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On (a, d)-distance antimagic labeling of 2-regular graphs containing 6-cycles

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Abstract

A graph $G = (V(G), E(G))$ of order n is called (a, d)-distance antimagic if there exists a bijection $\ell : V(G) \to \{1, 2, \ldots, n\}$ such that the set of vertex weights is $\{a, a+d, a+2d, \ldots, a+(n-1)d\}$, where $a > 0$ and $d \geq 0$. In such a case, ℓ is called an (a, d)-distance antimagic labeling of G. In this paper, we provide necessary and sufficient conditions of the graphs $C_n \cup tC_6$ where $n \in \{5, 7, 9, 11\}$, $C_{2t+1} \cup tC_6$, $m(C_n \cup C_6)$, $mC_n \cup \frac{1}{2}(mn-1)C_6$, and $C_{10} \cup mC_3 \cup tC_6$ to be (a, d)-distance antimagic for some positive integers m, n, and t.

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1. Introduction

For standard definitions and notation in graph theory, we refer the reader to Hartsfield and Ringel [8]. Let $G = (V(G), E(G))$ be a simple graph of order n without isolated vertices. A bijection $\ell : V(G) \to \{1, 2, \ldots, n\}$ is called a *vertex labeling* of G. The *weight* of a vertex

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$x \in V(G)$ with respect to ℓ is defined as

$$w(x) = \sum_{y \in N(x)} \ell(y),$$

where $N(x) = \{y \in V(G) : xy \in E(G)\}$. The labeling ℓ is called a *distance magic* (DM) *labeling* if $|\{w(x) : x \in V(G)\}| = 1$, and it is called a *distance antimagic* (DA) *labeling* if $|\{w(x) : x \in V(G)\}| = n$. In particular, if $\{w(x) : x \in V(G)\} = \{a + (i-1)d : 1 \leq i \leq n\}$, where $a > 0$ and $d \geq 0$, then ℓ is called an (a,d)-*DA labeling* and G is called an (a,d)-*DA graph*.

The concept of DM labeling was introduced independently by Vilfred in 1994 [28] and by Miller et al. in 2003 [18]. In [28], this labeling was referred to as a Σ-labeling, while Miller et al. [18] used the term 1-*vertex magic vertex labeling*. The term DM labeling was introduced by Sugeng et al. in 2009 [27]. Meanwhile, the concept of (a,d)-DA labeling was introduced by Arumugam and Kamatchi [1] in 2012 and DA labeling was introduced by Kamatchi and Arumugam [11] in 2013.

Graph labelings of 2-regular graphs have been extensively studied; see, for example, [2, 10, 15, 16, 17]. The results on DM and DA labelings of 2-regular graphs are studied in [11, 20, 21], and for other classes of graphs are presented in [3, 4, 5, 6, 9, 12, 13, 14, 19, 22, 24, 26, 29]. Additionally, generalizations of DM and DA labelings are discussed in [23, 25]. For further results on this topic and other types of graph labelings, we refer the reader to Gallians dynamic survey [7].

In [1], Arumugam and Kamatchi showed that the cycle C_n is (a,d)-DA iff n is odd and $d = 1$. They further proposed the following problem.

Problem 1. [1] *Characterize the disconnected 2-regular (a,d)-DA graphs.*

Ngurah and Simanjuntak [20] gave partial answers to the Problem 1 by establishing a necessary condition for a 2-regular graph to admit an (a,d)-DA labeling.

Lemma 1.1. [20] *For every integer $n \geq 3$, if G is a 2-regular (a,d)-DA graph of order n, then $d = 1$, n is odd, and G does not contain C_4.*

They also gave the following results.

Theorem 1.1. [20] *For every integer $m \geq 1, n \geq 3$, the graph mC_n is (a,d)-DA iff $d = 1$ and both m, n are odd.*

Theorem 1.2. [20] *For every integer $n \geq 7$, the graph $C_n \cup C_6$ is (a,d)-DA iff $d = 1$ and n is odd.*

Motivated by these resuls, in this paper, we investigate (a,d)-DA labelings of the graphs $C_n \cup tC_6$ for an integer $t \geq 2$ and $n \in \{5, 7, 9, 11\}$; $C_{2t+1} \cup tC_6$ for an integer $t \geq 6$; $m(C_n \cup C_6)$ for odd integers $m \geq 3$ and $n \geq 7$; $mC_n \cup \frac{1}{2}(mn-1)C_6$ for odd integers $m, n \geq 3$; and $C_{10} \cup mC_3 \cup tC_6$, for an odd integer $m \geq 1$ and a positive integer t.

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2. Main Results

This section presents a partial answer to Problem 1 through a characterization of certain classes of disconnected 2-regular (a,d)-DA graphs. Throughout this paper, in each figure, the black numbers denote the vertex labels, while the red numbers denote the corresponding vertex weights.

To prove the first theorem, we use the shifting Kötzig matrix $\mathcal{A}=[a_{i,j}]$ of order $3\times 2t$ whose row entries are permutations of $\{-t,-t+1,\ldots,-2,-1,1,2,\ldots,t\}$ and $\mathcal{B}=[b_{i,j}]$ of order $3\times 2t$ whose row entries are permutations of $\{-t-1,-t,\ldots,t,t+1\}$ such that $b_{1,j}\neq -1,0,$ and 1, $b_{2,j}\neq -t,0,$ and t, and $b_{3,j}\neq -t-1,0,$ and $t+1$, and each column has sum 0. The existence of $\mathcal{A}$ and $\mathcal{B}$ can be found in [10, Lemma 2] and [17, Lemma 9], respectively. For $t=3$, the matrices $\mathcal{A}$ and $\mathcal{B}$ are given as follows:

$$\mathcal{A}=\begin{bmatrix}-3&-2&-1&1&2&3\\1&3&-2&2&-3&-1\\2&-1&3&-3&1&-2\end{bmatrix},\quad \mathcal{B}=\begin{bmatrix}-4&-3&-2&2&3&4\\2&4&-1&1&-4&-2\\2&-1&3&-3&1&-2\end{bmatrix}.$$

The matrix $\mathcal{A}$ in [10, Lemma 2] satisfy

$$\{a_{2,j}+a_{3,j}\}=\{a_{3,j}+a_{1,j}\}=\{a_{1,j}+a_{2,j}\}=\{-t,-t+1,\ldots,-2,-1,1,2,\ldots,t-1,t\},$$

and the matrix $\mathcal{B}$ in [17, Lemma 9] satisfy

$$\begin{aligned}\{b_{1,j}+b_{2,j}\}&=\{-t,-t+1,\ldots,-2,-1,1,2,\ldots,t-1,t\}\\ \{b_{2,j}+b_{3,j}\}&=\{-t-1,-t,\ldots,-2,2,\ldots,t,t+1\}\\ \{b_{3,j}+b_{1,j}\}&=\{-t-1,-t+1,-t+2,\ldots,-2,-1,1,2,\ldots,t-2,t-1,t+1\}.\end{aligned}$$

The next theorem characterize (a,d)-DA labeling of $C_n\cup tC_6$ for $n\in\{5,7,9,11\}$ and $t\geq 2$.

Theorem 2.1. *For every integer $t\geq 2$ and $n\in\{5,7,9,11\}$, the graph $G\cong C_n\cup tC_6$ is (a,d)-DA iff $d=1$.*

Proof. Lemma 1.1 implies that, if G is (a,d)-DA, then $d=1$. Conversely, let

$$V(G)=\{z_i:1\leq i\leq n\}\cup\{y_{i,j}:1\leq i\leq 6,1\leq j\leq t\},$$

and

$$E(G)=\{z_iz_{i+1},\ z_nz_1:1\leq i\leq n-1\}\cup\{y_{i,j}y_{i+1,j},\ y_{6,j}y_{1,j}:1\leq i\leq 5,1\leq j\leq t\}.$$

For $j=1,2,\ldots,t$, let $Y_{2j-1}=\{y_{1,j},y_{3,j},y_{5,j}\}$ and $Y_{2j}=\{y_{2,j},y_{4,j},y_{6,j}\}$. Clearly, $Y_1,Y_2,\ldots,Y_{2t}$, and $\{z_i:1\leq i\leq n\}$ form a partition of $V(G)$. To show that G is (a,d)-DA, we consider the following cases.

Case 1. $n=5,7,9$.

Subcase 1.1. $t=2$.

Label the graph $C_5\cup 2C_6$ by $(1,2,9,13,15)$, $(3,4,10,8,16,12)$, $(5,7,6,11,17,14)$, graph $C_7\cup 2C_6$

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by $(2,19,11,4,18,17,9)$, $(1,3,10,12,15,14)$, $(5,6,7,8,13,16)$, and graph $C_9 \cup 2C_6$ by $(1,13,19,5,2,9,21,16,3$ These labelings are an $(\frac{n+15}{2},1)$-DA labelings of $C_5 \cup 2C_6$, $C_7 \cup 2C_6$, and $C_9 \cup 2C_6$, respectively.

Subcase 1.2. $t \geq 3$.

Consider a matrix $\mathcal{A}^* = [a^*_{i,j}]$ of order $3 \times 2t$ defined by

$$\begin{aligned} a^*_{1,j} &= t + 2 + a_{1,j}, \\ a^*_{2,j} &= 3t + \frac{1}{2}(n+1) + a_{2,j}, \text{ and} \\ a^*_{3,j} &= 5t + \frac{1}{2}(n+5) + a_{3,j}, \end{aligned}$$

where $j = 1, 2, \ldots, 2t$.

Define a labeling $\ell : V(G) \to \{1, 2, \ldots, 6t+n\}$ as follows.

For $n = 5$, label $\ell(z_1) = 1, \ell(z_2) = t+2, \ell(z_3) = 3t+3, \ell(z_4) = 4t+4$, and $\ell(z_5) = 5t+5$.

For $n = 7$, label $\ell(z_1) = 1, \ell(z_2) = t+2, \ell(z_3) = 5t+6, \ell(z_4) = 2t+3, \ell(z_5) = 4t+5, \ell(z_6) = 6t+7$, and $\ell(z_7) = 3t+4$.

For $n = 9$, label $\ell(z_1) = 1, \ell(z_2) = 2t+3, \ell(z_3) = 5t+7, \ell(z_4) = 2t+4, \ell(z_5) = 4t+6, \ell(z_6) = t+2, \ell(z_7) = 3t+5, \ell(z_8) = 6t+8$, and $\ell(z_9) = 6t+9$.

