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Outer multiset dimensions of (extended) corona product graphs

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Abstract

The outer multiset dimension of graph G, $dim_{ms}(G)$, is the cardinality of the smallest subset S of vertices that uniquely recognizes each vertex outside S using the multiset of distances between the vertex and the vertices in S. This paper considers the outer multiset dimensions of both the corona product and the extended corona product of a connected graph with one of the following: a complete graph, the complement of a complete graph, or a star.

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1. Introduction

The process of identifying vertices in a graph based on their unique structural characteristics, known as distinguishability or resolvability, has several practical applications. In computer networks, nodes can be located using structural features rather than explicit identifiers; in social networks, this idea helps analyze an individual's position or role relative to others with similar connections. Moreover, even simple structural properties, such as the degree of a vertex or the

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pattern formed by its neighbors, have been effectively used to re-identify users who were assumed to be anonymous in social graphs.

In graph theory, metric dimension provides a mathematical framework for measuring vertex distinguishability. It is based on choosing a specific set of reference vertices, called a resolving set, so that every vertex in the graph can be uniquely identified by its distances to these references. This concept plays a significant role in combinatorics as it helps to understand the structural relationships and symmetries of graphs. Furthermore, studies on the metric dimension have important applications in various fields, including network design, robot navigation, and information retrieval [2], where it is essential to determine unique positions or roles based on structural properties.

Slater [11] (1975) and Harary and Melter [5] (1976) independently introduced the study of metric dimension, defining a resolving set as a set of vertices that uniquely identifies vertices in a graph using distance information. Since then, metric dimension has been widely investigated across various classes of graphs. Let $G = (V, E)$ be a simple connected graph of order $n(G)$ and diameter $diam(G)$. Given an ordered set of vertices $S = \{w_1, \ldots, w_t\} \subseteq V(G)$, the *metric representation* of a vertex $u \in V$ with respect to S is the t-vectors $r(u|S) = (d_G(u, w_1), \ldots, d_G(u, w_t))$. If every two vertices u and w have different metric representations, i.e., $r(u|S) \neq r(w|S)$, then S is a *resolving set*. The *metric dimension* of G, $dim(G)$, is the cardinality of the smallest resolving set.

When we weaken the requirement that the resolving set be ordered, we obtain the concept of multiset dimension. For $S = \{w_1, \ldots, w_t\} \subseteq V(G)$, the *multiset representation* of $u \in V(G)$ with respect to S is the multiset $m(u|S) = \{d_G(u, w_1), \ldots, d_G(u, w_t)\}$. A set S such that two vertices of G have different multiset representations is called a *multiset resolving set*. If G has a multiset resolving set, then the *multiset dimension* of G, $md(G)$, is the cardinality of the smallest multiset resolving set; otherwise, G has an infinite multiset dimension [1, 10]. To avoid the problem of the infiniteness of the multiset version of metric dimension, the outer version of multiset resolving sets was introduced. A set S such that two vertices outside of S have different multiset representations is called an *outer multiset resolving set*. The cardinality of the smallest outer multiset resolving set is called the *outer multiset dimension* of G, denoted as $dim_{ms}(G)$ [4]. It is obvious that for any connected simple graph G, $1 \leq dim_{ms}(G) \leq n(G) - 1$. The only graphs that achieve the bounds are listed in the following two theorems.

Theorem 1.1. *[4] A graph G satisfies $dim_{ms}(G) = 1$ if and only if it is a path graph.*

Theorem 1.2. *[6] A graph G satisfies $dim_{ms}(G) = n(G) - 1$ if and only if G is regular with $diam(G) \leq 2$.*

As a result of Theorem 1.2, the outer multiset dimensions of non-regular graphs, or regular graphs with large diameter, remain largely unknown. In [9], we study the outer multiset dimensions of non-regular graphs of diameter 2 obtained from the join product. In this paper, we consider graphs with larger diameter obtained from both the corona and extended corona products.

In Section 2, we consider the outer multiset dimensions of the corona product with complete graphs, complements of complete graphs, and stars. Furthermore, the outer multiset dimensions of the extended corona product with complete graphs, the complement of complete graphs, and stars are considered in Section 3. The definitions of corona and the extended corona product are presented below.

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Definition 1. *[3] Let G be a graph with n vertices and let H be a graph. The* corona product *of G and H, denoted by $G \odot H$, is defined as the graph obtained by taking one copy of G and n copies of H, and then joining the i-th vertex of G with an edge to every vertex in the i-th copy of H.*

For example, the corona product of C_5 and K_2 can be seen in Figure 1.

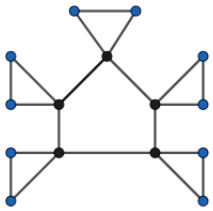


Figure 1. The corona product $C_5 \odot K_2$

Definition 2. *[8] Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be an ordered vertex set of a graph G and $\mathcal{H} = \{H_i \mid 1 \leq i \leq n\}$ be an ordered collection of graphs. Note that H_i can be an empty graph. We define* the extended corona product of G with $\mathcal{H}$, *$G \odot \mathcal{H}$, as a graph derived from joining each vertex v_i to all vertices in H_i ($1 \leq i \leq n$).*

The extended corona product is a generalization of the corona product, in the sense that if $\mathcal{H} = \{H_i \mid H_i \cong H, 1 \leq i \leq n\}$, the extended corona product $G \odot \mathcal{H}$ is isomorphic to the original corona product $G \odot H$. In addition, we denote the vertex set of $G \odot \mathcal{H}$ as follows. For $H_i \in \mathcal{H}$, let $V(H_i) = \{u_j \mid 1 \leq j \leq n_i\}$ and $V(G \odot \mathcal{H}) = V(G) \cup \{u_{k,j} \mid 1 \leq k \leq n,\ 1 \leq j \leq n_i\}$.

For example, Figure 2 illustrates $C_n \odot \mathcal{H}_\infty$, where $\mathcal{H}_\infty = \{\overline{K_1}, \overline{K_2}, \ldots, \overline{K_n}\}$, and $P_n \odot \mathcal{H}_\in$, where $\mathcal{H}_\in = \{\emptyset, \overline{K_2}, \overline{K_3}, \ldots, \overline{K_n}\}$.

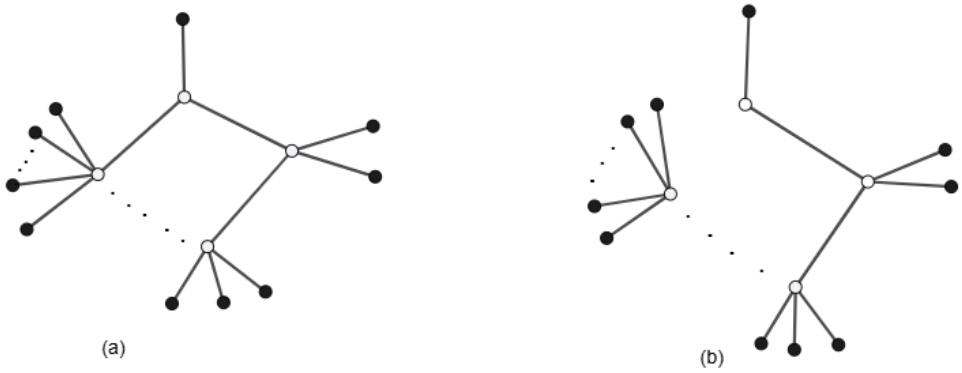


Figure 2. (a) The extended corona product $C_n \odot \{\overline{K_i} | 1 \leq i \leq n\}$ and (b) The extended corona product $P_n \odot \{\emptyset, \overline{K_i} | 2 \leq i \leq n\}$

In the rest of the paper, we use the following terminology and notation. For a positive integer k, $[k] = \{1, \ldots, k\}$. If an element d appears k times in a multiset, we may abbreviate it to d^k. For example, $\{1, 1, 1, 2, 5, 5\} = \{1^3, 2, 5^2\}$.

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The eccentricity of v, $ecc(v)$, is the largest distance between v and all other vertices in G. A vertex u in G is said to be *diametral* if there exists a vertex $v \in V(G)$ such that $d_G(u,v) = diam(G)$. We also say that v is a *vertex diametral to* u. For any vertex u of G, the set $N_G(u) = \{v \in V(G) | uv \in E(G)\}$ is called the *open neighbourhood* of the vertex u in G and $N_G[u] = N_G(u) \bigcup \{u\}$ is called the *closed neighbourhood* of u in G. The *degree* of a vertex u is $|N_G(u)|$. Two vertices u and v of G are *true twins* if $N_G[u] = N_G[v]$ and *false twins* if $N_G(u) = N_G(v)$. In general, u and v are called *twins* if they are true or false twins. The existence of twins is important for determining the outer multiset resolving set, as shown below.

Proposition 1.1. *[4] Let G be a nontrivial graph, and $S \subseteq V(G)$ be an outer multiset resolving set of G. If u and v are a pair of twin vertices, then $u \in S$ or $v \in S$.*

A property of graphs that will be frequently used in the following sections is multiset distance irregular.

Definition 3. *[6] A graph G is* multiset distance irregular *if for every two vertices $u, v \in V(G)$, the multisets $m_G(u|V(G))$ and $m_G(v|V(G))$ are different.*

An example of a multiset distance irregular graph is presented in Figure 3.

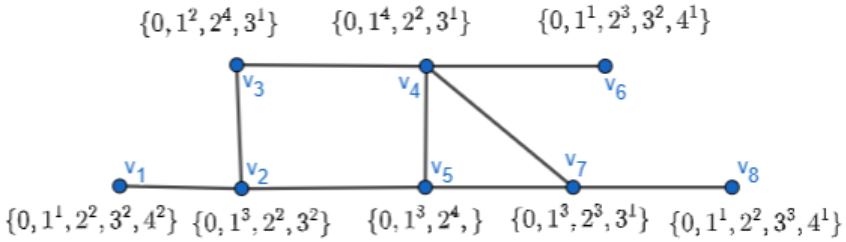


Figure 3. A multiset distance irregular graph.

2. Outer Multiset Dimension of Corona Product Graphs

In this section, we examine the corona product of an arbitrary connected graph with a complete graph, its complement, or a star. We use the following lemma to prove the lower bounds in Theorems 2.1 and 2.6.

Lemma 2.1. *Let G be a connected graph and $H \in \{K_k, \overline{K_k}, S_k\}$. Let $S \subseteq V(G \odot H)$ such that $S \bigcap V(G) = \emptyset$. Let $g_i, g_j \in V(G)$ where $m(g_i|V(G)) \neq m(g_j|V(G))$. If one of the conditions below holds, then $m(g_i|S) \neq m(g_j|S)$.*

1. *$H \in \{K_k, \overline{K_k}\}, k \geq 2$ and S consists of $k-1$ vertices from each copy of H.*
2. *$H = S_k, k \geq 3$ and S consists of $k-2$ leaves from each copy of H.*

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Proof. For $i \in \{1, \ldots, n\}$ and $t \in \{0, 1, \ldots, diam(G)\}$, let $r_i(t)$ be the number of vertices at a distance t from $g_i \in V(G)$. Since $m(g_i|V(G)) \neq m(g_j|V(G))$, there exists t such that $r_i(t) \neq r_j(t)$. Let $g_a \in V(G)$ where $d_G(g_i, g_a) = t$. In $G \odot H$, every path from g_i to a vertex of H_a passes through g_a. So, if $x \in V(H_a)$, then $d(x, g_i) = d(g_i, g_a) + 1 = t + 1$. Now, we consider two cases.

