigma = (12)(3)(4)$, then v_4 and u_4 have the representations $(1,0,2,1)$ and $(2,1,1,0)$, and if $\sigma = (12)(34)$, they have the representations $(1,0,3,1)$ and $(3,1,1,0)$. In both cases, the representations are distinct and each of them contains a 1, so $\eta_p(B_\sigma(4,1)) \le 4$.

For the lower bound, suppose that $\Pi = \{S_1, S_2, S_3\}$ is a dominating resolving partition of $B_\sigma(4,1)$. The diameter of $B_\sigma(4,1)$ is at most 3, so $p \le 3$. Moreover, $p \ge 2$, since with $p = 1$

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only three representations are available for eight vertices. Hence $p \in \{2, 3\}$. If $p = 2$, then each class has at most three possible representations. Since $B_\sigma(4, 1)$ is also a cubic graph with eight vertices that contains no C_3, the same argument as for M_8 in the proof of Theorem 1.1 gives a contradiction. If $p = 3$, then every vertex at a distance of at most 1 from z_0 belongs to H_p, so $4 \leq |H_p| \leq 2(p-1) = 4$. The argument for $5 \leq n \leq 9$ above gives a contradiction. Therefore $\eta_p(B_\sigma(n, 1)) = 4$ for every $n \geq 4$, which completes the proof. □

4. Concluding remarks

In this paper, we showed that the dominating partition dimension of a Möbius ladder M_n with even $n \geq 4$ and of a burger graph $B_\sigma(n, 1)$ with range one and $n \geq 4$ is equal to 4. In both cases, the upper bound comes from an explicit dominating resolving partition with four classes, and the lower bound comes from counting the representations that are available to a partition with three classes.

It is worth comparing these results with the partition dimension. Since $\beta_p(G) \leq \eta_p(G) \leq \beta_p(G)+1$ for every connected graph G [13], Theorems 1.1 and 1.2 imply that $\beta_p(M_n) \in \{3, 4\}$ and $\beta_p\big(B_\sigma(n, 1)\big) \in \{3, 4\}$ in the corresponding ranges of n. Thus, for both families, the domination requirement increases the number of classes needed by at most one, and the dominating partition dimension does not depend on n.

It remains to decide whether $\beta_p = 3$ or $\beta_p = 4$ for these graphs, that is, whether the domination requirement forces an additional class.

Problem 1. *Determine $\beta_p(M_n)$ and $\beta_p\big(B_\sigma(n, 1)\big)$. In particular, for which values of n do we have $\eta_p = \beta_p + 1$?*

A natural next step is to consider burger graphs $B_\sigma(n, q)$ with range $q \geq 2$. For these graphs, the four-class partition used in the proof of Theorem 1.2 is still dominating, but in general it is no longer resolving, since the edges $v_i u_{\sigma(i)}$ may now join vertices whose indices differ by more than one. On the other hand, the counting argument in Case 2 of the proof of Theorem 1.2 still applies, whereas Case 1 depends on the diameter of $B_\sigma(n, q)$ and on the number of vertices at maximum distance from a given vertex, both of which change with q. We therefore pose the following problem.

Problem 2. *Determine $\eta_p\big(B_\sigma(n, q)\big)$ for every $q \geq 2$. In particular, does $\eta_p\big(B_\sigma(n, q)\big) = 4$ hold for every sufficiently large n?*

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