Next, label the vertices of tC_6 as follows:

$$\ell(Y_j) = \mathbf{a}^*_{\pi(j)}, \quad 1 \leq j \leq 2t,$$

where π is any permutation of $\{1, 2, \ldots, 2t\}$ and $\mathbf{a}^*_{\pi(j)}$ is the $\pi(j)$-th column of $\mathcal{A}^*$. Since π is a permutation of $\{1, 2, \ldots, 2t\}$, every column of $\mathcal{A}^*$ is assigned to exactly one of the sets $Y_1, Y_2, \ldots, Y_{2t}$. First, we show that ℓ is a bijection. By the definition of matrix $\mathcal{A}$ and matrix $\mathcal{A}^*$, we obtain

$$\begin{aligned} \{a^*_{1,j} : 1 \leq j \leq 2t\} &= \{2, 3, \ldots, 2t+2\} \setminus \{t+2\}, \\ \{a^*_{2,j} : 1 \leq j \leq 2t\} &= \left\{2t + \frac{n+1}{2}, 2t + \frac{n+3}{2}, \ldots, 4t + \frac{n+1}{2}\right\} \setminus \left\{3t + \frac{n+1}{2}\right\}, \text{ and} \\ \{a^*_{3,j} : 1 \leq j \leq 2t\} &= \left\{4t + \frac{n+5}{2}, 4t + \frac{n+7}{2}, \ldots, 6t + \frac{n+5}{2}\right\} \setminus \left\{5t + \frac{n+5}{2}\right\}. \end{aligned}$$

These three sets are pairwise disjoint and each has cardinality $2t$. Thus,

$$\{\ell(y_{i,j}) : 1 \leq i \leq 6, 1 \leq j \leq t\} = \{1, 2, \ldots, 6t+n\} \setminus \{\ell(z_i) : 1 \leq i \leq n\},$$

or equivalently

$$\{\ell(y_{i,j}) : 1 \leq i \leq 6, 1 \leq j \leq t\} \cup \{\ell(z_i) : 1 \leq i \leq n\} = \{1, 2, \ldots, 6t+n\},$$

and hence ℓ is a bijection.

Next, we show that the set of vertex weights are consecutive integers with the smallest weight is $3t + \frac{1}{2}(n+3)$. The weights of the vertices of C_5, C_7, and C_9 are given by

$$\{3t+4, 4t+5, 5t+6, 6t+7, 8t+8\},$$

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$$\{3t+5, 4t+6, 5t+7, 6t+8, 7t+9, 8t+10, 9t+11\},$$

and

$$\{3t+6, 4t+7, 5t+8, 6t+9, 7t+10, 7t+11, 8t+12, 9t+13, 9t+14\},$$

respectively. We now determine the weights of all vertices of tC_6. For each $j = 1, 2, \ldots, t$,

$$\{w(y_{2,j}), w(y_{4,j}), w(y_{6,j})\} = \{\ell(y_{1,j}) + \ell(y_{3,j}), \ell(y_{3,j}) + \ell(y_{5,j}), \ell(y_{5,j}) + \ell(y_{1,j})\},$$
$$\{w(y_{1,j}), w(y_{3,j}), w(y_{5,j})\} = \{\ell(y_{2,j}) + \ell(y_{6,j}), \ell(y_{2,j}) + \ell(y_{4,j}), \ell(y_{4,j}) + \ell(y_{6,j})\}.$$

Consequently,

$$\{w(y_{i,j}) : 1 \le i \le 6,\ 1 \le j \le t\} = \bigcup_{j=1}^{2t} \{a^*_{1,j} + a^*_{2,j}, a^*_{2,j} + a^*_{3,j}, a^*_{3,j} + a^*_{1,j}\}.$$

Since

$$\{a^*_{1,j} + a^*_{2,j} : 1 \le j \le 2t\} = \left\{3t + \frac{n+5}{2}, 3t + \frac{n+7}{2}, \ldots, 5t + \frac{n+5}{2}\right\} \setminus \left\{4t + \frac{n+5}{2}\right\},$$
$$\{a^*_{2,j} + a^*_{3,j} : 1 \le j \le 2t\} = \{7t+n+3, 7t+n+4, \ldots, 9t+n+3\} \setminus \{8t+n+3\}, \text{ and}$$
$$\{a^*_{3,j} + a^*_{1,j} : 1 \le j \le 2t\} = \left\{5t + \frac{n+9}{2}, 5t + \frac{n+11}{2}, \ldots, 7t + \frac{n+9}{2}\right\} \setminus \left\{6t + \frac{n+9}{2}\right\},$$

then

$$\{w(y_{i,j}) : 1 \le i \le 6,\ 1 \le j \le t\} = \left\{\frac{6t+n+3}{2}, \frac{6t+n+5}{2}, \ldots, \frac{18t+3n+1}{2}\right\} \setminus \{w(z_i) : 1 \le i \le n\}.$$

Case 2. $n = 11$.
Consider a matrix $\mathcal{B}^* = [b^*_{i,j}]$ of order $3 \times 2t$ defined by

$$b^*_{1,j} = t + 2 + b_{1,j},$$
$$b^*_{2,j} = 3t + 6 + b_{2,j}, \text{ and}$$
$$b^*_{3,j} = 5t + 10 + b_{3,j},$$

where $j = 1, 2, \ldots, 2t$.

Define a labeling $\ell : V(G) \to \{1, 2, \ldots, 6t+11\}$ as follows.

$$\ell(z_1) = t+1, \ell(z_2) = t+3, \ell(z_3) = 4t+8, \ell(z_4) = 4t+9, \ell(z_5) = t+2, \ell(z_6) = 4t+6,$$

$$\ell(z_7) = 3t+6, \ell(z_8) = 2t+6, \ell(z_9) = 5t+10, \ell(z_{10}) = 6t+11, \text{ and } \ell(z_{11}) = 2t+4.$$

Next, label the vertices of tC_6 as follows:

$$\ell(Y_j) = \mathbf{b}^*_{\pi(j)}, \quad 1 \le j \le 2t,$$

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where π is any permutation of $\{1,2,\ldots,2t\}$ and $\mathbf{b}^*_{\pi(j)}$ is the $\pi(j)$-th column of $\mathcal{B}^*$.

We first verify that ℓ is a bijection. Observe that

$$\begin{aligned}
\{b^*_{1,j} : 1 \le j \le 2t\} &= \{1,2,\ldots,2t+3\} \setminus \{t+1, t+2, t+3\},\\
\{b^*_{2,j} : 1 \le j \le 2t\} &= \{2t+5\} \cup \{2t+7,\ldots,4t+7\} \setminus \{3t+6, 4t+6\},\\
\{b^*_{3,j} : 1 \le j \le 2t\} &= \{4t+10, 4t+11,\ldots,6t+11\} \setminus \{5t+10, 6t+11\}.
\end{aligned}$$

These three sets are pairwise disjoint and each has cardinality $2t$. Thus, $\mathcal{B}^*$ contains $6t$ pairwise distinct entries and

$$\{\ell(y_{i,j}) : 1 \le i \le 6, 1 \le j \le t\} = \{1,2,\ldots,6t+11\} \setminus \{\ell(z_i) : 1 \le i \le 11\}$$

or equivalently

$$\{\ell(y_{i,j}) : 1 \le i \le 6, 1 \le j \le t\} \cup \{\ell(z_i) : 1 \le i \le n\} = \{1,2,\ldots,6t+n\},$$

and hence ℓ is a bijection.

The weights of the vertices of C_{11} are

$$\{3t+7, 4t+8, 5t+9, 5t+10, 5t+12, 6t+12, 7t+12, 7t+14, 8t+15, 8t+16, 8t+17\}.$$

As in Case 1, the weights of all vertices of tC_6 are obtained from the three pairwise sums in each column of $\mathcal{B}^*$,

$$\{w(y_{i,j}) : 1 \le i \le 6,\ 1 \le j \le t\} = \bigcup_{j=1}^{2t} \{b^*_{1,j} + b^*_{2,j}, b^*_{2,j} + b^*_{3,j}, b^*_{3,j} + b^*_{1,j}\}.$$

Since,

$$\begin{aligned}
\{b^*_{1,j} + b^*_{2,j} : 1 \le j \le 2t\} &= \{3t+8, 3t+9,\ldots,5t+8\} \setminus \{4t+8\},\\
\{b^*_{2,j} + b^*_{3,j} : 1 \le j \le 2t\} &= \{7t+15, 7t+16,\ldots,9t+17\} \setminus \{8t+15, 8t+16, 8t+17\},\\
\{b^*_{3,j} + b^*_{1,j} : 1 \le j \le 2t\} &= \{5t+11\} \cup \{5t+13, 5t+14,\ldots,6t+11\}\\
&\quad \cup \{6t+13, 6t+14,\ldots,7t+11\} \cup \{7t+13\},
\end{aligned}$$

then

$$\{w(y_{i,j}) : 1 \le i \le 6,\ 1 \le j \le t\} = \{3t+7, 3t+8,\ldots,9t+17\} \setminus \{w(z_i) : 1 \le i \le 11\}.$$

Thus, for all cases, all vertex weights are distinct and

$$\{w(x) : x \in V(G)\} = \left\{\frac{6t+n+3}{2}, \frac{6t+n+5}{2}, \ldots, \frac{18t+3n+1}{2}\right\}.$$

Hence, ℓ is a $(\frac{6t+n+3}{2}, 1)$-DA labeling of G. □

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As examples, Figure 1 shows the labeling of $C_5 \cup 3C_6$ and $C_{11} \cup 3C_6$. In this case,

$$\mathcal{A}^* = \begin{bmatrix} 2 & 3 & 4 & 6 & 7 & 8 \\ 13 & 15 & 10 & 14 & 9 & 11 \\ 22 & 19 & 23 & 17 & 21 & 18 \end{bmatrix}, \quad \mathcal{B}^* = \begin{bmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 17 & 19 & 14 & 16 & 11 & 13 \\ 27 & 24 & 28 & 22 & 26 & 23 \end{bmatrix}.$$

For $C_5 \cup 3C_6$, if we set $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix}$, then

$$\ell(Y_1) = \mathbf{a}_1^*, \ell(Y_2) = \mathbf{a}_4^*, \ell(Y_3) = \mathbf{a}_2^*, \ell(Y_4) = \mathbf{a}_5^*, \ell(Y_5) = \mathbf{a}_3^*, \text{ and } \ell(Y_6) = \mathbf{a}_6^*.$$

as shown in Figure 1(a). If we set $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 2 & 4 & 6 \end{pmatrix}$, then

$$\ell(Y_1) = \mathbf{a}_3^*, \ell(Y_2) = \mathbf{a}_5^*, \ell(Y_3) = \mathbf{a}_1^*, \ell(Y_4) = \mathbf{a}_2^*, \ell(Y_5) = \mathbf{a}_4^*, \text{ and } \ell(Y_6) = \mathbf{a}_6^*,$$

as shown in Figure 1(b). For $C_{11} \cup 3C_6$, if we set $\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$, then

$$\ell(Y_1) = \mathbf{b}_1^*, \ell(Y_2) = \mathbf{b}_2^*, \ell(Y_3) = \mathbf{b}_3^*, \ell(Y_4) = \mathbf{b}_4^*, \ell(Y_5) = \mathbf{b}_5^*, \text{ and } \ell(Y_6) = \mathbf{b}_6^*,$$

as shown in Figure 1(c).