Case 1: $H \in \{K_k, \overline{K_k}\}, k \geq 2$**.**

Since S consists of $k - 1$ vertices from each copy of H, we have $m(g_i|S) = \{\ldots, (t + 1)^{r_i(t)(k-1)}, \ldots\}$ and $m(g_j|S) = \{\ldots, (t+1)^{r_j(t)(k-1)}, \ldots\}$. However, $r_i(t) \neq r_j(t)$, so $m(g_i|S) \neq m(g_j|S)$.

Case 2: $H = S_k, k \geq 3$**.**

Since S consists of $k - 2$ leaves from each copy of H, we have $m(g_i|S) = \{\ldots, (t + 1)^{r_i(t)(k-2)}, \ldots\}$ and $m(g_j|S) = \{\ldots, (t+1)^{r_j(t)(k-2)}, \ldots\}$. However, $r_i(t) \neq r_j(t)$, so $m(g_i|S) \neq m(g_j|S)$.

□

2.1. Outer Multiset Dimension of Corona Product with a Complete Graph or Its Complement

We start by providing a sharp lower bound on the outer multiset dimension of the corona product of an arbitrary connected graph with a complete graph or its complement.

Theorem 2.1. *For $n \geq 2$ and $k \geq 2$, if G is a connected graph of order n and $H \in \{K_k, \overline{K_k}\}$, then*

$$dim_{ms}(G \odot H) \geq n(k - 1).$$

Moreover, $dim_{ms}(G \odot H) = n(k - 1)$ if and only if G is multiset distance irregular.

Proof. Let $V(G) = \{g_1, \ldots, g_n\}$, $V(H) = \{h_1, \ldots, h_k\}$ and $V(G \odot H) = \{g_1, \ldots, g_n, h_{1,1}, \ldots, h_{1,k}, \ldots, h_{n,1}, \ldots, h_{n,k}\}$, where $h_{i,j}$ is the jth vertex in H_i, $1 \leq i \leq n, 1 \leq j \leq k$. Clearly, for any $1 \leq i \leq n, 1 \leq j_1, j_2 \leq k$, h_{i,j_1} and h_{i,j_2} are twins. By Proposition 1.1, every outer multiset resolving set of $G \odot H$ must contain at least one of h_{i,j_1} and h_{i,j_2}. Inductively, every outer multiset resolving set of $G \odot H$ contains at least $(k - 1)$ vertices of H_i, $1 \leq i \leq n$. Thus, $dim_{ms}(G \odot H) \geq n(k - 1)$.

For the second part of the theorem, let G be a multiset distance irregular graph. We shall prove that $dim_{ms}(G \odot H) \leq n(k - 1)$ by showing that $S = \{h_{1,1}, \ldots, h_{1,k-1}, h_{2,1}, \ldots, h_{2,k-1}, \ldots, h_{n,1}, \ldots, h_{n,k-1}\}$ is an outer multiset resolving set.

First, we show that for $1 \leq i, j \leq n$, $m(g_i|S) \neq m(h_{j,k}|S)$. When $H = K_k$, $2 \in m(g_i|S)$, but $2 \notin m(h_{j,k}|S)$. When $H = \overline{K_k}$, $1 \in m(g_i|S)$, but $1 \notin m(h_{j,k}|S)$. By this, $m(g_i|S) \neq m(h_{j,k}|S)$.

Next, we prove that for $1 \leq i, j \leq n$, $i \neq j$, $m(g_i|S) \neq m(g_j|S)$ and $m(h_{i,k}|S) \neq m(h_{j,k}|S)$. Since G is multiset distance irregular, $m(g_i|V(G)) \neq m(g_j|V(G))$, for two distinct vertices g_i, g_j. By Lemma 2.1, $m(g_i|S) \neq m(g_j|S$. Similarly, $m(h_{i,k}|S) \neq m(h_{j,k}|S)$, and S is an outer multiset resolving set. Hence, $dim_{ms}(G \odot H) = n(k - 1)$.

Conversely, suppose that $dim_{ms}(G \odot H) = n(k - 1)$, and let S be an outer multiset basis of $(G \odot H)$. Since the vertices of each H_i are twins, only one vertex of each H_i is not in S. This means that since $|S| = n(k - 1)$, the vertices of G are also not in S. Without loss of generality, assume that the vertices $h_{i,k}$ and g_i are not in S, $1 \leq i \leq n$. We shall prove that G is a multiset distance irregular graph by contradiction. Assume that G is not multiset distance irregular, then there exist

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g_i, g_j where $m(g_i|V(G) = m(g_j|V(G)) = \{0^{r_0}, 1^{r_1}, \ldots, diam(G)^{r_{diam(G)}}\}$, where $r_t \geq 0, 0 \leq t \leq diam(G)$. Thus, $m(g_i|S) = \{1^{r_0(k-1)}, 2^{r_1(k-1)}, \ldots, (diam(G)+1)^{r_{diam(G)}(k-1)}\} = m(g_j|S)$, a contradiction. □

The next two theorems provide examples of graphs whose corona products with a complete graph and its complement have outer multiset dimensions close to the lower bound in Theorem 2.1; that is, the outer multiset dimensions are one or two more than the lower bound.

Theorem 2.2. *For $n \geq 2$ and $k \geq 2$, if $H \in \{K_k, \overline{K_k}\}$, then*

$$dim_{ms}(P_n \odot H) = n(k-1) + 1.$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, $E(P_n) = \{v_1v_2, \ldots, v_{n-1}v_n\}$, $V(H) = \{h_1, h_2, \ldots, h_k\}$, and $V(P_n \odot H) = \{v_1, v_2, \ldots, v_n, h_{1,1}, h_{1,2}, \ldots, h_{1,k}, \ldots, h_{n,1}, h_{n,2}, \ldots, h_{n,k}\}$, where $h_{i,j}$ is the jth vertex in H_i, $1 \leq i \leq n, 1 \leq j \leq k$.

Clearly, for any $1 \leq i \leq n, 1 \leq j_1, j_2 \leq k$, h_{i,j_1} and h_{i,j_2} are twins. By Proposition 1.1, every outer multiset resolving set of $P_n \odot H$ must contain at least one of h_{i,j_1} and h_{i,j_2}. Inductively, every outer multiset resolving set S of $P_n \odot H$ contains at least $(k-1)$ vertices of H_i, $1 \leq i \leq n$. Without loss of generality, let $\{h_{1,1}, h_{1,2}, \ldots, h_{1,k-1}, h_{2,1}, h_{2,2}, \ldots, h_{2,k-1}, \ldots, h_{n,1}, h_{n,2}, \ldots, h_{n,k-1}\} \subseteq S$. However, $S' = \{h_{1,1}, \ldots, h_{1,k-1}, \ldots, h_{n,1}, h_{n,2}, \ldots, h_{n,k-1}\}$ is not an outer multiset resolving set since $m(v_i|S') = m(v_{n+1-i}|S')$. And so, $|S| \geq n(k-1)+1$.

We shall prove that $|S| \leq n(k-1)+1$ by showing that $S = S' \bigcup \{h_{(n,k)}\}$ is an outer multiset resolving set. First, we show that for $1 \leq i < j \leq n$, $m(v_i|S) \neq m(v_j|S)$ and $m(h_{i,k}|S) \neq m(h_{j,k}|S)$. Let $ecc(v_i)$ be the eccentricity of v_i in P_n. Then $m(v_i|S) = \{1^{i_1}, 2^{i_2}, \ldots, (ecc(v_i)+1)^{i_{ecc(v_i)+1}}\}$, where i_j is the number of vertices of S at distance j from v_i, for $1 \leq j \leq ecc(v_i)+1$. If every two vertices v_i and v_j satisfy $ecc(v_i) \neq ecc(v_j)$, then $m(v_i|S) \neq m(v_j|S)$.

Now, assume that there exist v_i and v_j such that $ecc(v_i) = ecc(v_j)$. Then $j = n - i + 1$, where $1 \leq i \leq \frac{n}{2}$. Then v_n is the vertex in P_n with $d(v_i, v_n) = ecc(v_i)$, on the other hand, v_1 is the vertex in P_n with $d(v_j, v_1) = ecc(v_j)$. Since v_i is adjacent to the $(k-1)$ vertices in S, v_j has $(k-1)$ vertices with distance $ecc(v_j)+1$. But v_n is adjacent to the k vertices in S implying that v_i has k vertices with distance $ecc(v_i)+1$. Thus, $m(v_i|S) = \{\ldots, (ecc(v_i)+1)^k\} \neq \{\ldots, (ecc(v_j)+1)^{k-1}\} = m(v_j|S)$. Similarly, $m(h_{(i,k)}|S) \neq m(h_{(j,k)}|S)$.

Next, we prove that for $1 \leq i, j \leq n$, $m(v_i|S) \neq m(h_{j,k}|S)$. When $H = K_k$, $2 \in m(v_i|S)$, but $2 \notin m(h_{j,k}|S)$. When $H = \overline{K_k}$, $1 \in m(v_i|S)$, but $1 \notin m(h_{j,k}|S)$. By this, $m(v_i|S) \neq m(h_{j,k}|S)$. Thus, S is an outer multiset resolving set. □

Theorem 2.3. *For $n \geq 3$ and $k \geq 2$, if $H \in \{K_k, \overline{K_k}\}$, then*

$$dim_{ms}(C_n \odot H) = n(k-1) + 2.$$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, $E(C_n) = \{v_1v_2, \ldots, v_nv_1\}$, $V(H) = \{h_1, h_2, \ldots, h_k\}$, and $V(C_n \odot H) = \{v_1, \ldots, v_n, h_{1,1}, \ldots, h_{1,k}, \ldots, h_{n,1}, \ldots, h_{n,k}\}$, where $h_{i,j}$ is the jth vertex in H_i, $1 \leq i \leq n, 1 \leq j \leq k$. To prove the lower bound, suppose that S is an outer multiset resolving set with $|S| \leq n(k-1)+1$.

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Clearly, for any $1 \leq i \leq n, 1 \leq j_1, j_2 \leq k$, h_{i,j_1} and h_{i,j_2} are twins. From Proposition 1.1, every outer multiset resolving set of $C_n \odot H$ must contain at least one of h_{i,j_1} and h_{i,j_2}. Inductively, every outer multiset resolving set of $C_n \odot H$ contains at least $(k-1)$ vertices of H_i, $1 \leq i \leq n$. Without loss of generality, let $S = \{h_{1,1}, \ldots, h_{1,k-1}, h_{2,1}, \ldots, h_{2,k-1}, \ldots, h_{n,1}, \ldots, h_{n,k-1}\} \bigcup A$.

If $A = \emptyset$, then for any $i, 1 \leq i \leq n$, $m(v_i|S) = \{1^{(k-1)}, 2^{2(k-1)}, \ldots, \frac{n}{2}^{2(k-1)}, (\frac{n}{2}+1)^{(k-1)}\}$, if n is even, and $m(v_i|S) = \{1^{k-1}, 2^{2(k-1)}, \ldots, (\frac{n+1}{2})^{2(k-1)}\}$, if n is odd.