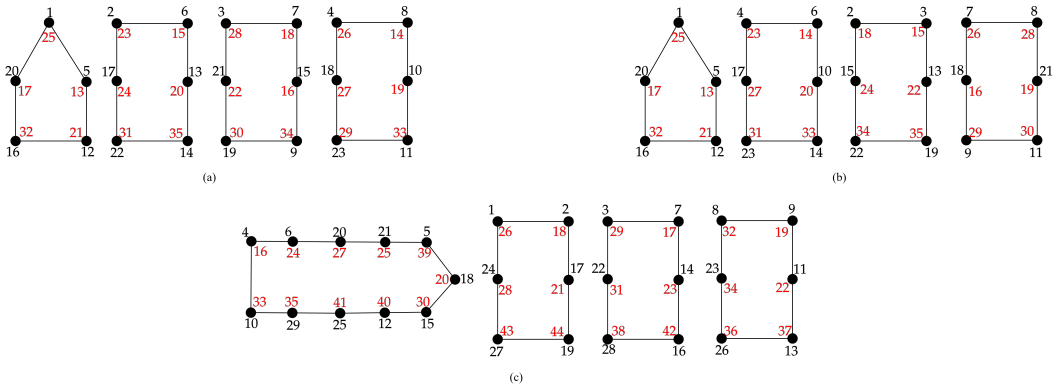


Figure 1: (a). A $(13,1)$-DA labeling of $C_5 \cup 3C_6$ based on π_1. (b). A $(13,1)$-DA labeling of $C_5 \cup 3C_6$ based on π_2. (c). A $(16,1)$-DA labeling of $C_{11} \cup 3C_6$ based on π_3.

In the following theorem, we investigate a necessary and sufficient condition for the graph $C_{2t+1} \cup tC_6$ to be (a,d)-DA. The case $C_3 \cup C_6$ was proved by Ngurah and Simanjuntak [20], whereas the cases $C_{2t+1} \cup tC_6$ for $t = 2,3,4,5$ were proved in the previous theorem. Thus, in the next theorem, we consider the case $t \geq 6$.

Theorem 2.2. *For every integer $t \geq 6$, the graph $G \cong C_{2t+1} \cup tC_6$ is (a,d)-DA iff $d = 1$.*

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Proof. Lemma 1.1 implies that, if G is (a, d)-DA, then $d = 1$. Conversely, let

$$V(G) = \{z_i : 1 \leq i \leq 2t + 1\} \cup \{y_{i,j} : 1 \leq i \leq 6, 1 \leq j \leq t\},$$

and

$$E(G) = \{z_i z_{i+1} : 1 \leq i \leq 2t\} \cup \{z_{2t+1} z_1\} \cup \{y_{i,j} y_{i+1,j}, y_{6,j} y_{1,j} : 1 \leq i \leq 5, 1 \leq j \leq t\}.$$

Define a labeling $\ell : V(G) \to \{1, 2, \ldots, 8t + 1\}$ as follows. For $j = 1, 2, \ldots, t$, label the vertices of tC_6 with

$$\ell(y_{i,j}) = \begin{cases} 4t + 2j, & \text{if } i = 1, \\ 8t - 2j + 2, & \text{if } i = 2, \\ 8t - 4j + 3, & \text{if } i = 3, \\ 4t - 2j + 2, & \text{if } i = 4, \\ 2j, & \text{if } i = 5, \\ 4j - 1, & \text{if } i = 6. \end{cases}$$

For $i = 1, 2, \ldots, 2t + 1$, label the vertices of C_{2t+1} with

$$\ell(z_i) = 4f(z_i) - 3,$$

where f is a $(t + 2, 1)$-DA labeling of C_{2t+1} [1, Theorem 2.7]. To verify that ℓ is a bijection, observe that

$$\begin{aligned} \{\ell(y_{1,j}) : 1 \leq j \leq t\} &= \{4t + 2, 4t + 4, \ldots, 6t\}, \\ \{\ell(y_{2,j}) : 1 \leq j \leq t\} &= \{6t + 2, 6t + 4, \ldots, 8t\}, \\ \{\ell(y_{3,j}) : 1 \leq j \leq t\} &= \{4t + 3, 4t + 7, \ldots, 8t - 1\}, \\ \{\ell(y_{4,j}) : 1 \leq j \leq t\} &= \{2t + 2, 2t + 4, \ldots, 4t\}, \\ \{\ell(y_{5,j}) : 1 \leq j \leq t\} &= \{2, 4, \ldots, 2t\}, \\ \{\ell(y_{6,j}) : 1 \leq j \leq t\} &= \{3, 7, \ldots, 4t - 1\}. \end{aligned}$$

Thus,

$$\{\ell(y_{i,j}) : 1 \leq i \leq 6,\ 1 \leq j \leq t\} = \{2, 4, \ldots, 8t\} \cup \{3, 7, \ldots, 8t - 1\}.$$

Since $f : V(C_{2t+1}) \to \{1, 2, \ldots, 2t + 1\}$ is a bijection, then

$$\{\ell(z_i) : 1 \leq i \leq 2t + 1\} = \{4f(z_i) - 3 : 1 \leq i \leq 2t + 1\} = \{1, 5, 9, \ldots, 8t + 1\}.$$

Therefore,

$$\{\ell(x) : x \in V(G)\} = \{2, 4, \ldots, 8t\} \cup \{3, 7, \ldots, 8t - 1\} \cup \{1, 5, \ldots, 8t + 1\} = \{1, 2, \ldots, 8t + 1\},$$

and hence ℓ is a bijection.

We now determine the weights of all vertices of tC_6. For $j = 1, 2, \ldots, t$,

$$w(y_{1,j}) = \ell(y_{2,j}) + \ell(y_{6,j}) = 8t + 2j + 1.$$

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Therefore,

$$\{w(y_{1,j}) : 1 \leq j \leq t\} = \{8t + 3, 8t + 5, \ldots, 10t + 1\}.$$

Analogously, the weights of the remaining vertices of tC_6 are given by

$$\begin{aligned}
\{w(y_{2,j}) : 1 \leq j \leq t\} &= \{10t + 3, 10t + 5, \ldots, 12t + 1\},\\
\{w(y_{3,j}) : 1 \leq j \leq t\} &= \{8t + 4, 8t + 8, \ldots, 12t\},\\
\{w(y_{4,j}) : 1 \leq j \leq t\} &= \{6t + 3, 6t + 5, \ldots, 8t + 1\},\\
\{w(y_{5,j}) : 1 \leq j \leq t\} &= \{4t + 3, 4t + 5, \ldots, 6t + 1\},\\
\{w(y_{6,j}) : 1 \leq j \leq t\} &= \{4t + 4, 4t + 8, \ldots, 8t\}.
\end{aligned}$$

For each $i = 1, 2, \ldots, 2t + 1$, the weight of the vertex z_i is

$$w(z_i) = \ell(z_{i-1}) + \ell(z_{i+1}) = 4w_f(z_i) - 6,$$

where $w_f(z_i)$ denotes the weight of z_i under f. Since f is a $(t + 2, 1)$-DA labeling, then $\{w_f(z_i) : 1 \leq i \leq 2t + 1\} = \{t + 2, t + 3, \ldots, 3t + 2\}$. So,

$$\{w(z_i) : 1 \leq i \leq 2t + 1\} = \{4t + 2,\ 4t + 6,\ 4t + 10,\ \ldots,\ 12t + 2\}.$$

Hence, ℓ is a $(4t + 2, 1)$-DA labeling of G. □

Figure 2 (a) shows the $(8, 1)$-DA labeling of C_{13}, whereas Figure 2 (b) shows the $(26, 1)$-DA labeling of $C_{13} \cup 6C_6$ based on Figure 2 (a).

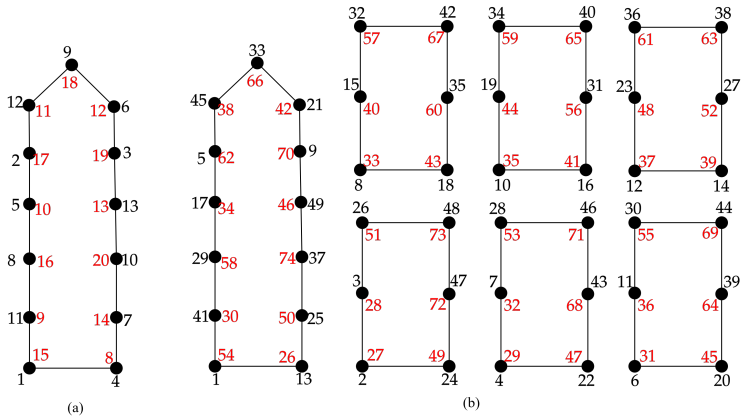


Figure 2: (a). A $(8, 1)$-DA labeling of C_{13}. (b). A $(26, 1)$-DA labeling of $C_{13} \cup 6C_6$ based on (a).

In Theorem 1.2, Ngurah and Simanjuntak [20] proved that $C_n \cup C_6$ is (a, d)-DA iff $n \geq 7$ is odd and $d = 1$. To prove Theorem 1.2, they considered several cases. We generalize this result to m disjoint copies of $C_n \cup C_6$. We have attempted to obtain a simpler labeling for our result. However, no such labeling has yet been found. Therefore, to prove Theorem 2.3 we consider the same cases as in the proof of Theorem 1.2.

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For an odd integer m, let us consider the Kötzig matrix $\mathcal{C} = [c_{ij}]$ of order $3 \times m$ given by

$$\mathcal{C} = \begin{bmatrix} 1 & 2 & \cdots & \frac{m-1}{2} & \frac{m+1}{2} & \cdots & m-1 & m \\ \frac{m+3}{2} & \frac{m+5}{2} & \cdots & m & 1 & \cdots & \frac{m-1}{2} & \frac{m+1}{2} \\ m-1 & m-3 & \cdots & 2 & m & \cdots & 3 & 1 \end{bmatrix}.$$

The matrix $\mathcal{C}$ satisfy

$$\{c_{1,j}+c_{2,j}\} = \{c_{1,j}+c_{3,j}\} = \{c_{2,j}+c_{3,j}\} = \left\{\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{3m+1}{2}\right\}.$$

Theorem 2.3. *For every integer $m \geq 3$ and $n \geq 7$, the graph $G \cong m(C_n \cup C_6)$ is (a,d)-DA iff $d = 1$ and both m and n are odd.*

Proof. Lemma 1.1 implies that, if G is (a,d)-DA, then $d = 1$. Since G is (a,d)-DA, $6m + mn$ must be odd, which implies that both m and n are odd. Conversely, let m and n be odd,

$$V(G) = \{z_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_{i,j} : 1 \leq i \leq 6, 1 \leq j \leq m\},$$

and

$$\begin{aligned} E(G) =& \{z_{i,j}z_{i+1,j}, z_{n,j}z_{1,j} : 1 \leq i \leq n-1, 1 \leq j \leq m\} \\ & \cup \{y_{i,j}y_{i+1,j}, y_{6,j}y_{1,j} : 1 \leq i \leq 5, 1 \leq j \leq m\}. \end{aligned}$$

Define a labeling $\ell : V(G) \to \{1, 2, \ldots, m(n+6)\}$ as follows:

Case 1. $n \equiv 1 \pmod 4$.

For each $j = 1, 2, \ldots, m$, define the labels of the vertices in mC_6 by

$$\ell(y_{i,j}) = \begin{cases} \frac{1}{2}m(n-3) + c_{1,j}, & \text{if } i = 1, \\ \frac{1}{4}m(3n+5+2i) + c_{\frac{i}{2},j}, & \text{if } i = 2, 4, 6, \\ \frac{1}{2}m(n+i-2) + c_{\frac{i+1}{2},j}, & \text{if } i = 3, 5. \end{cases}$$

The labels of the vertices in mC_n are defined according to the following subcases.

Subcase 1.1. $n \equiv 1 \pmod 8$.

For $n = 9$ and for each $j = 1, 2, \ldots, m$, define

$$\ell(z_{i,j}) = \begin{cases} m(14i-14) + c_{1,j}, & \text{if } i = 1, 2, \\ m(42-10i) + c_{2,j}, & \text{if } i = 3, 4, \\ m(7i-34) + c_{1,j}, & \text{if } i = 5, 6, \\ m(55-6i) + c_{2,j}, & \text{if } i = 7, 8, \\ 4m + c_{3,j}, & \text{if } i = 9. \end{cases}$$

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For $n \neq 9$ and for each $j = 1, 2, \ldots, m$, define

$$\ell(z_{i,j}) = \begin{cases} \frac{1}{4}m(i-1) + c_{1,j}, & \text{if } i \equiv 1 \pmod 4,\ i \neq n-4, n, \\ \frac{1}{4}m(3n+5) + c_{1,j}, & \text{if } i = n-4, \\ \frac{1}{4}m(2n+7+i) + c_{2,j}, & \text{if } i \equiv 3 \pmod 4,\ i \neq n-2, \\ \frac{1}{4}m(n-1) + c_{2,j}, & \text{if } i = n-2, \\ \frac{1}{4}m(n-7+i) + c_{1,j}, & \text{if } i \equiv 2 \pmod 8,\ i \neq n-7, \\ \frac{1}{2}m(n-1) + c_{1,j}, & \text{if } i = n-7, \\ \frac{1}{4}m(3n+17+i) + c_{2,j}, & \text{if } i \equiv 4 \pmod 8,\ i \neq n-5, \\ m(n+4) + c_{2,j}, & \text{if } i = n-5, \\ \frac{1}{4}m(n+1+i) + c_{1,j}, & \text{if } i \equiv 6 \pmod 8,\ i \neq n-3, \\ \frac{1}{2}m(n-7) + c_{1,j}, & \text{if } i = n-3, \\ \frac{1}{4}m(3n+25+i) + c_{2,j}, & \text{if } i \equiv 0 \pmod 8,\ i \neq n-1, n-9, \\ \frac{1}{4}m(5n+11-i) + c_{2,j}, & \text{if } i = n-9, n-1, \\ \frac{1}{4}m(3n+25) + c_{3,j}, & \text{if } i = n. \end{cases}$$

Subcase 1.2. $n \equiv 5 \pmod 8$.
For each $j = 1, 2, \ldots, m$, define

$$\ell(z_{i,j}) = \begin{cases} \frac{1}{4}m(i-1) + c_{1,j}, & \text{if } i \equiv 1 \pmod 4,\ i \neq n-4, n, \\ \frac{1}{4}m(3n+5) + c_{1,j}, & \text{if } i = n-4, \\ \frac{1}{4}m(2n+7+i) + c_{2,j}, & \text{if } i \equiv 3 \pmod 4,\ i \neq n-2, \\ \frac{1}{4}m(n-1) + c_{2,j}, & \text{if } i = n-2, \\ \frac{1}{4}m(n-7+i) + c_{1,j}, & \text{if } i \equiv 2 \pmod 8, \\ \frac{1}{4}m(3n+25+i) + c_{2,j}, & \text{if } i \equiv 4 \pmod 8,\ i \neq n-1, \\ m(n+3) + c_{2,j}, & \text{if } i = n-1, \\ \frac{1}{4}m(n+1+i) + c_{1,j}, & \text{if } i \equiv 6 \pmod 8,\ i \neq n-7, \\ \frac{1}{2}m(n-1) + c_{1,j}, & \text{if } i = n-7, \\ \frac{1}{4}m(3n+17+i) + c_{2,j}, & \text{if } i \equiv 0 \pmod 8,\ i \neq n-5, \\ m(n+5) + c_{2,j}, & \text{if } i = n-5, \\ \frac{1}{4}m(3n+21) + c_{3,j}, & \text{if } i = n. \end{cases}$$

We first verify that ℓ is a bijection. For $j = 1, 2, \ldots, m$, by the definition of $\mathcal{C}$ the six sets of

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labels of mC_6 are

$$\{\ell(y_{1,j})\} = \left\{\frac{m(n-3)}{2}+1, \frac{m(n-3)}{2}+2, \ldots, \frac{m(n-1)}{2}\right\},$$
$$\{\ell(y_{2,j})\} = \left\{\frac{m(3n+9)}{4}+1, \frac{m(3n+9)}{4}+2, \ldots, \frac{m(3n+13)}{4}\right\},$$
$$\{\ell(y_{3,j})\} = \left\{\frac{m(n+1)}{2}+1, \frac{m(n+1)}{2}+2, \ldots, \frac{m(n+3)}{2}\right\},$$
$$\{\ell(y_{4,j})\} = \left\{\frac{m(3n+13)}{4}+1, \frac{m(3n+13)}{4}+2, \ldots, \frac{m(3n+17)}{4}\right\},$$
$$\{\ell(y_{5,j})\} = \left\{\frac{m(n+3)}{2}+1, \frac{m(n+3)}{2}+2, \ldots, \frac{m(n+5)}{2}\right\},$$
$$\{\ell(y_{6,j})\} = \left\{\frac{m(3n+17)}{4}+1, \frac{m(3n+17)}{4}+2, \ldots, \frac{m(3n+21)}{4}\right\}.$$

For $n = 9$, the nine sets of labels of mC_9 are

$$\{\ell(z_{1,j})\} = \{1, 2, \ldots, m\},$$
$$\{\ell(z_{2,j})\} = \{14m+1, 14m+2, \ldots, 15m\},$$
$$\{\ell(z_{3,j})\} = \{12m+1, 12m+2, \ldots, 13m\},$$
$$\{\ell(z_{4,j})\} = \{2m+1, 2m+2, \ldots, 3m\},$$
$$\{\ell(z_{5,j})\} = \{m+1, m+2, \ldots, 2m\},$$
$$\{\ell(z_{6,j})\} = \{8m+1, 8m+2, \ldots, 9m\},$$
$$\{\ell(z_{7,j})\} = \{13m+1, 13m+2, \ldots, 14m\},$$
$$\{\ell(z_{8,j})\} = \{7m+1, 7m+2, \ldots, 8m\},$$
$$\{\ell(z_{9,j})\} = \{4m+1, 4m+2, \ldots, 5m\}.$$

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For $n \neq 9$, $n \equiv 1 \pmod 8$, and for $j = 1, 2, \ldots, m$, the sets of labels of mC_n are

$$\{\ell(z_{i,j})\} = \begin{cases} \left\{\frac{m(i-1)}{4}+1, \frac{m(i-1)}{4}+2, \ldots, \frac{m(i+3)}{4}\right\}, & i \equiv 1 \pmod 4, i \neq n-4, n, \\ \left\{\frac{m(3n+5)}{4}+1, \frac{m(3n+5)}{4}+2, \ldots, \frac{m(3n+9)}{4}\right\}, & i = n-4, \\ \left\{\frac{m(2n+7+i)}{4}+1, \frac{m(2n+7+i)}{4}+2, \ldots, \frac{m(2n+11+i)}{4}\right\}, & i \equiv 3 \pmod 4, i \neq n-2, \\ \left\{\frac{m(n-1)}{4}+1, \frac{m(n-1)}{4}+2, \ldots, \frac{m(n+3)}{4}\right\}, & i = n-2, \\ \left\{\frac{m(n-7+i)}{4}+1, \frac{m(n-7+i)}{4}+2, \ldots, \frac{m(n-3+i)}{4}\right\}, & i \equiv 2 \pmod 8, i \neq n-7, \\ \left\{\frac{m(n-1)}{2}+1, \frac{m(n-1)}{2}+2, \ldots, \frac{m(n+1)}{2}\right\}, & i = n-7, \\ \left\{\frac{m(3n+17+i)}{4}+1, \frac{m(3n+17+i)}{4}+2, \ldots, \frac{m(3n+21+i)}{4}\right\}, & i \equiv 4 \pmod 8, i \neq n-5, \\ \{m(n+4)+1, m(n+4)+2, \ldots, m(n+5)\}, & i = n-5, \\ \left\{\frac{m(n+1+i)}{4}+1, \frac{m(n+1+i)}{4}+2, \ldots, \frac{m(n+5+i)}{4}\right\}, & i \equiv 6 \pmod 8, i \neq n-3, \\ \left\{\frac{m(n-7)}{2}+1, \frac{m(n-7)}{2}+2, \ldots, \frac{m(n-5)}{2}\right\}, & i = n-3, \\ \left\{\frac{m(3n+25+i)}{4}+1, \frac{m(3n+25+i)}{4}+2, \ldots, \frac{m(3n+29+i)}{4}\right\}, & i \equiv 0 \pmod 8, i \neq n-1, n-9, \\ \left\{\frac{m(5n+11-i)}{4}+1, \frac{m(5n+11-i)}{4}+2, \ldots, \frac{m(5n+15-i)}{4}\right\}, & i = n-9, n-1, \\ \left\{\frac{m(3n+25)}{4}+1, \frac{m(3n+25)}{4}+2, \ldots, \frac{m(3n+29)}{4}\right\}, & i = n. \end{cases}$$

For $n \equiv 5 \pmod 8$ and $j = 1, 2, \ldots, m$, the sets of labels of mC_n are

$$\{\ell(z_{i,j})\} = \begin{cases} \left\{\frac{m(i-1)}{4}+1, \frac{m(i-1)}{4}+2, \ldots, \frac{m(i+3)}{4}\right\}, & i \equiv 1 \pmod 4, i \neq n-4, n, \\ \left\{\frac{m(3n+5)}{4}+1, \frac{m(3n+5)}{4}+2, \ldots, \frac{m(3n+9)}{4}\right\}, & i = n-4, \\ \left\{\frac{m(2n+7+i)}{4}+1, \frac{m(2n+7+i)}{4}+2, \ldots, \frac{m(2n+11+i)}{4}\right\}, & i \equiv 3 \pmod 4, i \neq n-2, \\ \left\{\frac{m(n-1)}{4}+1, \frac{m(n-1)}{4}+2, \ldots, \frac{m(n+3)}{4}\right\}, & i = n-2, \\ \left\{\frac{m(n-7+i)}{4}+1, \frac{m(n-7+i)}{4}+2, \ldots, \frac{m(n-3+i)}{4}\right\}, & i \equiv 2 \pmod 8, \\ \left\{\frac{m(3n+25+i)}{4}+1, \frac{m(3n+25+i)}{4}+2, \ldots, \frac{m(3n+29+i)}{4}\right\}, & i \equiv 4 \pmod 8, i \neq n-1, \\ \{m(n+3)+1, m(n+3)+2, \ldots, m(n+4)\}, & i = n-1, \\ \left\{\frac{m(n+1+i)}{4}+1, \frac{m(n+1+i)}{4}+2, \ldots, \frac{m(n+5+i)}{4}\right\}, & i \equiv 6 \pmod 8, i \neq n-7, \\ \left\{\frac{m(n-1)}{2}+1, \frac{m(n-1)}{2}+2, \ldots, \frac{m(n+1)}{2}\right\}, & i = n-7, \\ \left\{\frac{m(3n+17+i)}{4}+1, \frac{m(3n+17+i)}{4}+2, \ldots, \frac{m(3n+21+i)}{4}\right\}, & i \equiv 0 \pmod 8, i \neq n-5, \\ \{m(n+5)+1, m(n+5)+2, \ldots, m(n+6)\}, & i = n-5, \\ \left\{\frac{m(3n+21)}{4}+1, \frac{m(3n+21)}{4}+2, \ldots, \frac{m(3n+25)}{4}\right\}, & i = n. \end{cases}$$

The n sets of labels of mC_n together with the six sets of labels of mC_6 form exactly $n+6$ pairwise disjoint sets and each consisting of m consecutive labels. Hence, there are $m(n+6)$ distinct labels

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in total. Since the smallest label is 1 and the largest is $m(n+6)$, it follows that

$$\{\ell(x) : x \in V(G)\} = \{1, 2, \ldots, m(n+6)\}.$$

Therefore, ℓ is a bijection.