Now, assume that $A \neq \emptyset$, and so $|A| = 1$. Let $1 \leq i \leq n$. The vertices of the cycle adjacent to v_i are v_{i-1} and v_{i+1}, where the indices are read modulo n. Suppose that $v_i \in A$ or $h_{i,k} \in A$. If $v_i \in S$, $m(v_{i-1}|S) = \{1^k, 2^{2(k-1)}, \ldots, \frac{n}{2}^{2(k-1)}, (\frac{n}{2}+1)^{(k-1)}\} = m(v_{i+1}|S)$, if n is even, and $m(v_{i-1}|S) = \{1^k, 2^{2(k-1)}, \ldots, (\frac{n+1}{2})^{2(k-1)}\} = m(v_{i+1}|S)$, if n is odd. If $h_{i,k} \in S$, $m(v_{i-1}|S) = \{1^{k-1}, 2^{2(k-1)+1}, \ldots, \frac{n}{2}^{2(k-1)}, (\frac{n}{2}+1)^{(k-1)}\} = m(v_{i+1}|S)$, if n is even, and $m(v_{i-1}|S) = \{1^{k-1}, 2^{2(k-1)+1}, \ldots, (\frac{n+1}{2})^{2(k-1)}\} = m(v_{i+1}|S)$, if n is odd.

Thus $m(v_{i-1}|S) = m(v_{i+1}|S)$, so S is not an outer multiset resolving set, and $|S| \geq n(k-1)+2$.

Now we shall prove that $|S| \leq n(k-1)+2$, by showing that

$$S = \{h_{1,1}, \ldots, h_{1,k-1}, \ldots, h_{n,1}, \ldots, h_{n,k-1}\} \bigcup \{v_1, h_{2,k}\}$$

is an outer multiset resolving set. Let $S_1 = \{h_{1,1}, \ldots, h_{1,k-1}, \ldots, h_{n,1}, h_{n,2}, \ldots, h_{n,k-1}\}$ and $S_2 = \{v_1, h_{2,k}\}$.

First, we show that for $1 \leq j < l \leq n$, $m(v_j|S) \neq m(v_l|S)$ and $m(h_{j,k}|S) \neq m(h_{l,k}|S)$. In cycle C_n, each v_j and v_l have two distinct vertices at distance t, $t \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$. Additionally, v_j and v_l have $(k-1)$ neighbors in S_1. Now consider two cases: (i) $d(v_j, v_1) = d(v_l, v_1)$ and (ii) $d(v_j, v_1) \neq d(v_l, v_1)$.

Case (i): $d(v_j, v_1) = d(v_l, v_1)$**.**

In this case, $d(v_j, v_1) = j - 1 = d(v_l, v_1), 2 \leq j \leq \lceil \frac{n}{2} \rceil$, and so, $l = n + 2 - j$. If $j = 2$, v_2 has $(k-1)$ neighbors in S_1 and two neighbors in S_2, however v_n has $(k-1)$ neighbors in S_1 and one neighbor in S_2. If $2 < j \leq \lceil \frac{n}{2} \rceil$, v_j has $2(k-1)$ vertices in S_1 and two vertices in S_2 at distance $(j-1)$, however v_l has $2(k-1)$ vertices in S_1 and one vertex in S_2 at distance $(j-1)$. Thus, $m(v_j|S) \neq m(v_l|S)$.

Case (ii): $d(v_j, v_1) \neq d(v_l, v_1)$**.**

Let

$$d(v_l, v_1) = t = \begin{cases} l - 1, & \text{if } l \leq \lfloor \frac{n}{2} \rfloor + 1, \\ n - l + 1, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 < l \leq n. \end{cases}$$

If $l \leq \lfloor \frac{n}{2} \rfloor + 1$, v_l has $2(k-1)$ vertices in S_1 and two vertices in S_2 at distance t, however v_j has $2(k-1)$ vertices in S_1 and no vertex in S_2 at distance t. Thus $m(v_j|S) \neq m(v_l|S)$.

If $\lfloor \frac{n}{2} \rfloor + 1 < l < n$, v_l has $2(k-1)$ vertices in S_1 and one vertex in S_2 at distance t, however v_j has $2(k-1)$ vertices in S_1 and no vertex in S_2 at distance t. If $l = n$, v_l has $(k-1)$ neighbors in S_1 and one neighbor in S_2, however v_j has $(k-1)$ neighbors in S_1 and no neighbors in S_2. Thus $m(v_j|S) \neq m(v_l|S)$. Similarly, $m(h_{j,k}|S) \neq m(h_{l,k}|S)$.

Next, we prove that for $1 \leq j, l \leq n$, $m(v_j|S) \neq m(h_{l,k}|S)$. When n is even, $\max_{s \in S} d(v_j, s) = \frac{n}{2}+1$ and $\max_{s \in S} d(h_{l,k}, s) = \frac{n}{2}+2$. When n is odd, $\max_{s \in S} d(v_j, s) = \frac{n+1}{2}$ and $\max_{s \in S} d(h_{l,k}, s) =$

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$\frac{n+1}{2} + 1$. Since $\max_{s\in S} d(v_j, s) \neq \max_{s\in S} d(h_{l,k}, s)$, thus $m(v_j|S) \neq m(h_{l,k}|S)$. Thus, S is an outer multiset resolving set. □

The following Theorems 2.4 and 2.5 show that there are graphs, one dense and the other sparse, whose corona products with a complete graph and its complement have outer multiset dimensions closer to their order than the lower bound in Theorem 2.1.

Theorem 2.4. *For $n \geq 3$ and $k \geq 2$, if K_n is a complete graph of order n and $H \in \{K_k, \overline{K_k}\}$, then*

$$dim_{ms}(K_n \odot H) = nk + n - 4.$$

Proof. Let $V(K_n) = \{g_1, g_2, \ldots, g_n\}$, $V(H) = \{h_1, h_2, \ldots, h_k\}$, and $V(K_n \odot H) = \{g_1, \ldots, g_n, h_{1,1}, \ldots, h_{1,k}, \ldots, h_{n,1}, \ldots, h_{n,k}\}$, where $h_{i,j}$ is the jth vertex in H_i, $1 \leq i \leq n, 1 \leq j \leq k$. To prove the lower bound, suppose that S is an outer multiset resolving set with $|S| \leq nk + n - 5$.

Clearly, for any $1 \leq i \leq n, 1 \leq j_1, j_2 \leq k$, h_{i,j_1} and h_{i,j_2} are twins. By Proposition 1.1, every outer multiset resolving set of $K_n \odot H$ must contain at least one of h_{i,j_1} and h_{i,j_2}. Inductively, every outer multiset resolving set of $K_n \odot H$ contains at least $(k-1)$ vertices of $H_i, 1 \leq i \leq n$. Without loss of generality, let $\{h_{1,1}, \ldots, h_{1,k-1}, h_{2,1}, \ldots, h_{2,k-1}, \ldots, h_{n,1}, \ldots, h_{n,k-1}\} \subseteq S$. Define $A = \{g_i | 1 \leq i \leq n\}$ and $B = \{h_{i,k} | 1 \leq i \leq n\}$. There are at least 5 vertices of A, B outside of S, so we have the following two possibilities:

1. At least three vertices of A are not in S ($|A \setminus S| \geq 3$)
   Let g_i, g_j, g_l be three distinct vertices not in S, for $1 \leq i, j, l \leq n$. According to the Pigeonhole principle, there exist two vertices g_i, g_j such that $h_{i,k}, h_{j,k} \in S$ or $h_{i,k}, h_{j,k} \notin S$. Then we obtain $m(g_i|S) = m(g_j|S)$.
2. At most two vertices of A are not in S ($|A \setminus S| \leq 2$).
   Here, at least three vertices of B are not in S. Let $h_{i,k}$, $h_{j,k}$, $h_{l,k}$ be three distinct vertices in $B \setminus S$, for $1 \leq i, j, l \leq n$. Consider the g_i, g_j, g_l vertices of A. According to the Pigeonhole Principle, two of them belong to S or are outside S. For $g_i, g_j \in S$ or $g_i, g_j \notin S$, we then obtain $m(h_{i,k}|S) = m(h_{j,k}|S)$.

In both cases, S is not an outer multiset resolving set. Thus $|S| \geq nk + n - 4$.

Now we shall prove that $|S| \leq nk + n - 4$ by showing that $S = V(K_n \odot H) \setminus \{g_2, g_3, h_{1,k}, h_{3,k}\}$ is an outer multiset resolving set.

When $H = K_k$,

$$m(g_2|S) = \{1^{k+(n-2)}, 2^{|S|-(k+n-2)}\}, m(g_3|S) = \{1^{(k-1)+(n-2)}, 2^{|S|-(k+n-3)}\}, \text{ and}$$
$$m(h_{1,k}|S) = \{1^{k}, 2^{(n-3)}, 3^{|S|-(k+n-3)}\}, m(h_{3,k}|S) = \{1^{k-1}, 2^{(n-2)}, 3^{|S|-(k+n-3)}\}.$$

When $H = \overline{K_k}$,

$$m(g_2|S) = \{1^{k+(n-2)}, 2^{|S|-(k+n-2)}\}, m(g_3|S) = \{1^{(k-1)+(n-2)}, 2^{|S|-(k+n-3)}\}, \text{ and}$$
$$m(h_{1,k}|S) = \{1, 2^{(k-1)+(n-3)}, 3^{|S|-(k+n-3)}\}, m(h_{3,k}|S) = \{2^{(k-1)+(n-2)}, 3^{|S|-(k+n-3)}\}.$$

Since every vertex outside of S has a distinct multiset representation, S is an outer multiset resolving set. Hence, $dim_{ms}(K_n \odot H) = nk + n - 4$. □

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Theorem 2.5. *For $n \geq 4$ and $k \geq 2$, if S_n is a star of order n and $H \in \{K_k, \overline{K_k}\}$, then*

$$dim_{ms}(S_n \odot H) = nk + n - 6.$$

Proof. Note that $S_n = K_1 + \overline{K_{n-1}}$. Let $V(K_1) = \{g_1\}$, $V(\overline{K_{n-1}}) = \{g_2, \ldots, g_n\}$, $V(H) = \{h_1, h_2, \ldots, h_k\}$, and $V(S_n \odot H) = \{g_1, g_2, \ldots, g_n, h_{1,1}, h_{1,2}, \ldots, h_{1,k}, \ldots, h_{n,1}, h_{n,2}, \ldots, h_{n,k}\}$, where $h_{i,j}$ is the jth vertex in H_i, $1 \leq i \leq n, 1 \leq j \leq k$. To prove the lower bound, suppose that S is an outer multiset resolving set with $|S| \leq nk + n - 7$.

Clearly, for any $1 \leq i \leq n, 1 \leq j_1, j_2 \leq k$, h_{i,j_1} and h_{i,j_2} are twins. By Proposition 1.1, every outer multiset resolving set of $S_n \odot H$ must contain at least one of h_{i,j_1} and h_{i,j_2}. Inductively, every outer multiset resolving set of $S_n \odot H$ contains at least $(k-1)$ vertices of H_i. Without loss of generality, let $\{h_{1,1}, \ldots, h_{1,k-1}, h_{2,1}, \ldots, h_{2,k-1}, \ldots, h_{n,1}, \ldots, h_{n,k-1}\} \subseteq S$.