We now determine the weights of the vertices of G. For $j = 1, 2, \ldots, m$, the weights of the vertices of mC_6 are

$$w(y_{i,j}) = \begin{cases} \dfrac{m\big((16-i)n + 78 - 13i\big)}{10} + c_{1,j} + c_{3,j}, & i = 1, 6, \\ \dfrac{m\big(i(n+13) - 28\big)}{2} + c_{1,j} + c_{2,j}, & i = 2, 3, \\ \dfrac{m\big((i-2)n + 11i - 40\big)}{2} + c_{2,j} + c_{3,j}, & i = 4, 5. \end{cases}$$

By the pair-sum property of $\mathcal{C}$, for $j = 1, 2, \ldots, m$, the six sets of weights of mC_6 are

$$\{w(y_{1,j})\} = \left\{ \frac{m(3n+14)+3}{2}, \frac{m(3n+14)+5}{2}, \ldots, \frac{m(3n+16)+1}{2} \right\},$$
$$\{w(y_{2,j})\} = \left\{ \frac{m(2n-1)+3}{2}, \frac{m(2n-1)+5}{2}, \ldots, \frac{m(2n+1)+1}{2} \right\},$$
$$\{w(y_{3,j})\} = \left\{ \frac{m(3n+12)+3}{2}, \frac{m(3n+12)+5}{2}, \ldots, \frac{m(3n+14)+1}{2} \right\},$$
$$\{w(y_{4,j})\} = \left\{ \frac{m(2n+5)+3}{2}, \frac{m(2n+5)+5}{2}, \ldots, \frac{m(2n+7)+1}{2} \right\},$$
$$\{w(y_{5,j})\} = \left\{ \frac{m(3n+16)+3}{2}, \frac{m(3n+16)+5}{2}, \ldots, \frac{m(3n+18)+1}{2} \right\},$$
$$\{w(y_{6,j})\} = \left\{ \frac{m(2n+1)+3}{2}, \frac{m(2n+1)+5}{2}, \ldots, \frac{m(2n+3)+1}{2} \right\}.$$

For $n = 9$, the weights of the vertices of mC_n are

$$w(z_{i,j}) = \begin{cases} 18m + c_{1,j} + c_{3,j}, & i = 1, \\ \dfrac{m(i+22)}{2} + c_{1,j} + c_{2,j}, & i = 2, 4, 6, \\ \dfrac{m(67-i)}{4} + c_{1,j} + c_{2,j}, & i = 3, 7, \\ \dfrac{m(55-3i)}{4} + c_{1,j} + c_{2,j}, & i = 5, 9, \\ 17m + c_{2,j} + c_{3,j}, & i = 8, \end{cases}$$

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and the sets of weights of mC_n are

$$\{w(z_{1,j})\} = \left\{\frac{37m+3}{2}, \frac{37m+5}{2}, \ldots, \frac{39m+1}{2}\right\},$$
$$\{w(z_{2,j})\} = \left\{\frac{25m+3}{2}, \frac{25m+5}{2}, \ldots, \frac{27m+1}{2}\right\},$$
$$\{w(z_{3,j})\} = \left\{\frac{33m+3}{2}, \frac{33m+5}{2}, \ldots, \frac{35m+1}{2}\right\},$$
$$\{w(z_{4,j})\} = \left\{\frac{27m+3}{2}, \frac{27m+5}{2}, \ldots, \frac{29m+1}{2}\right\},$$
$$\{w(z_{5,j})\} = \left\{\frac{21m+3}{2}, \frac{21m+5}{2}, \ldots, \frac{23m+1}{2}\right\},$$
$$\{w(z_{6,j})\} = \left\{\frac{29m+3}{2}, \frac{29m+5}{2}, \ldots, \frac{31m+1}{2}\right\},$$
$$\{w(z_{7,j})\} = \left\{\frac{31m+3}{2}, \frac{31m+5}{2}, \ldots, \frac{33m+1}{2}\right\},$$
$$\{w(z_{8,j})\} = \left\{\frac{35m+3}{2}, \frac{35m+5}{2}, \ldots, \frac{37m+1}{2}\right\},$$
$$\{w(z_{9,j})\} = \left\{\frac{15m+3}{2}, \frac{15m+5}{2}, \ldots, \frac{17m+1}{2}\right\}.$$

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For $n \neq 9$, $n \equiv 1 \pmod 8$, the weights of the vertices of mC_n are

$$w(z_{i,j}) = \begin{cases} m(n+5) + c_{1,j} + c_{3,j}, & i = 1, \\ \dfrac{m(2n+i+9)}{2} + c_{1,j} + c_{2,j}, & i \equiv 1 \pmod 4, i \neq 1, n-8, n-4, n, \\ \dfrac{m(2n+i+5)}{2} + c_{1,j} + c_{2,j}, & i \equiv 3 \pmod 8, i \neq n-6, \\ \dfrac{m(2n+i+13)}{2} + c_{1,j} + c_{2,j}, & i \equiv 7 \pmod 8, i \neq n-10, n-2, \\ \dfrac{m(n+i+3)}{2} + c_{1,j} + c_{2,j}, & i \equiv 0 \pmod 2, i \neq n-5, n-3, n-1, \\ \dfrac{m(3n+5)}{2} + c_{1,j} + c_{2,j}, & i = n-10, \\ \dfrac{m(3n+9)}{2} + c_{1,j} + c_{2,j}, & i = n-8, \\ \dfrac{m(3n+7)}{2} + c_{1,j} + c_{2,j}, & i = n-6, \\ \dfrac{m(3n+3)}{2} + c_{1,j} + c_{2,j}, & i = n-5, \\ \dfrac{m(3n+1)}{2} + c_{1,j} + c_{2,j}, & i = n-4, \\ m(n+1) + c_{1,j} + c_{2,j}, & i = n-3, \\ \dfrac{m(3n-1)}{2} + c_{1,j} + c_{2,j}, & i = n-2, \\ m(n+6) + c_{2,j} + c_{3,j}, & i = n-1, \\ m(n+3) + c_{1,j} + c_{2,j}, & i = n, \end{cases}$$

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and the sets of weights of mC_n are

$$\{w(z_{i,j})\} = \begin{cases} \left\{\frac{m(2n+11)+3}{2}, \frac{m(2n+11)+5}{2}, \ldots, \frac{m(2n+13)+1}{2}\right\}, & i=1, \\ \left\{\frac{m(2n+i+10)+3}{2}, \frac{m(2n+i+10)+5}{2}, \ldots, \frac{m(2n+i+12)+1}{2}\right\}, & i \equiv 1 \pmod 4, \\ & i \neq 1, n-8, n-4, n, \\ \left\{\frac{m(2n+i+6)+3}{2}, \frac{m(2n+i+6)+5}{2}, \ldots, \frac{m(2n+i+8)+1}{2}\right\}, & i \equiv 3 \pmod 8, i \neq n-6, \\ \left\{\frac{m(2n+i+14)+3}{2}, \frac{m(2n+i+14)+5}{2}, \ldots, \frac{m(2n+i+16)+1}{2}\right\}, & i \equiv 7 \pmod 8, \\ & i \neq n-10, n-2, \\ \left\{\frac{m(n+i+4)+3}{2}, \frac{m(n+i+4)+5}{2}, \ldots, \frac{m(n+i+6)+1}{2}\right\}, & i \equiv 0 \pmod 2, \\ & i \neq n-5, n-3, n-1, \\ \left\{\frac{m(3n+6)+3}{2}, \frac{m(3n+6)+5}{2}, \ldots, \frac{m(3n+8)+1}{2}\right\}, & i=n-10, \\ \left\{\frac{m(3n+10)+3}{2}, \frac{m(3n+10)+5}{2}, \ldots, \frac{m(3n+12)+1}{2}\right\}, & i=n-8, \\ \left\{\frac{m(3n+8)+3}{2}, \frac{m(3n+8)+5}{2}, \ldots, \frac{m(3n+10)+1}{2}\right\}, & i=n-6, \\ \left\{\frac{m(3n+4)+3}{2}, \frac{m(3n+4)+5}{2}, \ldots, \frac{m(3n+6)+1}{2}\right\}, & i=n-5, \\ \left\{\frac{m(3n+2)+3}{2}, \frac{m(3n+2)+5}{2}, \ldots, \frac{m(3n+4)+1}{2}\right\}, & i=n-4, \\ \left\{\frac{m(2n+3)+3}{2}, \frac{m(2n+3)+5}{2}, \ldots, \frac{m(2n+5)+1}{2}\right\}, & i=n-3, \\ \left\{\frac{m(3n)+3}{2}, \frac{m(3n)+5}{2}, \ldots, \frac{m(3n+2)+1}{2}\right\}, & i=n-2, \\ \left\{\frac{m(2n+13)+3}{2}, \frac{m(2n+13)+5}{2}, \ldots, \frac{m(2n+15)+1}{2}\right\}, & i=n-1, \\ \left\{\frac{m(2n+7)+3}{2}, \frac{m(2n+7)+5}{2}, \ldots, \frac{m(2n+9)+1}{2}\right\}, & i=n. \end{cases}$$