Define $A = \{g_i | 1 \leq i \leq n\}$ and $B = \{h_{i,k} | 1 \leq i \leq n\}$. There are at least 7 vertices of A, B outside of S, so we have the following two possibilities:

1. At least four vertices of A are not in S ($|A \setminus S| \geq 4$).
   For either $g_1 \notin S$ or $g_1 \in S$, let g_j, g_l, g_m be three distinct vertices not in S that correspond to the leaves of the star, for $2 \leq j, l, m \leq n$. According to the Pigeonhole Principle, there exist two vertices g_j, g_l such that either $h_{j,k}, h_{l,k} \in S$ or $h_{j,k}, h_{l,k} \notin S$. Then we obtain $m(g_j|S) = m(g_l|S)$.
2. At most three vertices of A are not in S ($|A \setminus S| \leq 3$).
   Here, at least four vertices of B are not in S. For $h_{1,k} \notin S$ or $h_{1,k} \in S$, let $h_{j,k}, h_{l,k}, h_{m,k}$ be three distinct vertices in $B \setminus S$, for $2 \leq j, l, m \leq n$. Consider the g_j, g_l, g_m vertices of A. According to the Pigeonhole Principle, two of them either belong to S or are outside S. For $g_j, g_l \in S$ or $g_j, g_l \notin S$, we then obtain $m(h_{j,k}|S) = m(h_{l,k}|S)$.

In both cases, S is not an outer multiset resolving set. Thus $|S| \geq nk + n - 6$.

Now we shall prove that $|S| \leq nk + n - 6$ by showing that

$$S = V(S_n \odot H) \setminus \{g_1, g_3, g_4, h_{1,k}, h_{2,k}, h_{4,k}\}$$

is an outer multiset resolving set.

When $H = K_k$,

$$m(g_3|S) = \{1^k, 2^{(k-1)+(n-3)}, 3^{|S|-(2k+n-4)}\}, m(g_4|S) = \{1^{k-1}, 2^{(k-1)+(n-3)}, 3^{|S|-(2k+n-5)}\},$$
$$m(g_1|S) = \{1^{(k-1)+(n-3)}, 2^{|S|-(k+n-4)}\}, m(h_{1,k}|S) = \{1^{k-1}, 2^{(n-3)}, 3^{|S|-(k+n-4)}\},$$
$$m(h_{2,k}|S) = \{1^k, 3^{(k-1)+(n-4)}, 4^{|S|-(2k+n-5)}\}, m(h_{4,k}|S) = \{1^{k-1}, 3^{(k-1)+(n-3)}, 4^{|S|-(2k+n-5)}\}.$$

When $H = \overline{K_k}$,

$$m(g_3|S) = \{1^k, 2^{(k-1)+(n-3)}, 3^{|S|-(2k+n-4)}\}, m(g_4|S) = \{1^{k-1}, 2^{(k-1)+(n-3)}, 3^{|S|-(2k+n-5)}\},$$
$$m(g_1|S) = \{1^{(k-1)+(n-3)}, 2^{|S|-(k+n-4)}\}, m(h_{1,k}|S) = \{2^{(k-1)+(n-3)}, 3^{|S|-(k+n-4)}\},$$
$$m(h_{2,k}|S) = \{1, 2^{(k-1)}, 3^{(k-1)+(n-4)}, 4^{|S|-(2k+n-5)}\},$$
$$m(h_{4,k}|S) = \{2^{(k-1)}, 3^{(k-1)+(n-3)}, 4^{|S|-(2k+n-5)}\}.$$

Since every vertex outside of S has a distinct multiset representation, S is an outer multiset resolving set. Hence, $dim_{ms}(S_n \odot H) = nk + n - 6$. □

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In Theorems 2.4 and 2.5, the outer multiset dimensions of the corona products of a complete graph and a star with a complete graph or its complement are four and six less than the orders of the graphs. In light of Theorem 1.2, it is natural to ask the following question: For $n \geq 4$ and $k \geq 2$, let $H \in \{K_k, \overline{K_k}\}$. Is there a connected graph G of order n such that $dim_{ms}(G \odot H) = n+nk-2$? If the answer to the previous question is negative, then a more general question can be asked. For $k \geq 2$, let $H \in \{K_k, \overline{K_k}\}$. If G is a connected graph, find a sharp upper bound for $dim_{ms}(G \odot H)$.

Finally, we conclude this subsection with the following question.

Problem 1. *Let $k \geq 2$. If G is an arbitrary connected graph and $H \in \{K_k, \overline{K_k}\}$, determine $dim_{ms}(G \odot H)$.*

2.2. Outer Multiset Dimension of Corona Product with a Star

Next, we provide a sharp lower bound for the outer multiset dimension of the corona product of a connected graph with a star.

Theorem 2.6. *Let $n \geq 2$ and $k \geq 3$. If G is a connected graph of order n and S_k is a star of order k, then*

$$dim_{ms}(G \odot S_k) \geq n(k-2).$$

Moreover, $dim_{ms}(G \odot S_k) = n(k-2)$ if and only if G is multiset distance irregular.

Proof. Note that $S_k = K_1 + \overline{K_{k-1}}$. Let $V(K_1) = \{h_1\}$, $V(\overline{K_{k-1}}) = \{h_2, \ldots, h_k\}$, $V(G) = \{g_1, \ldots, g_n\}$, and $V(G \odot S_k) = \{g_1, \ldots, g_n, h_{1,1}, \ldots, h_{1,k}, \ldots, h_{n,1}, \ldots, h_{n,k}\}$, where $h_{i,j}$ is the jth vertex in S_{k_i}, $1 \leq i \leq n, 1 \leq j \leq k$.

Clearly, for any $1 \leq i \leq n, 2 \leq j_1, j_2 \leq k$, h_{i,j_1} and h_{i,j_2} are twins. By Proposition 1.1, every outer multiset resolving set of $G \odot S_k$ must contain at least one of h_{i,j_1} and h_{i,j_2}. Inductively, every outer multiset resolving set of $G \odot S_k$ contains at least $(k-2)$ vertices of S_{k_i}, $1 \leq i \leq n$. Thus, $dim_{ms}(G \odot S_k) \geq n(k-2)$.

For the second part of the theorem, let G be a multiset distance irregular graph. We shall prove that $dim_{ms}(G \odot S_k) \leq n(k-2)$ by showing that $S = \{h_{1,2}, \ldots, h_{1,k-1}, \ldots, h_{n,2}, \ldots, h_{n,k-1}\}$ is an outer multiset resolving set.

First, we show that for $1 \leq i, j \leq n$, $i \neq j$, $m(g_i|S) \neq m(g_j|S)$, $m(h_{i,1}|S) \neq m(h_{j,1}|S)$, and $m(h_{i,k}|S) \neq m(h_{j,k}|S)$. Since G is multiset distance irregular, then for two distinct vertices g_i, g_j we have $m(g_i|V(G)) \neq m(g_j|V(G))$. By Lemma 2.1, $m(g_i|S) \neq m(g_j|S)$. Similarly, $m(h_{i,1}|S) \neq m(h_{j,1}|S)$, and $m(h_{i,k}|S) \neq m(h_{j,k}|S)$.

Next, we prove that for $1 \leq i, j \leq n$, $m(g_i|S) \neq m(h_{j,1}|S)$, $m(g_i|S) \neq m(h_{j,k}|S)$, and $m(h_{i,1}|S) \neq m(h_{j,k}|S)$. Since distance 2 occurs in $m(g_i|S)$ but not in $m(h_{j,1}|S)$, so $m(g_i|S) \neq m(h_{j,1}|S)$. The distance 1 occurs in $m(g_i|S)$ but not in $m(h_{j,k}|S)$, so $m(g_i|S) \neq m(h_{j,k}|S)$. Further, the distance 1 occurs in $m(h_{i,1}|S)$ but not in $m(h_{j,k}|S)$, so $m(h_{i,1}|S) \neq m(h_{j,k}|S)$. Thus, S is an outer multiset resolving set, and hence $dim_{ms}(G \odot S_k) = n(k-2)$.

Conversely, suppose that $dim_{ms}(G \odot S_k) = n(k-2)$, and let S be an outer multiset basis of $(G \odot S_k)$. Since the vertices $h_{i,2}, \ldots, h_{i,k}$ of each S_{k_i} are twins for $1 \leq i \leq n$, only one of these vertices $h_{i,2}, \ldots, h_{i,k}$ does not belong to S. This means that since $|S| = n(k-2)$, the vertices of G are also not in S. Without loss of generality, assume that the vertices $h_{i,1}$, $h_{i,k}$

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and g_i are not in S, $1 \leq i \leq n$. We shall prove that G is a multiset distance irregular graph by contradiction. Assume that G is not multiset distance irregular, then there exist g_i, g_j where $m(g_i|V(G) = m(g_j|V(G)) = \{0^{r_0}, 1^{r_1}, \ldots, diam(G)^{r_{diam(G)}}\}$, where $r_t \geq 0, 0 \leq t \leq diam(G)$. Thus, $m(g_i|S) = \{1^{r_0(k-2)}, 2^{r_1(k-2)}, \ldots, (diam(G)+1)^{r_{diam(G)}(k-2)}\} = m(g_j|S)$, a contradiction. □

The next two theorems provide examples of graphs whose corona products with a star have outer multiset dimensions close to the lower bound in Theorem 2.6; that is, the outer multiset dimensions are at most 2 more than the lower bound.

Theorem 2.7. *For $n \geq 2$ and $k \geq 3$, if P_n is a path of order n and S_k is a star of order k, then*

$$dim_{ms}(P_n \odot S_k) = n(k-2) + 1.$$

Proof. Note that $S_k = K_1 + \overline{K_{k-1}}$. Let $V(K_1) = \{h_1\}$, $V(\overline{K_{k-1}}) = \{h_2, \ldots, h_k\}$, $V(P_n) = \{v_1, \ldots, v_n\}$, $E(P_n) = \{v_1v_2, \ldots, v_{n-1}v_n\}$ and $V(P_n \odot S_k) = \{v_1, \ldots, v_n, h_{1,1}, \ldots, h_{1,k}, \ldots, h_{n,1}, \ldots, h_{n,k}\}$, where $h_{i,j}$ is the jth vertex in S_{k_i}, $1 \leq i \leq n, 1 \leq j \leq k$.

Clearly, for any $1 \leq i \leq n, 2 \leq j_1, j_2 \leq k$, h_{i,j_1} and h_{i,j_2} are twins. By Proposition 1.1, every outer multiset resolving set of $P_n \odot S_k$ must contain at least one of h_{i,j_1} and h_{i,j_2}. Inductively, every outer multiset resolving set of $P_n \odot S_k$ contains at least $(k-2)$ vertices of S_{k_i}, $1 \leq i \leq n$. Without loss of generality, let $\{h_{1,2}, \ldots, h_{1,k-1}, \ldots, h_{n,2}, \ldots, h_{n,k-1}\} \subseteq S$. However, $S' = \{h_{1,2}, \ldots, h_{1,k-1}, \ldots, h_{n,2}, h_{n,3}, \ldots, h_{n,k-1}\}$ is not an outer multiset resolving set since $m(v_i|S') = m(v_{n+1-i}|S')$. And so, $|S| \geq n(k-2)+1$.