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For $n \equiv 5 \pmod 8$, the weights of the vertices of mC_n are

$$w(z_{i,j}) = \begin{cases} m(n+4) + c_{1,j} + c_{3,j}, & i = 1, \\ \frac{m(2n+i+5)}{2} + c_{1,j} + c_{2,j}, & i \equiv 1 \pmod 8, i \neq 1, n-4, \\ \frac{m(2n+i+9)}{2} + c_{1,j} + c_{2,j}, & i \equiv 3 \pmod 4, i \neq n-6, n-2, \\ \frac{m(2n+i+13)}{2} + c_{1,j} + c_{2,j}, & i \equiv 5 \pmod 8, i \neq n-8, n, \\ \frac{m(n+i+3)}{2} + c_{1,j} + c_{2,j}, & i \equiv 0 \pmod 2, i \neq n-5, n-3, n-1, \\ \frac{m(3n+7)}{2} + c_{1,j} + c_{2,j}, & i = n-8, \\ \frac{m(3n+9)}{2} + c_{1,j} + c_{2,j}, & i = n-6, \\ \frac{m(3n+3)}{2} + c_{1,j} + c_{2,j}, & i = n-5, \\ \frac{m(3n+5)}{2} + c_{1,j} + c_{2,j}, & i = n-4, \\ m(n+1) + c_{1,j} + c_{2,j}, & i = n-3, \\ \frac{m(3n+1)}{2} + c_{1,j} + c_{2,j}, & i = n-2, \\ m(n+5) + c_{2,j} + c_{3,j}, & i = n-1, \\ m(n+3) + c_{1,j} + c_{2,j}, & i = n, \end{cases}$$

and the sets of weights of mC_n are

$$\{w(z_{i,j})\} = \begin{cases} \left\{ \frac{m(2n+9)+3}{2}, \frac{m(2n+9)+5}{2}, \ldots, \frac{m(2n+11)+1}{2} \right\}, & i = 1, \\ \left\{ \frac{m(2n+i+6)+3}{2}, \ldots, \frac{m(2n+i+8)+1}{2} \right\}, & i \equiv 1 \pmod 8, i \neq 1, n-4, \\ \left\{ \frac{m(2n+i+10)+3}{2}, \ldots, \frac{m(2n+i+12)+1}{2} \right\}, & i \equiv 3 \pmod 4, i \neq n-6, n-2, \\ \left\{ \frac{m(2n+i+14)+3}{2}, \ldots, \frac{m(2n+i+16)+1}{2} \right\}, & i \equiv 5 \pmod 8, i \neq n-8, n, \\ \left\{ \frac{m(n+i+4)+3}{2}, \frac{m(n+i+4)+5}{2}, \ldots, \frac{m(n+i+6)+1}{2} \right\}, & i \equiv 0 \pmod 2, i \neq n-5, n-3, n-1, \\ \left\{ \frac{m(3n+8)+3}{2}, \frac{m(3n+8)+5}{2}, \ldots, \frac{m(3n+10)+1}{2} \right\}, & i = n-8, \\ \left\{ \frac{m(3n+10)+3}{2}, \frac{m(3n+10)+5}{2}, \ldots, \frac{m(3n+12)+1}{2} \right\}, & i = n-6, \\ \left\{ \frac{m(3n+4)+3}{2}, \frac{m(3n+4)+5}{2}, \ldots, \frac{m(3n+6)+1}{2} \right\}, & i = n-5, \\ \left\{ \frac{m(3n+6)+3}{2}, \frac{m(3n+6)+5}{2}, \ldots, \frac{m(3n+8)+1}{2} \right\}, & i = n-4, \\ \left\{ \frac{m(2n+3)+3}{2}, \frac{m(2n+3)+5}{2}, \ldots, \frac{m(2n+5)+1}{2} \right\}, & i = n-3, \\ \left\{ \frac{m(3n+2)+3}{2}, \frac{m(3n+2)+5}{2}, \ldots, \frac{m(3n+4)+1}{2} \right\}, & i = n-2, \\ \left\{ \frac{m(2n+11)+3}{2}, \frac{m(2n+11)+5}{2}, \ldots, \frac{m(2n+13)+1}{2} \right\}, & i = n-1, \\ \left\{ \frac{m(2n+7)+3}{2}, \frac{m(2n+7)+5}{2}, \ldots, \frac{m(2n+9)+1}{2} \right\}, & i = n. \end{cases}$$

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The n sets of vertex weights of mC_n together with the six sets of vertex weights of mC_6 form exactly $n+6$ pairwise disjoint sets and each consisting of m consecutive integers with the smallest and largest vertex weights are, respectively,

$$\frac{m(n+6)+3}{2} \quad \text{and} \quad \frac{m(3n+18)+1}{2}.$$

Hence, for $n \equiv 1 \pmod 4$,

$$\{w(x) : x \in V(G)\} = \left\{ \frac{m(n+6)+3}{2}, \frac{m(n+6)+5}{2}, \ldots, \frac{m(3n+18)+1}{2} \right\}.$$

Case 2. $n \equiv 3 \pmod 4$.
For each $j = 1, 2, \ldots, m$, define the labels of the vertices in mC_6 by

$$\ell(y_{i,j}) = \begin{cases} \frac{1}{2}m(n+i-2) + c_{\frac{i+1}{2},j}, & \text{if } i = 1,3,5, \\ \frac{1}{4}m(3n+3+2i) + c_{\frac{i}{2},j}, & \text{if } i = 2,4,6. \end{cases}$$

For each $j = 1, 2, \ldots, m$, define the labels of the vertices in mC_n by

$$\ell(z_{i,j}) = \begin{cases} \frac{1}{4}m(i-1) + c_{1,j}, & \text{if } i \equiv 1 \pmod 4, \\ \frac{1}{4}m(2n+11+i) + c_{2,j}, & \text{if } i \equiv 3 \pmod 4,\ i \neq n-4, n, \\ \frac{1}{4}m(3n+19) + c_{2,j}, & \text{if } i = n-4, \\ \frac{1}{4}m(3n+23) + c_{2,j}, & \text{if } i = n, \\ \frac{1}{4}m(n-1+i) + c_{1,j}, & \text{if } i \equiv 2 \pmod 4,\ i \neq n-1, \\ \frac{1}{2}m(n+5) + c_{3,j}, & \text{if } i = n-1, \\ \frac{1}{4}m(3n+23+i) + c_{2,j}, & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

Next, we show that ℓ is a bijection. By the definition of $\mathcal{C}$, for $j = 1, 2, \ldots, m$, the six sets of labels of mC_6 are

$$\{\ell(y_{1,j})\} = \left\{ \frac{m(n-1)}{2}+1, \frac{m(n-1)}{2}+2, \ldots, \frac{m(n+1)}{2} \right\},$$
$$\{\ell(y_{2,j})\} = \left\{ \frac{m(3n+7)}{4}+1, \frac{m(3n+7)}{4}+2, \ldots, \frac{m(3n+11)}{4} \right\},$$
$$\{\ell(y_{3,j})\} = \left\{ \frac{m(n+1)}{2}+1, \frac{m(n+1)}{2}+2, \ldots, \frac{m(n+3)}{2} \right\},$$
$$\{\ell(y_{4,j})\} = \left\{ \frac{m(3n+11)}{4}+1, \frac{m(3n+11)}{4}+2, \ldots, \frac{m(3n+15)}{4} \right\},$$
$$\{\ell(y_{5,j})\} = \left\{ \frac{m(n+3)}{2}+1, \frac{m(n+3)}{2}+2, \ldots, \frac{m(n+5)}{2} \right\},$$
$$\{\ell(y_{6,j})\} = \left\{ \frac{m(3n+15)}{4}+1, \frac{m(3n+15)}{4}+2, \ldots, \frac{m(3n+19)}{4} \right\}.$$

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The sets of labels of mC_n are

$$\{\ell(z_{i,j})\} = \begin{cases} \left\{\frac{m(i-1)}{4}+1, \frac{m(i-1)}{4}+2, \ldots, \frac{m(i+3)}{4}\right\}, & i \equiv 1 \pmod 4, \\ \left\{\frac{m(2n+11+i)}{4}+1, \frac{m(2n+11+i)}{4}+2, \ldots, \frac{m(2n+15+i)}{4}\right\}, & i \equiv 3 \pmod 4, i \neq n-4, n, \\ \left\{\frac{m(3n+19)}{4}+1, \frac{m(3n+19)}{4}+2, \ldots, \frac{m(3n+23)}{4}\right\}, & i = n-4, \\ \left\{\frac{m(3n+23)}{4}+1, \frac{m(3n+23)}{4}+2, \ldots, \frac{m(3n+27)}{4}\right\}, & i = n, \\ \left\{\frac{m(n-1+i)}{4}+1, \frac{m(n-1+i)}{4}+2, \ldots, \frac{m(n+3+i)}{4}\right\}, & i \equiv 2 \pmod 4, i \neq n-1, \\ \left\{\frac{m(n+5)}{2}+1, \frac{m(n+5)}{2}+2, \ldots, \frac{m(n+7)}{2}\right\}, & i = n-1, \\ \left\{\frac{m(3n+23+i)}{4}+1, \frac{m(3n+23+i)}{4}+2, \ldots, \frac{m(3n+27+i)}{4}\right\}, & i \equiv 0 \pmod 4. \end{cases}$$

By similar arguments as in the Case 1, it follows that

$$\{\ell(x) : x \in V(G)\} = \{1, 2, \ldots, m(n+6)\}.$$

Therefore, ℓ is a bijection.

The sets of weights of mC_6 are

$$\begin{aligned} \{w(y_{1,j})\} &= \left\{\frac{m(3n+12)+3}{2}, \frac{m(3n+12)+5}{2}, \ldots, \frac{m(3n+14)+1}{2}\right\}, \\ \{w(y_{2,j})\} &= \left\{\frac{m(2n+1)+3}{2}, \frac{m(2n+1)+5}{2}, \ldots, \frac{m(2n+3)+1}{2}\right\}, \\ \{w(y_{3,j})\} &= \left\{\frac{m(3n+10)+3}{2}, \frac{m(3n+10)+5}{2}, \ldots, \frac{m(3n+12)+1}{2}\right\}, \\ \{w(y_{4,j})\} &= \left\{\frac{m(2n+5)+3}{2}, \frac{m(2n+5)+5}{2}, \ldots, \frac{m(2n+7)+1}{2}\right\}, \\ \{w(y_{5,j})\} &= \left\{\frac{m(3n+14)+3}{2}, \frac{m(3n+14)+5}{2}, \ldots, \frac{m(3n+16)+1}{2}\right\}, \\ \{w(y_{6,j})\} &= \left\{\frac{m(2n+3)+3}{2}, \frac{m(2n+3)+5}{2}, \ldots, \frac{m(2n+5)+1}{2}\right\}. \end{aligned}$$

The sets of weights of mC_n are

$$\{w(z_{i,j})\} = \begin{cases} \left\{\frac{m(2n+i+12)+3}{2}, \ldots, \frac{m(2n+i+14)+1}{2}\right\}, & i \equiv 1 \pmod 2, i \neq n-2, n, \\ \left\{\frac{m(n+i+6)+3}{2}, \ldots, \frac{m(n+i+8)+1}{2}\right\}, & i \equiv 0 \pmod 2, i \neq n-5, n-3, n-1, \\ \left\{\frac{m(n+i+12)+3}{2}, \ldots, \frac{m(n+i+14)+1}{2}\right\}, & i = n-5, n-3, n-1, \\ \left\{\frac{m(3n+16)+3}{2}, \ldots, \frac{m(3n+18)+1}{2}\right\}, & i = n-2, \\ \left\{\frac{m(n+6)+3}{2}, \frac{m(n+6)+5}{2}, \ldots, \frac{m(n+8)+1}{2}\right\}, & i = n. \end{cases}$$

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By the similar argument as in the Case 1, it follows that

$$\{w(x) : x \in V(G)\} = \left\{ \frac{m(n+6)+3}{2}, \frac{m(n+6)+5}{2}, \ldots, \frac{m(3n+18)+1}{2} \right\}.$$

Hence, ℓ is a $(\frac{m(n+6)+3}{2}, 1)$-DA labeling of G. □

Figure 3 illustrates the labeling constructed in the proof of Theorem 2.3 for the graph $3(C_7 \cup C_6)$. In this case, we use the Kötzig matrix

$$\mathcal{C} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}.$$

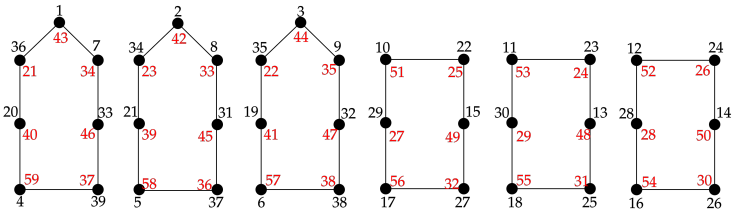


Figure 3: A $(21,1)$-DA labeling of $3(C_7 \cup C_6)$.

The following theorem characterize (a,d)-DA labeling of $mC_n \cup \frac{1}{2}(mn-1)C_6$ for all integers $m, n \geq 3$.

Theorem 2.4. *For all integers $m, n \geq 3$, the graph $G \cong mC_n \cup \frac{1}{2}(mn-1)C_6$ is (a,d)-DA iff $d = 1$ and both m and n are odd.*

Proof. Lemma 1.1 implies that, if G is (a,d)-DA, then $d = 1$. Since $\frac{1}{2}(mn-1)$ is an integer for all integers $m, n \geq 3$, $mn-1$ is even which implies m and n are odd. Conversely, let m and n be odd,

$$V(G) = \{z_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \left\{ y_{i,j} : 1 \leq i \leq 6, 1 \leq j \leq \frac{1}{2}(mn-1) \right\},$$

and

$$\begin{aligned} E(G) =& \{z_{i,j}z_{i+1,j}, z_{n,j}z_{1,j} : 1 \leq i \leq n-1, 1 \leq j \leq m\} \\ & \cup \left\{ y_{i,j}y_{i+1,j}, y_{6,j}y_{1,j} : 1 \leq i \leq 5, 1 \leq j \leq \frac{1}{2}(mn-1) \right\}. \end{aligned}$$

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Define a labeling $\ell : V(G) \to \{1, 2, \ldots, 4mn-3\}$ as follows. For $j = 1, 2, \ldots, \frac{1}{2}(mn-1)$, label each vertex of $\frac{1}{2}(mn-1)C_6$ with

$$\ell(y_{i,j}) = \begin{cases} 2j, & \text{if } i = 1, \\ 4j-1, & \text{if } i = 2, \\ 2mn-2+2j, & \text{if } i = 3, \\ 4mn-2j-2, & \text{if } i = 4, \\ 4mn-4j-1, & \text{if } i = 5, \\ 2mn-2j, & \text{if } i = 6. \end{cases}$$

For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, define

$$\ell(z_{i,j}) = \begin{cases} 4mf(z_i) - 4m - 3 + 4c_{1,j}, & \text{if } i \equiv 1, 2 \pmod 4,\ i \neq n-1, \text{ and } n \equiv 3 \pmod 4, \\ 4mf(z_i) - 4m - 3 + 4c_{1,j}, & \text{if } i \equiv 1, 2 \pmod 4,\ i \neq n, \text{ and } n \equiv 1 \pmod 4, \\ 4mf(z_i) - 4m - 3 + 4c_{2,j}, & \text{if } i \equiv 0, 3 \pmod 4, \\ 4mf(z_i) - 4m - 3 + 4c_{3,j}, & \text{if } i = n-1, n \equiv 3 \pmod 4, \text{ or } i = n, n \equiv 1 \pmod 4. \end{cases}$$

where $c_{1,j}, c_{2,j}, c_{3,j}$ are entries of $\mathcal{C}$, and f is a $(\frac{n+3}{2}, 1)$-DA labeling of C_n [1, Theorem 2.7].

The bijectivity of ℓ is verified by the same argument as in the proof of Theorem 2.2. From the definition of ℓ, we have

$$\begin{aligned} \left\{\ell(y_{1,j}) : 1 \le j \le \frac{1}{2}(mn-1)\right\} &= \{2, 4, \ldots, mn-1\}, \\ \left\{\ell(y_{2,j}) : 1 \le j \le \frac{1}{2}(mn-1)\right\} &= \{3, 7, \ldots, 2mn-3\}, \\ \left\{\ell(y_{3,j}) : 1 \le j \le \frac{1}{2}(mn-1)\right\} &= \{2mn, \ldots, 3mn-3\}, \\ \left\{\ell(y_{4,j}) : 1 \le j \le \frac{1}{2}(mn-1)\right\} &= \{3mn-1, \ldots, 4mn-4\}, \\ \left\{\ell(y_{5,j}) : 1 \le j \le \frac{1}{2}(mn-1)\right\} &= \{2mn+1, \ldots, 4mn-5\}, \\ \left\{\ell(y_{6,j}) : 1 \le j \le \frac{1}{2}(mn-1)\right\} &= \{mn+1, \ldots, 2mn-2\}. \end{aligned}$$

For every fixed i, by the definition of $\mathcal{C}$,

$$\{\ell(z_{i,j}) : 1 \le j \le m\} = \big\{4mf(z_i) - 4m - 3 + 4j : 1 \le j \le m\big\}.$$

Since $f : V(C_n) \to \{1, 2, \ldots, n\}$ is a bijection, we have

$$\{\ell(z_{i,j}) : 1 \le i \le n,\ 1 \le j \le m\} = \{1, 5, 9, \ldots, 4mn-3\}.$$

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Consequently,

$$\begin{aligned}\{\ell(x) : x \in V(G)\} &= \{1,5,\ldots,4mn-3\} \cup \{2,4,\ldots,4mn-4\} \cup \{3,7,\ldots,4mn-5\}\\ &= \{1,2,\ldots,4mn-3\}.\end{aligned}$$

Hence, ℓ is a bijection.

For $j = 1,2,\ldots,\frac{1}{2}(mn-1)$, the vertex weights of $\frac{1}{2}(mn-1)C_6$ are

$$\begin{aligned}\{w(y_{1,j})\} &= \{2mn+1,\ 2mn+3,\ \ldots,\ 3mn-2\},\\ \{w(y_{2,j})\} &= \{2mn+2,\ 2mn+6,\ \ldots,\ 4mn-4\},\\ \{w(y_{3,j})\} &= \{4mn-1,\ 4mn+1,\ \ldots,\ 5mn-4\},\\ \{w(y_{4,j})\} &= \{5mn-2,\ 5mn,\ \ldots,\ 6mn-5\},\\ \{w(y_{5,j})\} &= \{4mn,\ 4mn+4,\ \ldots,\ 6mn-6\},\ \text{and}\\ \{w(y_{6,j})\} &= \{3mn,\ 3mn+2,\ \ldots,\ 4mn-3\}.\end{aligned}$$

For $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$, the vertex weights of $z_{i,j}$ are given by

$$w(z_{i,j}) = \begin{cases} 4mw_f(z_i) - 2m - 4c_{1,j}, & \text{if } i = n-2,\ n \equiv 3 \pmod 4, \text{ or } i = n-1,\ n \equiv 1 \pmod 4,\\ 4mw_f(z_i) - 2m - 4c_{2,j}, & \text{if } i = n,\ n \equiv 3 \pmod 4, \text{ or } i = 1,\ n \equiv 1 \pmod 4,\\ 4mw_f(z_i) - 2m - 4c_{3,j}, & \text{otherwise,}\end{cases}$$

where $w_f(z_i)$ denotes the weight of z_i under f.

Since f is a $(\frac{n+3}{2},1)$-DA labeling [1, Theorem 2.7], then for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$, the vertex weights of mC_n are

$$\{w(z_{i,j})\} = \{2mn, 2mn+4, \ldots, 6mn-4\}.$$

Hence, ℓ is an $(2mn,1)$-DA labeling of G. □

Figure 4 (a) is a $(4,1)$-DA labeling of C_5 and Figure 4 (b) illustrates the $(30,1)$-DA labeling of $3C_5 \cup 7C_6$ based on Figure 4 (a).

Next, we investigate an (a,d)-DA labeling of 2-regular graphs whose components are cycles of three different lengths. In particular, we focus on $C_{10} \cup mC_3 \cup tC_6$. To prove this result, we use the shifting Kötzig matrix $\mathcal{D} = [d_{i,j}]$ of order $3 \times (2r+1)$ whose row entries are permutations of $\{-r-1,\ldots,r+1\}$ such that $d_{1,j} \neq -r-1$ and 1, $d_{2,j} \neq 0$ and -1, and $d_{3,j} \neq r+1$ and 0, and each column has sum 0. Such a matrix $\mathcal{D}$ can be found in [17, Lemma 7]. For $r = 4$, the matrix $\mathcal{D}$ is given as follows:

$$\mathcal{D} = \begin{bmatrix} -4 & -3 & -2 & -1 & 0 & 2 & 3 & 4 & 5\\ 1 & 2 & 5 & -3 & 4 & 3 & -5 & -2 & -4\\ 3 & 1 & -3 & 4 & -4 & -5 & 2 & -2 & -1 \end{bmatrix}.$$

By the definition of $\mathcal{D}$ [17, Lemma 7], we obtain

$$\begin{aligned}\{d_{1,j} + d_{2,j}\} &= \{-r, -r+1, \ldots, -2, -1, 1, 2, \ldots, r+1\},\\ \{d_{2,j} + d_{3,j}\} &= \{-r-1, \ldots, -2, 0, 1, 2, \ldots, r\},\ \text{and}\\ \{d_{3,j} + d_{1,j}\} &= \{-r-1, \ldots, -2, -1, 2, 3, \ldots, r+1\}.\end{aligned}$$

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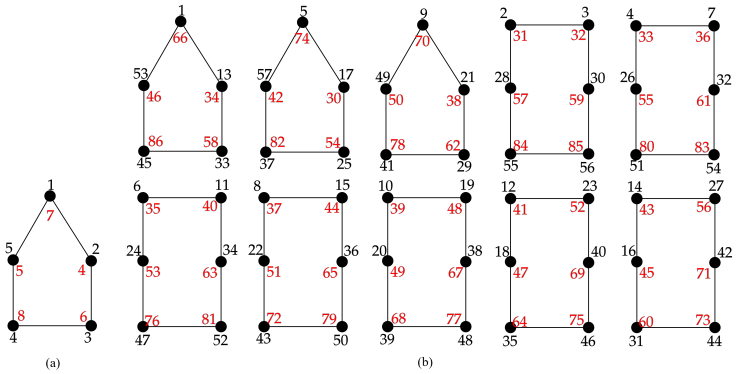


Figure 4: (a). A $(4,1)$-DA labeling of C_5. (b). A $(30,1)$-DA labeling of $3C_5 \cup 7C_6$ based on (a).