We shall prove that $|S| \leq n(k-2)+1$ by showing that $S = S' \bigcup \{h_{n,k}\}$ is an outer multiset resolving set. First, we show that for $1 \leq i < j \leq n$, $m(v_i|S) \neq m(v_j|S)$, $m(h_{i,1}|S) \neq m(h_{j,1}|S)$, and $m(h_{i,k}|S) \neq m(h_{j,k}|S)$. Let $ecc(v_i)$ be the eccentricity of v_i in P_n. Then $m(v_i|S) = \{1^{i_1}, 2^{i_2}, \ldots, (ecc(v_i)+1)^{i_{ecc(v_i)+1}}\}$, where i_j is the number of vertices of S at distance j from v_i, for $1 \leq j \leq ecc(v_i)+1$. If every two vertices v_i and v_j satisfy $ecc(v_i) \neq ecc(v_j)$, then $m(v_i|S) \neq m(v_j|S)$.

Now, assume that there exist v_i and v_j such that $ecc(v_i) = ecc(v_j)$. Then $j = n - i + 1$, where $1 \leq i \leq \frac{n}{2}$. Then v_n is the vertex in P_n with $d(v_i, v_n) = ecc(v_i)$; on the other hand, v_1 is the vertex in P_n with $d(v_j, v_1) = ecc(v_j)$. Since v_i is adjacent to $(k-2)$ vertices in S, then v_j has $(k-2)$ vertices with distance $ecc(v_j)+1$. But v_n is adjacent to $k-1$ vertices in S, which implies v_i has $k-1$ vertices with distance $ecc(v_i)+1$. Thus, $m(v_i|S) = \{\ldots, (ecc(v_i)+1)^{k-1}\} \neq \{\ldots, (ecc(v_j)+1)^{k-2}\} = m(v_j|S)$. Similarly, $m(h_{(i,1)}|S) \neq m(h_{(j,1)}|S)$ and $m(h_{(i,k)}|S) \neq m(h_{(j,k)}|S)$.

Next, we prove that for $1 \leq i, j \leq n$, $m(v_i|S) \neq m(h_{j,1}|S)$, $m(v_i|S) \neq m(h_{j,k}|S)$, and $m(h_{i,1}|S) \neq m(h_{j,k}|S)$. Since distance 2 occurs in $m(v_i|S)$ but not in $m(h_{j,1}|S)$, so $m(v_i|S) \neq m(h_{j,1}|S)$. The distance 1 occurs in $m(v_i|S)$ but not in $m(h_{j,k}|S)$, so $m(v_i|S) \neq m(h_{j,k}|S)$. Further, the distance 1 occurs in $m(h_{i,1}|S)$ but not in $m(h_{j,k}|S)$, so $m(h_{i,1}|S) \neq m(h_{j,k}|S)$. Thus, S is an outer multiset resolving set, and hence $dim_{ms}(P_n \odot S_k) = n(k-2)+1$. □

Theorem 2.8. *For $n \geq 3$ and $k \geq 4$, if C_n is a cycle of order n and S_k is a star of order k, then*

$$dim_{ms}(C_n \odot S_k) = n(k-2) + 2.$$

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Proof. Note that $S_k = K_1 + \overline{K_{k-1}}$. Let $V(K_1) = \{h_1\}$, $V(\overline{K_{k-1}}) = \{h_2, \ldots, h_k\}$, $V(C_n) = \{v_1, \ldots, v_n\}$, $E(C_n) = \{v_1v_2, \ldots, v_nv_1\}$, and $V(C_n \odot S_k) = \{v_1, \ldots, v_n, h_{1,1}, h_{1,2}, \ldots, h_{1,k}, \ldots, h_{n,1}, h_{n,2}, \ldots, h_{n,k}\}$, where $h_{i,j}$ is the jth vertex in S_{k_i}, $1 \leq i \leq n, 1 \leq j \leq k$. To prove the lower bound, suppose that S is an outer multiset resolving set with $|S| \leq n(k-2)+1$.

Clearly, for any $1 \leq i \leq n, 2 \leq j_1, j_2 \leq k$, h_{i,j_1} and h_{i,j_2} are twins. By Proposition 1.1, every outer multiset resolving set of $C_n \odot S_k$ must contain at least one of h_{i,j_1} and h_{i,j_2}. Inductively, every outer multiset resolving set of $C_n \odot S_k$ contains at least $(k-2)$ vertices of S_{k_i}, $1 \leq i \leq n$. Without loss of generality, let $S = \{h_{1,2}, \ldots, h_{1,k-1}, \ldots, h_{n,2}, \ldots, h_{n,k-1}\} \bigcup A$.

If $A = \emptyset$, then for any $i, 1 \leq i \leq n$, $m(v_i|S) = \{1^{(k-2)}, 2^{2(k-2)}, \ldots, \frac{n}{2}^{2(k-2)}, (\frac{n}{2}+1)^{(k-2)}\}$, if n is even, and $m(v_i|S) = \{1^{k-2}, 2^{2(k-2)}, \ldots, (\frac{n+1}{2})^{2(k-2)}\}$, if n is odd.

Now, assume that $A \neq \emptyset$, and so $|A| = 1$. Let $1 \leq i \leq n$. The vertices of the cycle adjacent to v_i are v_{i-1} and v_{i+1}, where the indices are read modulo n, so that $v_0 = v_n$ and $v_{n+1} = v_1$. Then $A = \{v_i\}$, $A = \{h_{i,1}\}$, or $A = \{h_{i,k}\}$. If $v_i \in S$, $m(v_{i-1}|S) = \{1^{k-1}, 2^{2(k-2)}, \ldots, \frac{n}{2}^{2(k-2)}, (\frac{n}{2}+1)^{(k-2)}\} = m(v_{i+1}|S)$, if n is even, and $m(v_{i-1}|S) = \{1^{k-1}, 2^{2(k-2)}, \ldots, (\frac{n+1}{2})^{2(k-2)}\} = m(v_{i+1}|S)$, if n is odd. If $h_{i,1} \in S$ or $h_{i,k} \in S$, $m(v_{i-1}|S) = \{1^{k-2}, 2^{2(k-2)+1}, \ldots, \frac{n}{2}^{2(k-2)}, (\frac{n}{2}+1)^{(k-2)}\} = m(v_{i+1}|S)$, if n is even, and $m(v_{i-1}|S) = \{1^{k-2}, 2^{2(k-2)+1}, \ldots, (\frac{n+1}{2})^{2(k-2)}\} = m(v_{i+1}|S)$, if n is odd. Thus, $m(v_{i-1}|S) = m(v_{i+1}|S)$, a contradiction, and so, $|S| \geq n(k-2)+2$.

Now we shall prove that $|S| \leq n(k-2)+2$ by showing that $S = S_1 \bigcup S_2$ is an outer multiset resolving set, where $S_1 = \{h_{1,2}, \ldots, h_{1,k-1}, \ldots, h_{n,2}, \ldots, h_{n,k-1}\}$ and $S_2 = \{v_1, h_{2,k}\}$.

First, we show that for $1 \leq j < l \leq n$, $m(v_j|S) \neq m(v_l|S)$, $m(h_{j,1}|S) \neq m(h_{l,1}|S)$, and $m(h_{j,k}|S) \neq m(h_{l,k}|S)$. In C_n, each v_j and v_l has two distinct vertices at a distance of $t, t \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$. Additionally, v_j and v_l have $(k-2)$ neighbors in S_1. Now consider two cases: (i) $d(v_j, v_1) = d(v_l, v_1)$ and (ii) $d(v_j, v_1) \neq d(v_l, v_1)$.

Case (i): $d(v_j, v_1) = d(v_l, v_1)$**.**

In this case, $d(v_j, v_1) = j - 1 = d(v_l, v_1), 2 \leq j \leq \lceil \frac{n}{2} \rceil$, and so, $l = n + 2 - j$. If $j = 2$, v_2 has $(k-2)$ neighbors in S_1 and two neighbors in S_2; however, v_n has $(k-2)$ neighbors in S_1 and one neighbor in S_2. If $2 < j \leq \lceil \frac{n}{2} \rceil$, v_j has $2(k-2)$ vertices in S_1 and two vertices in S_2 at distance $(j-1)$; however, v_l has $2(k-2)$ vertices in S_1 and one vertex in S_2 at distance $(j-1)$. Thus, $m(v_j|S) \neq m(v_l|S)$.

Case (ii): $d(v_j, v_1) \neq d(v_l, v_1)$**.**

Let

$$d(v_l, v_1) = t = \begin{cases} l - 1, & \text{if } l \leq \lfloor \frac{n}{2} \rfloor + 1, \\ n - l + 1, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 < l \leq n. \end{cases}$$

If $l \leq \lfloor \frac{n}{2} \rfloor + 1$, v_l has $2(k-2)$ vertices in S_1 and two vertices in S_2 at distance t, however v_j has $2(k-2)$ vertices in S_1 and no vertex in S_2 at distance t. Thus $m(v_j|S) \neq m(v_l|S)$.

If $\lfloor \frac{n}{2} \rfloor + 1 < l < n$, v_l has $2(k-2)$ vertices in S_1 and one vertex in S_2 at distance t, however v_j has $2(k-2)$ vertices in S_1 and no vertex in S_2 at distance t. If $l = n$, v_l has $(k-2)$ neighbors in S_1 and one neighbor in S_2, however v_j has $(k-2)$ neighbors in S_1 and no neighbors in S_2. Thus $m(v_j|S) \neq m(v_l|S)$. Similarly, $m(h_{(j,1)}|S) \neq m(h_{(l,1)}|S)$ and $m(h_{(j,k)}|S) \neq m(h_{(l,k)}|S)$.

Next, we prove that for $1 \leq j, l \leq n$, $m(v_j|S) \neq m(h_{l,1}|S)$ and $m(v_j|S) \neq m(h_{l,k}|S)$. When n is even, $\max_{s \in S} d(v_j, s) = \frac{n}{2} + 1$, $\max_{s \in S} d(h_{l,k}, s) = \frac{n}{2} + 2$ and $\max_{s \in S} d(h_{l,1}, s) = \frac{n}{2} + 2$. When n is odd, $\max_{s \in S} d(v_j, s) = \frac{n+1}{2}$, $\max_{s \in S} d(h_{l,k}, s) = \frac{n+1}{2} + 1$ and $\max_{s \in S} d(h_{l,1}, s) =$

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$\frac{n+1}{2}+1$. Since $\max_{s\in S} d(v_j,s) \neq \max_{s\in S} d(h_{l,1},s)$, $\max_{s\in S} d(v_j,s) \neq \max_{s\in S} d(h_{l,k},s)$. Thus, $m(v_j|S) \neq m(h_{l,1}|S)$ and $m(v_j|S) \neq m(h_{l,k}|S)$.

Finally, we show that $m(h_{j,1}|S) \neq m(h_{l,k}|S)$ for $1 \leq j,l \leq n$. If $j \in \{2,n\}$ and $l=1$, then $h_{j,1}$ has exactly one vertex of S at distance 2, while $h_{1,k}$ has $k-2$ vertices of S at distance 2, since $k \geq 4$. If $j \notin \{2,n\}$ and $l=1$, then $2 \notin m(h_{j,1}|S)$, but $2 \in m(h_{1,k}|S)$. If $l \neq 1$, then $1 \in m(h_{j,1}|S)$, but $1 \notin m(h_{l,k}|S)$. In each case, $m(h_{j,1}|S) \neq m(h_{l,k}|S)$.

Thus, S is an outer multiset resolving set, and hence $dim_{ms}(C_n \odot S_k) = n(k-2)+2$. □

We conclude this subsection by asking the following question.

Problem 2. *For $k \geq 3$, let S_k be a star of order k. If G is an arbitrary connected graph, determine $dim_{ms}(G \odot S_k)$.*

3. Outer Multiset Dimension of Extended Corona Products

In this section, we generalize the previous corona product results to the extended corona product.

3.1. Outer Multiset Dimension of Extended Corona Product with Complete Graphs or Their Complements

Theorem 3.1. *Let G be a connected graph of order $n \geq 2$. If $\mathcal{H} = \{H_1,\ldots,H_n\}$, where $H_i \in \{K_{k_i}, \overline{K_{k_i}}\}$ and $k_i \geq 2$, then*

$$\dim_{ms}(G \odot \mathcal{H}) \geq \sum_{i=1}^{n}(k_i-1).$$

Moreover,

1. *if all k_i are pairwise distinct, then equality holds;*
2. *if all k_i are equal and G is multiset distance irregular, then equality holds.*

Proof. Let $V(G) = \{v_1,v_2,\ldots,v_n\}$ and $\mathcal{H} = \{H_i | 1 \leq i \leq n\}$. Let $|V(H_i)| = k_i$, $V(H_i) = \{u_l \mid 1 \leq l \leq k_i\}$. Let $V(G \odot \mathcal{H}) = V(G) \cup \{u_{i,l} \mid 1 \leq i \leq n, 1 \leq l \leq k_i\}$, where $u_{i,l}$ is the lth vertex in H_i.

Clearly, for any $1 \leq i \leq n, 1 \leq l_1,l_2 \leq k_i$, u_{i,l_1} and u_{i,l_2} are twins. By Proposition 1.1, every outer multiset resolving set of $(G \odot \mathcal{H})$ must contain at least one of u_{i,l_1} and u_{i,l_2}. Inductively, every outer multiset resolving set of $(G \odot \mathcal{H})$ contains at least (k_i-1) vertices of H_i, $1 \leq i \leq n$. Thus, $dim_{ms}(G \odot \mathcal{H}) \geq \sum_{i=1}^{n}(k_i-1)$.

For the second part of the theorem, we shall prove that $dim_{ms}(G \odot \mathcal{H}) \leq \sum_{i=1}^{n}(k_i-1)$, by showing that $S = \{u_{i,x} | 1 \leq i \leq n, 1 \leq x \leq k_i-1\}$ is an outer multiset resolving set. We have the following two cases:

Case 1: All k_i are pairwise distinct.

First, we show that for $1 \leq i,j \leq n$, $i \neq j$, $m(v_i|S) \neq m(v_j|S)$ and $m(u_{i,k_i}|S) \neq m(u_{j,k_j}|S)$. The numbers of vertices adjacent to v_i and v_j in S are (k_i-1) and (k_j-1). Note that $m(v_i|S) = \{1^{k_i-1},\ldots\}$ and $m(v_j|S) = \{1^{k_j-1},\ldots\}$. Since $k_i \neq k_j$ then $k_i-1 \neq k_j-1$ which implies that $m(v_i|S) \neq m(v_j|S)$. Similarly $m(u_{i,k_i}|S) \neq m(u_{j,k_j}|S)$.

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Now, we prove that for $1 \leq s,t \leq n$, $m(v_s|S) \neq m(u_{t,k_t}|S)$. If H_t is a complete graph, distance 2 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, we have $m(v_s|S) \neq m(u_{t,k_t}|S)$. If H_t is the complement of a complete graph, distance 1 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, and so $m(v_s|S) \neq m(u_{t,k_t}|S)$. Thus, S is an outer multiset resolving set.

Case 2: All k_i are equal and G is multiset distance irregular.

First, we show that for $1 \leq i,j \leq n$, $i \neq j$, $m(v_i|S) \neq m(v_j|S)$ and $m(u_{i,k_i}|S) \neq m(u_{j,k_j}|S)$. If both H_i and H_j are complete graphs or complements of complete graphs, then by Theorem 2.1, $m(v_i|S) \neq m(v_j|S)$, $m(u_{i,k_i}|S) \neq m(u_{j,k_j}|S)$. If H_i is a complete graph and H_j is the complement of a complete graph, we apply the same arguments as in Theorem 2.1 to obtain $m(v_i|S) \neq m(v_j|S)$. In addition, distance 1 occurs in $m(u_{i,k_i}|S)$, but not in $m(u_{j,k_j}|S)$, so we have $m(u_{i,k_i}|S) \neq m(u_{j,k_j}|S)$.

Now, we prove that for $1 \leq s,t \leq n$, $m(v_s|S) \neq m(u_{t,k_t}|S)$. If H_t is a complete graph, distance 2 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, we have $m(v_s|S) \neq m(u_{t,k_t}|S)$. If H_t is the complement of a complete graph, distance 1 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, and so $m(v_s|S) \neq m(u_{t,k_t}|S)$. Thus, S is an outer multiset resolving set. □

Unlike the corona product, where we have characterized the graphs G for which $\dim_{ms}(G \odot H)$, with $H \in \{K_k, \overline{K_k}\}$, equals the lower bound of Theorem 2.1, we have not yet managed to characterize such graphs for the extended corona product and the lower bound in Theorem 3.1. Equality is established here in two cases: when all k_i are pairwise distinct for every connected graph G, and when all k_i are equal and G is multiset distance irregular. For multiset distance irregular graphs G, the general intermediate case, where some but not all k_i coincide, is unresolved. In the following example, equality does not hold.

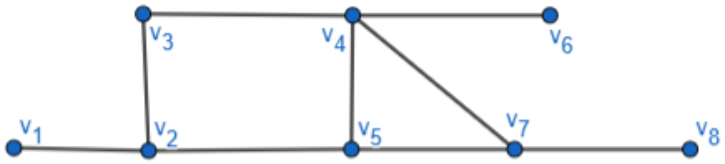


Figure 4. A multiset distance irregular graph G.

Example 3.1. *Let G be a multiset distance irregular graph of order 8 as illustrated in Figure 4. Let $\mathcal{H} = \{K_3, \overline{K_4}, \overline{K_2}, K_2, K_2, \overline{K_2}, K_2, \overline{K_3}\}$. If S were an outer multiset resolving set with $|S| = \sum_{i=1}^{8}(k_i - 1)$, the twin condition would force S to contain exactly $k_i - 1$ vertices of H_i, $1 \leq i \leq 8$. Then $m(v_3|S) = \{1, 2^4, 3^5, 4^2\} = m(v_4|S)$, a contradiction. Therefore $|S| \geq \sum_{i=1}^{8}(k_i - 1) + 1$. We now prove $|S| \leq \sum_{i=1}^{8}(k_i - 1) + 1$. by considering*

$$S = \{u_{1,1}, u_{1,2}, u_{2,1}, u_{2,2}, u_{2,3}, u_{3,1}, u_{4,1}, u_{5,1}, u_{6,1}, u_{7,1}, u_{8,1}, u_{8,2}\} \bigcup \{u_{3,2}\}.$$

The multiset representations of the vertices are depicted in the following tables.

Every vertex outside of S has a distinct multiset representation, so S is an outer multiset resolving set.

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Vertex z	$m(z\|S)$
v_1	$\{1^2, 2^3, 3^3, 4^2, 5^3\}$
v_2	$\{1^3, 2^5, 3^2, 4^3\}$
v_3	$\{1^2, 2^4, 3^5, 4^2\}$
v_4	$\{1, 2^5, 3^4, 4^3\}$
v_5	$\{1, 2^5, 3^7\}$
v_6	$\{1, 2, 3^4, 4^5, 5^2\}$
v_7	$\{1, 2^4, 3^6, 4^2\}$
v_8	$\{1^2, 2, 3^2, 4^6, 5^2\}$

Vertex z	$m(z\|S)$
$u_{1,3}$	$\{1^2, 3^3, 4^3, 5^2, 6^3\}$
$u_{2,4}$	$\{2^3, 3^5, 4^2, 5^3\}$
$u_{4,2}$	$\{1, 3^5, 4^4, 5^3\}$
$u_{5,2}$	$\{1, 3^5, 4^7\}$
$u_{6,2}$	$\{2, 3, 4^4, 5^5, 6^2\}$
$u_{7,2}$	$\{2, 3^4, 4^6, 5^2\}$
$u_{8,3}$	$\{2^2, 3, 4^2, 5^6, 6^2\}$

This example raises the following question: Let G be a multiset distance irregular graph of order n. Characterize all $\mathcal{H} = \{H_1, \ldots, H_n\}$, with $H_i \in \{K_{k_i}, \overline{K_{k_i}}\}$, where $dim_{ms}(G \odot \mathcal{H}) = \sum_{i=1}^{n}(k_i - 1)$.

In the next theorem, we determine the outer multiset dimension of the extended corona product between any path and any collection of $\mathcal{H} = \{H_1, \ldots, H_n\}$ with $H_i \in \{K_{k_i}, \overline{K_{k_i}}\}$. This theorem also shows that, in general, the condition of G being multiset distance irregular is not necessary for $dim_{ms}(G \odot \mathcal{H})$, where $\mathcal{H} = \{H_1, \ldots, H_n\}$ and $H_i \in \{K_{k_i}, \overline{K_{k_i}}\}$ are equal to the lower bound in Theorem 3.1. In particular, we show that if G is a path, that is not multiset distance irregular, there exists a collection $\mathcal{H} = \{H_1, \ldots, H_n\}$ with $H_i \in \{K_{k_i}, \overline{K_{k_i}}\}$ such that $dim_{ms}(G \odot \mathcal{H})$ is equal to the lower bound of Theorem 3.1.

Theorem 3.2. *For $n \geq 2$, let $\mathcal{H} = \{H_1, \ldots, H_n\}$, with $H_i \in \{K_{k_i}, \overline{K_{k_i}}\}$, $k_i \geq 2, 1 \leq i \leq n$. Then*

$$dim_{ms}(P_n \odot \mathcal{H}) = \begin{cases} 1 + \sum_{i=1}^{n}(k_i - 1), \textit{ if } k_i = k_{n+1-i}, \textit{ for all } i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}, \\ \sum_{i=1}^{n}(k_i - 1), \textit{ there exists } i \in \{1, 2, \ldots, \lfloor n/2 \rfloor\} \textit{ such that } k_i \neq k_{n+1-i}. \end{cases}$$

Proof. Let $\mathcal{H} = \{H_i | 1 \leq i \leq n\}$, $|V(H_i)| = k_i$, and $V(H_i) = \{u_l \mid 1 \leq l \leq k_i\}$. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$, and $V(P_n \odot \mathcal{H}) = V(G) \cup \{u_{i,l} \mid 1 \leq i \leq n,\ 1 \leq l \leq k_i\}$, where $u_{i,l}$ is the lth vertex in H_i.