Theorem 2.5. *For every integer $m,t \geq 1$, the graph $G \cong C_{10} \cup mC_3 \cup tC_6$ is (a,d)-DA iff $d=1$ and m is odd.*

Proof. Lemma 1.1 implies that, if G is (a,d)-DA, then $d=1$. Since G is (a,d)-DA, $6t+10+3m$ must be odd, which implies that m is odd. Conversely, let $m=2s+1$ for some nonnegative integer s. Define G as a graph with

$$V(G)=\{z_i : 1 \leq i \leq 10\} \cup \{x_{1,j}, x_{2,j}, x_{3,j} : 1 \leq j \leq 2s+1\} \cup \{y_{i,j} : 1 \leq i \leq 6, 1 \leq j \leq t\},$$

and

$$\begin{aligned} E(G) =& \{z_i z_{i+1}, z_{10}z_1 : 1 \leq i \leq 9\} \cup \{x_{1,j}x_{2,j}, x_{2,j}x_{3,j}, x_{3,j}x_{1,j} : 1 \leq j \leq 2s+1\} \\ & \cup \{y_{i,j}y_{i+1,j}, y_{6,j}y_{1,j} : 1 \leq i \leq 5, 1 \leq j \leq t\}. \end{aligned}$$

For $j=1,2,\ldots,t$, let $Y_{2j-1}=\{y_{1,j}, y_{3,j}, y_{5,j}\}$ and $Y_{2j}=\{y_{2,j}, y_{4,j}, y_{6,j}\}$. For $j=1,2,\ldots,2s+1$, let $Y_{2t+j}=\{x_{1,j}, x_{2,j}, x_{3,j}\}$. Clearly, the sets $Y_1, Y_2, \ldots, Y_{2(t+s)+1}$ and $\{z_i : 1 \leq i \leq 10\}$ form a partition of $V(G)$. Let $r=t+s$ and consider the matrix $\mathcal{D}=[d_{ij}]$ of order $3 \times (2r+1)$. To show that G is (a,d)-DA, we consider the following cases.

Case 1. $r=1$.

If $r=1$, then $(t,s)=(1,0)$. Label the graph $C_{10} \cup C_3 \cup C_6$ by

$$(19,17,9,14,8,10,18,4,6,1),(16,13,3),(15,21,5,11,7,2).$$

This labeling is an $(11,1)$-DA labeling of G.

Case 2. $r=2$.

If $r=2$, then $(t,s)=(1,1)$ or $(t,s)=(2,0)$. For $(t,s)=(1,1)$, label the graph $C_{10} \cup 3C_3 \cup C_6$ by

$$(25,21,10,8,20,9,1,19,13,5),(24,7,12),(23,4,11),(22,3,15),(18,17,14,16,2,6).$$

For $(t,s)=(2,0)$, label the graph $C_{10} \cup C_3 \cup 2C_6$ by

$$(25,21,10,8,20,9,1,19,13,5),(24,7,12),(23,22,4,3,11,15),(18,17,14,16,2,6).$$

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These labelings are a $(14,1)$-DA labelings of $C_{10} \cup 3C_3 \cup C_6$ and $C_{10} \cup C_3 \cup 2C_6$, respectively.

Case 3. $r \geq 3$.

Consider a matrix $\mathcal{D}^* = [d^*_{i,j}]$ of order $3 \times (2r+1)$ defined by

$$\begin{aligned} d^*_{1,j} &= r + 1 + d_{1,j}, \\ d^*_{2,j} &= 3r + 7 + d_{2,j}, \text{ and} \\ d^*_{3,j} &= 5r + 12 + d_{3,j}, \end{aligned}$$

where $j = 1, 2, \ldots, 2r+1$.

Define a labeling $\ell : V(G) \to \{1, 2, \ldots, 6r+13\}$ as follows.

$$\ell(z_1) = 3r+7, \ell(z_2) = r+2, \ell(z_3) = 6r+13, \ell(z_4) = 5r+12, \ell(z_5) = 2r+5,$$

$$\ell(z_6) = 2r+4, \ell(z_7) = 2r+3, \ell(z_8) = 3r+6, \ell(z_9) = 4r+10, \text{ and } \ell(z_{10}) = 4r+9.$$

Next, label the vertices of $(2s+1)C_3 \cup tC_6$ as follows:

$$\ell(Y_j) = \mathbf{d}^*_{\sigma(j)}, \quad 1 \leq j \leq 2r+1,$$

where σ is any permutation of $\{1, 2, \ldots, 2r+1\}$ and $\mathbf{d}^*_{\sigma(j)}$ is the $\sigma(j)$-th column of $\mathcal{D}^*$. Since σ is a permutation of $\{1, 2, \ldots, 2r+1\}$, every column of $\mathcal{D}^*$ is assigned to exactly one of the sets $Y_1, Y_2, \ldots, Y_{2r+1}$.

We first verify that ℓ is a bijection. From the definitons of $d^*_{i,j}$, we have

$$\begin{aligned} \{d^*_{1,j} : 1 \leq j \leq 2r+1\} &= \{1, 2, \ldots, 2r+2\} \setminus \{r+2\}, \\ \{d^*_{2,j} : 1 \leq j \leq 2r+1\} &= \{2r+6, 2r+7, \ldots, 4r+8\} \setminus \{3r+6, 3r+7\}, \\ \{d^*_{3,j} : 1 \leq j \leq 2r+1\} &= \{4r+11, 4r+12, \ldots, 6r+12\} \setminus \{5r+12\}. \end{aligned}$$

These three sets are pairwise disjoint, and each has cardinality $2r+1$. Thus,

$$\{\ell(x_{i,j})\} \cup \{\ell(y_{i,j})\} = \{1, 2, \ldots, 6r+13\} \setminus \{\ell(z_i) : 1 \leq i \leq 10\},$$

or

$$\{\ell(x) : x \in V(G)\} = \{1, 2, \ldots, 6r+13\}.$$

Hence, ℓ is a bijection.

The weights of the vertices of C_{10} are given by

$$\{4r+8, 5r+10, 5r+11, 6r+13, 6r+14, 7r+15, 7r+16, 7r+17, 8r+18, 9r+20\}.$$

For $j = 1, 2, \ldots, 2s+1$, the sets of weights of $(2s+1)C_3$ are

$$\{w(x_{1,j}), w(x_{2,j}), w(x_{3,j})\} = \{\ell(x_{2,j}) + \ell(x_{3,j}), \ell(x_{3,j}) + \ell(y_{1,j}), \ell(x_{1,j}) + \ell(y_{2,j})\}.$$

For $j = 1, 2, \ldots, t$, the sets of weights of tC_6 are

$$\begin{aligned} \{w(y_{2,j}), w(y_{4,j}), w(y_{6,j})\} &= \{\ell(y_{1,j}) + \ell(y_{3,j}), \ell(y_{3,j}) + \ell(y_{5,j}), \ell(y_{5,j}) + \ell(y_{1,j})\}, \\ \{w(y_{1,j}), w(y_{3,j}), w(y_{5,j})\} &= \{\ell(y_{2,j}) + \ell(y_{6,j}), \ell(y_{2,j}) + \ell(y_{4,j}), \ell(y_{4,j}) + \ell(y_{6,j})\}. \end{aligned}$$

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Consequently,

$$\{w(x_{i,j})\} \cup \{w(y_{i,j})\} = \bigcup_{j=1}^{2r+1} \{d^*_{1,j} + d^*_{2,j}, d^*_{2,j} + d^*_{3,j}, d^*_{3,j} + d^*_{1,j}\}.$$

Since

$$\begin{aligned}
\{d^*_{2,j} + d^*_{3,j} : 1 \le j \le 2r+1\} &= \{7r+18, 7r+19, \ldots, 8r+17\} \cup \{8r+19, 8r+20, \ldots, 9r+19\},\\
\{d^*_{3,j} + d^*_{1,j} : 1 \le j \le 2r+1\} &= \{5r+12, 5r+13, \ldots, 6r+12\} \cup \{6r+15, 6r+16, \ldots, 7r+14\},\\
\{d^*_{1,j} + d^*_{2,j} : 1 \le j \le 2r+1\} &= \{3r+8, 3r+9, \ldots, 4r+7\} \cup \{4r+9, 4r+10, \ldots, 5r+9\},
\end{aligned}$$

then, all vertex weights are distinct and

$$\begin{aligned}
\{w(x) : x \in V(G)\} &= \left\{ \frac{(6r+13)+3}{2}, \frac{(6r+13)+5}{2}, \ldots, \frac{(6r+13)+3+2(6r+12)}{2} \right\}\\
&= \{3r+8, 3r+9, \ldots, 9r+20\}.
\end{aligned}$$

Hence, ℓ is a $(3r+8, 1)$-DA labeling of G.

□

As an example, Figure 5 shows the labeling of $C_{10} \cup 3C_3 \cup 3C_6$. In this case, $m = 3$, $s = 1$, $t = 3$, $r = 4$, and

$$\mathcal{D}^* = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 10 \\ 20 & 21 & 24 & 16 & 23 & 22 & 14 & 17 & 15 \\ 35 & 33 & 29 & 36 & 28 & 27 & 34 & 30 & 31 \end{bmatrix}.$$

If we set $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \end{pmatrix}$, then

$$\ell(Y_1) = \mathbf{d}^*_4, \ell(Y_2) = \mathbf{d}^*_5, \ell(Y_3) = \mathbf{d}^*_6, \ell(Y_4) = \mathbf{d}^*_7, \ell(Y_5) = \mathbf{d}^*_8,$$

$$\ell(Y_6) = \mathbf{d}^*_9, \ell(Y_7) = \mathbf{d}^*_1, \ell(Y_8) = \mathbf{d}^*_2, \text{ and } \ell(Y_9) = \mathbf{d}^*_3.$$

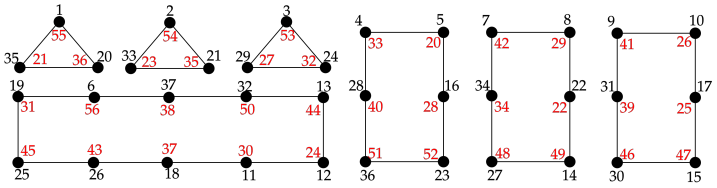


Figure 5: A $(20, 1)$-DA labeling of $C_{10} \cup 3C_3 \cup 3C_6$.

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3. Conclusion

In this paper, we have given necessary and sufficent conditions for the 2-regular graphs $C_n \cup tC_6$ where $n \in \{5, 7, 9, 11\}$ and $t \geq 2$, $C_{2t+1} \cup tC_6$ where $t \geq 6$, $m(C_n \cup C_6)$ for odd m and n, $mC_n \cup \frac{1}{2}(mn-1)C_6$ for odd integers $m, n \geq 3$, and $C_{10} \cup mC_3 \cup tC_6$ for odd m and $t \geq 1$ to be (a, d)-DA. Related to these results, we propose two open problems as follows:

1. Find an $(a, 1)$-DA labeling of $C_n \cup tC_6$ for $t \geq 2$ and odd $n \geq 13$.
2. Find an $(a, 1)$-DA labeling of $mC_n \cup tC_6$ for $t \geq 1$ and odd integers $m \geq 1$ and $n \geq 7$.

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