Clearly, for any $1 \leq i \leq n, 1 \leq l_1, l_2 \leq k_i$, u_{i,l_1} and u_{i,l_2} are twins. By Proposition 1.1, every outer multiset resolving set of $(P_n \odot \mathcal{H})$ must contain at least one of u_{i,l_1} and u_{i,l_2}. Inductively, every outer multiset resolving set of $(P_n \odot \mathcal{H})$ contains at least $(k_i - 1)$ vertices of H_i, $1 \leq i \leq n$. Thus,

$$dim_{ms}(P_n \odot \mathcal{H}) \geq \sum_{i=1}^{n}(k_i - 1). \tag{1}$$

We distinguish two cases:

Case 1: $k_i = k_{n+1-i}$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Suppose that S is an outer multiset resolving set with $|S| = \sum_{i=1}^{n}(k_i - 1)$. Then $m(v_i|S) = m(v_{n+1-i}|S)$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, a contradiction. So, if S is an outer multiset resolving set, it must be $|S| \geq 1 + \sum_{i=1}^{n}(k_i - 1)$. Now we shall prove that there exist S with $1 + \sum_{i=1}^{n}(k_i - 1)$ vertices, by defining

$$S = \{u_{i,x} | 1 \leq i \leq n, 1 \leq x \leq k_i - 1\} \bigcup \{u_{n,k_n}\}.$$

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First, we show that for $1 \leq i < j \leq n$, $m(v_i|S) \neq m(v_j|S)$. Let $ecc(v_i)$ be the eccentricity of v_i in P_n. We consider two subcases.

Subcase 1.i: $ecc(v_i) \neq ecc(v_j)$. Then $\max_{s\in S} d(v_i, s) \neq \max_{s\in S} d(v_j, s)$; hence $m(v_i|S) \neq m(v_j|S)$.

Subcase 1.ii: $ecc(v_i) = ecc(v_j)$. Then $j = n - i + 1$, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. In this case, v_n is the vertex in P_n with $d(v_i, v_n) = ecc(v_i)$. On the other hand, v_1 is the vertex in P_n with $d(v_j, v_1) = ecc(v_j)$. Since $k_i = k_j$, $m(v_i|S) = \{\ldots, (ecc(v_i) + 1)^{k_n}\} \neq \{\ldots, (ecc(v_j) + 1)^{k_1-1}\} = m(v_j|S)$.

Similarly, $m(u_{(i,k_i)}|S) \neq m(u_{(j,k_j)}|S)$.

Next, consider $1 \leq s, t \leq n$. When H_t is a complete graph, distance 2 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, we have $m(v_s|S) \neq m(u_{t,k_t}|S)$. When H_t is the complement of a complete graph, distance 1 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, which means $m(v_s|S) \neq m(u_{t,k_t}|S)$.

Thus, S is an outer multiset resolving set with $|S| = 1 + \sum_{i=1}^{n}(k_i - 1)$

Case 2: There exist $i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ such that $k_i \neq k_{n+1-i}$.

By Equation (1), $dim_{ms}(G \odot \mathcal{H}) \geq \sum_{i=1}^{n}(k_i - 1)$. Now, we shall prove $dim_{ms}(G \odot \mathcal{H}) \leq \sum_{i=1}^{n}(k_i - 1)$, showing that $S = \{u_{i,x}|1 \leq i \leq n, 1 \leq x \leq k_i - 1\}$ is an outer multiset resolving set.

First, we show that for $1 \leq j < l \leq n$, $m(v_j|S) \neq m(v_l|S)$. The multiplicity of distance 1 in $m(v_l|S)$ is $k_l - 1$ and in $m(v_j|S)$ is $k_j - 1$. If $k_j - 1 \neq k_l - 1$, which implies that $k_j \neq k_l$, then $m(v_j|S) \neq m(v_l|S)$. Now, assuming $k_j = k_l$, we consider two subcases.

Subcase 2.i: v_j, v_l are not symmetric pair, i.e. $l \neq n + 1 - j$. We have $\max_{s\in S} d(v_j, s) = \max\{j, n+1-j\}$. Since $l \neq j$ and $l \neq n+1-j$, it follows that $\max_{s\in S} d(v_j, s) \neq \max_{s\in S} d(v_l, s)$. Thus $m(v_j|S) \neq m(v_l|S$.

Subcase 2.ii: v_j, v_l are symmetric pair, i.e. $l = n+1-j$. Since there exists $i \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ with $k_i \neq k_{n+1-i}$, let s be the smallest index in $\{1, \ldots, \lfloor n/2 \rfloor\}$ with $k_s \neq k_{n+1-s}$, and so $t = n + 1 - s$. Then, for $j \leq \lfloor n/2 \rfloor$, $k_{2j-t} = k_{n+1-2j+t}$ for every $2j - t < s$ and indices outside $\{1, \ldots, n\}$ contribute 0. We show that $m(v_j|S) \neq m(v_{n+1-j}|S)$. Consider the distance $t - j + 1$. Since $d(v_j, x) = t - j + 1$ only if $x \in V(H_t) \cup V(H_{2j-t})$, then the multiplicity of $t - j + 1$ in $m(v_j|S)$ is $(k_t - 1) + (k_{2j-t} - 1)$. Similarly, $d(v_{n+1-j}, x) = t - j + 1$ only if $x \in V(H_s) \cup V(H_{n+1-2j+t})$, and so, the multiplicity of $t - j + 1$ in $m(v_{n+1-j}|S)$ is $(k_s - 1) + (k_{n+1-2j+t} - 1)$. Since $2j - t < s$, we have $k_{2j-t} = k_{n+1-(2j-t)}$, and the difference of the two multiplicities is $(k_t - 1) - (k_s - 1) = k_{n+1-s} - k_s \neq 0$. Hence $m(v_j|S) \neq m(v_{n+1-j}|S)$.

Similarly $m(u_{j,k_j}|S) \neq m(u_{l,k_l}|S)$.

Next, consider $1 \leq s, t \leq n$. When H_t is a complete graph, distance 2 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, we have $m(v_s|S) \neq m(u_{t,k_t}|S)$. When H_t is the complement of a complete graph, distance 1 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, we have $m(v_s|S) \neq m(u_{t,k_t}|S)$.

Thus, S is an outer multiset resolving set with $|S| = \sum_{i=1}^{n}(k_i - 1)$. □

In Theorem 3.2, we provide an example of a graph G that is not multiset distance irregular, together with collections $\mathcal{H} = \{H_1, \ldots, H_n\}$, where $H_i \in \{K_{k_i}, \overline{K_{k_i}}\}$, for which $\dim_{ms}(G \odot \mathcal{H})$ equals the lower bound in Theorem 3.1. However, when the k_i are not pairwise distinct, it remains open which other graphs have the same property. This leads to the following question: Characterize all connected graphs G of order n and all collections $\mathcal{H} = \{H_1, \ldots, H_n\}$, with $H_i \in \{K_{k_i}, \overline{K_{k_i}}\}$ and $k_i \geq 2$, for which $\dim_{ms}(G \odot \mathcal{H}) = \sum_{i=1}^{n}(k_i - 1)$.

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We conclude this subsection by asking a general question related to the outer multiset dimension of the extended corona product with complete graphs or their complements.

Problem 3. *Let G be a connected graph of order n and $\mathcal{H} = \{H_1, \ldots, H_n\}$ with $H_i \in \{K_{k_i}, \overline{K_{k_i}}\}$, determine $dim_{ms}(G \odot \mathcal{H})$.*

3.2. Outer Multiset Dimension of Extended Corona Product with Stars

Theorem 3.3. *For $n \geq 3$, let G be a connected graph of order n. If $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$, then*

$$dim_{ms}(G \odot \mathcal{H}) \geq \sum_{i=1}^{n} (k_i - 2).$$

Moreover,

1. *if all k_i are pairwise distinct, then equality holds;*
2. *if all k_i are equal and G is multiset distance irregular, then equality holds.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$. Let $|V(S_{k_i})| = k_i$, $V(S_{k_i}) = \{u_l \mid 1 \leq l \leq k_i\}$, where u_1 is the center of S_{k_i}. Let $V(G \odot \mathcal{H}) = V(G) \cup \{u_{i,l} \mid 1 \leq i \leq n, 1 \leq l \leq k_i\}$, where $u_{i,l}$ is the lth vertex in S_{k_i}.

Clearly, for any $1 \leq i \leq n, 2 \leq l_1, l_2 \leq k_i$, u_{i,l_1} and u_{i,l_2} are twins. By Proposition 1.1, every outer multiset resolving set of $(G \odot \mathcal{H})$ must contain at least one of u_{i,l_1} and u_{i,l_2}. Inductively, every outer multiset resolving set of $(G \odot \mathcal{H})$ contains at least $(k_i - 2)$ vertices of S_{k_i}, $1 \leq i \leq n$. Thus, $dim_{ms}(G \odot \mathcal{H}) \geq \sum_{i=1}^{n}(k_i - 2)$.

For the second part of the theorem, we shall prove that $dim_{ms}(G \odot \mathcal{H}) \leq \sum_{i=1}^{n}(k_i - 2)$. By showing that $S = \{u_{i,y} | 1 \leq i \leq n, 2 \leq y \leq k_i - 1\}$ is an outer multiset resolving set. We have the following two cases:

Case 1: All k_i are pairwise distinct.

First, we show that for $1 \leq i, j \leq n$, $i \neq j$, $m(v_i|S) \neq m(v_j|S)$, $m(u_{i,1}|S) \neq m(u_{j,1}|S)$, and $m(u_{i,k_i}|S) \neq m(u_{j,k_j}|S)$. The number of vertices adjacent to v_i and v_j in S are $(k_i - 2)$ and $(k_j - 2)$. Note that $m(v_i|S) = \{1^{k_i-2}, \ldots\}$ and $m(v_j|S) = \{1^{k_j-2}, \ldots\}$. Since $k_i \neq k_j$ for $1 \leq i \neq j \leq n$, then $k_i - 2 \neq k_j - 2$, which implies that $m(v_i|S) \neq m(v_j|S)$. Similarly, $m(u_{i,k_i}|S) \neq m(u_{j,k_j}|S)$ and $m(u_{i,1}|S) \neq m(u_{j,1}|S)$.

Next, we prove that for $1 \leq s, t \leq n$, $m(v_s|S) \neq m(u_{t,1}|S)$, $m(v_s|S) \neq m(u_{t,k_t}|S)$, and $m(u_{s,1}|S) \neq m(u_{t,k_t}|S)$. Since distance 2 occurs in $m(v_s|S)$, but not in $m(u_{t,1}|S)$, we have $m(v_s|S) \neq m(u_{t,1}|S)$. Distance 1 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, and so $m(v_s|S) \neq m(u_{t,k_t}|S)$. In addition, dsiatnce 1 occurs in $m(u_{s,1}|S)$, but not in $m(u_{t,k_t}|S)$, which means $m(u_{s,1}|S) \neq m(u_{t,k_t}|S)$.

Case 2: All k_i are equal and G is multiset distance irregular.

In this case, $\mathcal{H} = \{S_{k_i} \mid S_{k_i} \cong S_k, 1 \leq i \leq n\}$, the extended corona product $G \odot \mathcal{H}$ is isomorphic to the original corona product $G \odot S_k$. Thus, by Theorem 2.6, S is an outer multiset resolving set. □

Unlike for the corona product, where we have characterized the graph G whose $dim_{ms}(G \odot S_k)$ is equal to the lower bound of Theorem 2.6, we have not yet managed to characterize the graphs

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G whose $dim_{ms}(G \odot \mathcal{H})$, $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$, are equal to the lower bound in Theorem 3.3. Equality is established here in two cases: when all k_i are pairwise distinct for every connected graph G, and when all k_i are equal and G is multiset distance irregular. For multiset distance irregular graphs G, the general intermediate case, where some but not all k_i coincide, is unresolved. In the following example, equality does not hold.

Example 3.2. *Let G be a multiset-distance-irregular graph of order 8, as illustrated in Figure 4. Let $\mathcal{H} = \{S_4, S_5, S_3, S_3, S_3, S_3, S_3, S_4\}$. Let $V(G \odot \mathcal{H}) = V(G) \cup \{u_{i,l} \mid 1 \leq i \leq 8, 1 \leq l \leq k_i\}$ It is clear that $u_{i,1}$ is a central vertex and $u_{i,l}, 1 \leq i \leq 8, 2 \leq l \leq k_i$, are twin vertices. If S were an outer multiset resolving set with $|S| = \sum_{i=1}^{8}(k_i - 2)$, the twin condition would force S to contain exactly $k_i - 2$ vertices of S_{k_i}, $1 \leq i \leq 8$. Then $m(v_3|S) = \{1, 2^4, 3^5, 4^2\} = m(v_4|S)$, a contradiction. Therefore, $|S| \geq \sum_{i=1}^{8}(k_i - 2) + 1$. We shall prove that $|S| \leq \sum_{i=1}^{8}(k_i - 2) + 1$ by considering*

$$S = \{u_{1,2}, u_{1,3}, u_{2,2}, u_{2,3}, u_{2,4}, u_{3,2}, u_{4,2}, u_{5,2}, u_{6,2}, u_{7,2}, u_{8,2}, u_{8,3}\} \bigcup \{u_{3,3}\}.$$

The multiset representations of the vertices are depicted in the following tables.

Vertex z	$m(z\|S)$
v_1	$\{1^2, 2^3, 3^3, 4^2, 5^3\}$
v_2	$\{1^3, 2^5, 3^2, 4^3\}$
v_3	$\{1^2, 2^4, 3^5, 4^2\}$
v_4	$\{1, 2^5, 3^4, 4^3\}$
v_5	$\{1, 2^5, 3^7\}$
v_6	$\{1, 2, 3^4, 4^5, 5^2\}$
v_7	$\{1, 2^4, 3^6, 4^2\}$
v_8	$\{1^2, 2, 3^2, 4^6, 5^2\}$

Vertex z	$m(z\|S)$
$u_{1,1}$	$\{1^2, 3^3, 4^3, 5^2, 6^3\}$
$u_{2,1}$	$\{1^3, 3^5, 4^2, 5^3\}$
$u_{3,1}$	$\{1^2, 3^4, 4^5, 5^2\}$
$u_{4,1}$	$\{1, 3^5, 4^4, 5^3\}$
$u_{5,1}$	$\{1, 3^5, 4^7\}$
$u_{6,1}$	$\{1, 3, 4^4, 5^5, 6^2\}$
$u_{7,1}$	$\{1, 3^4, 4^6, 5^2\}$
$u_{8,1}$	$\{1^2, 3, 4^2, 5^6, 6^2\}$

Vertex z	$m(z\|S)$
$u_{1,4}$	$\{2^2, 3^3, 4^3, 5^2, 6^3\}$
$u_{2,5}$	$\{2^3, 3^5, 4^2, 5^3\}$
$u_{4,3}$	$\{2, 3^5, 4^4, 5^3\}$
$u_{5,3}$	$\{2, 3^5, 4^7\}$
$u_{6,3}$	$\{2, 3, 4^4, 5^5, 6^2\}$
$u_{7,3}$	$\{2, 3^4, 4^6, 5^2\}$
$u_{8,4}$	$\{2^2, 3, 4^2, 5^6, 6^2\}$

Every vertex outside of S has a distinct multiset representation, and so S is an outer multiset resolving set.

This example raises the following question: Let G be a multiset distance irregular graph of order n. Characterize all $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$, where $dim_{ms}(G \odot \mathcal{H}) = \sum_{i=1}^{n}(k_i - 2)$.

In Theorem 3.4, we determine the outer multiset dimension of the extended corona product between any paths and any collection of $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$. The theorem gives an example that the condition of G being multiset distance irregular is not necessary for $dim_{ms}(G \odot \mathcal{H})$, $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$, to be equal to the lower bound in Theorem 3.3. We show that if G is a path, that is not multiset distance irregular, there is a collection $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$ such that $dim_{ms}(G \odot \mathcal{H})$ is equal to the lower bound of Theorem 3.3.

Theorem 3.4. *For $n \geq 3$, $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$. Then*

$$dim_{ms}(P_n \odot \mathcal{H}) = \begin{cases} 1 + \sum_{i=1}^{n}(k_i - 2), \textit{ if } k_i = k_{n+1-i}, \textit{ for all } i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}, \\ \sum_{i=1}^{n}(k_i - 2), \textit{ there exist } i \in \{1, 2, \ldots, \lfloor n/2 \rfloor\} \textit{ such that } k_i \neq k_{n+1-i}. \end{cases}$$

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Proof. Let $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$, $|V(S_{k_i})| = k_i$, $V(S_{k_i}) = \{u_l \mid 1 \leq l \leq k_i\}$, where u_1 is the center of S_{k_i}. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$, and $V(P_n \odot \mathcal{H}) = V(P_n) \cup \{u_{i,l} \mid 1 \leq i \leq n,\ 1 \leq l \leq k_i\}$, where $u_{i,l}$ is the lth vertex in S_{k_i}.

Clearly, for any $1 \leq i \leq n, 2 \leq l_1, l_2 \leq k_i$, u_{i,l_1} and u_{i,l_2} are twins. By Proposition 1.1, every outer multiset resolving set of $(P_n \odot \mathcal{H})$ must contain at least one of u_{i,l_1} and u_{i,l_2}. Inductively, every outer multiset resolving set of $(P_n \odot \mathcal{H})$ contains at least $(k_i - 2)$ vertices of S_{k_i}, $1 \leq i \leq n$. Thus,

$$dim_{ms}(P_n \odot \mathcal{H}) \geq \sum_{i=1}^{n} (k_i - 2). \tag{2}$$

We distinguish two cases:

Case 1: $k_i = k_{n+1-i}$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Suppose that S is an outer multiset resolving set with $|S| = \sum_{i=1}^{n}(k_i - 2)$. Then $m(v_i|S) = m(v_{n+1-i}|S)$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, a contradiction. So, if S is an outer multiset resolving set, it must be $|S| \geq 1 + \sum_{i=1}^{n}(k_i - 2)$. Now we shall prove that there exists S with $1 + \sum_{i=1}^{n}(k_i - 2)$ vertices, by defining

$$S = \{u_{i,y} | 1 \leq i \leq n, 2 \leq y \leq k_i - 1\} \bigcup \{u_{n,k_n}\}.$$

First, we show that for $1 \leq i < j \leq n$, $m(v_i|S) \neq m(v_j|S)$. Let $ecc(v_i)$ be the eccentricity of v_i in P_n. We consider two subcases.

Subcase 1.i: $ecc(v_i) \neq ecc(v_j)$. Then $\max_{s \in S} d(v_i, s) \neq \max_{s \in S} d(v_j, s)$; hence $m(v_i|S) \neq m(v_j|S)$.

Subcase 1.ii: $ecc(v_i) = ecc(v_j)$. Then $j = n - i + 1$, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Inthis case, v_n is the vertex in P_n with $d(v_i, v_n) = ecc(v_i)$. On the other hand, v_1 is the vertex in P_n with $d(v_j, v_1) = ecc(v_j)$. Since $k_i = k_j$, $m(v_i|S) = \{\ldots, (ecc(v_i)+1)^{k_n - 1}\} \neq \{\ldots, (ecc(v_j)+1)^{k_1 - 2}\} = m(v_j|S)$.

Similarly, $m(u_{i,1}|S) \neq m(u_{j,1}|S)$ and $m(u_{i,k_i}|S) \neq m(u_{j,k_j}|S)$.

Next, consider $1 \leq s, t \leq n$. Since distance 1 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, we have $m(v_s|S) \neq m(u_{t,k_t}|S)$. Distance 2 occurs in $m(v_s|S)$, but not in $m(u_{t,1}|S)$, and so $m(v_s|S) \neq m(u_{t,1}|S)$. Finally, distance 1 occurs in $m(u_{s,1}|S)$, but not in $m(u_{t,k_t}|S)$, which means $m(u_{s,1}|S) \neq m(u_{t,k_t}|S)$.

Thus, S is an outer multiset resolving set.

Case 2: There exist $i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ such that $k_i \neq k_{n+1-i}$. By Equation (2), $dim_{ms}(P_n \odot \mathcal{H}) \geq \sum_{i=1}^{n}(k_i - 2)$. Now, we shall prove $dim_{ms}(P_n \odot \mathcal{H}) \leq \sum_{i=1}^{n}(k_i - 2)$, by showing that $S = \{u_{i,y} | 1 \leq i \leq n, 2 \leq y \leq k_i - 1\}$ is an outer multiset resolving set.

First, we show that for $1 \leq j < l \leq n$, $m(v_j|S) \neq m(v_l|S)$. The multiplicity of distance 1 in $m(v_l|S)$ is $k_l - 2$ and in $m(v_j|S)$ is $k_j - 2$. If $k_j - 2 \neq k_l - 2$, which implies that $k_j \neq k_l$, then $m(v_j|S) \neq m(v_l|S)$. Now, assuming $k_j = k_l$, we consider two subcases.

Subcase 2.i: v_j, v_l are not symmetric pairs, i.e., $l \neq n + 1 - j$. We have $\max_{s \in S} d(v_j, s) = \max\{j,\ n+1-j\}$. Since $l \neq j$ and $l \neq n+1-j$, it follows that $\max_{s \in S} d(v_j, s) \neq \max_{s \in S} d(v_l, s)$. Thus, $m(v_j|S) \neq m(v_l|S)$.

Subcase 2.ii: v_j, v_l are symmetric pairs, i.e., $l = n+1-j$. Since there exists $i \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ with $k_i \neq k_{n+1-i}$, let s be the smallest index in $\{1, \ldots, \lfloor n/2 \rfloor\}$ with $k_s \neq k_{n+1-s}$, and so $t = n + 1 - s$. Then, for $j \leq \lfloor n/2 \rfloor$, $k_{2j-t} = k_{n+1-2j+t}$ for every $2j - t < s$ and indices

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outside $\{1, \ldots, n\}$ contribute 0. We show that $m(v_j|S) \neq m(v_{n+1-j}|S)$. Consider the distance $t - j + 1$. Since $d(v_j, x) = t - j + 1$ only if $x \in V(H_t) \cup V(H_{2j-t})$, then the multiplicity of $t - j + 1$ in $m(v_j|S)$ is $(k_t - 2) + (k_{2j-t} - 2)$. Similarly, $d(v_{n+1-j}, x) = t - j + 1$ only if $x \in V(H_s) \cup V(H_{n+1-2j+t})$, and so, the multiplicity of $t - j + 1$ in $m(v_{n+1-j}|S)$ is $(k_s - 2) + (k_{n+1-2j+t} - 2)$. Since $2j - t < s$, we have $k_{2j-t} = k_{n+1-(2j-t)}$, and the difference of the two multiplicities is $(k_t - 2) - (k_s - 2) = k_{n+1-s} - k_s \neq 0$. Hence $m(v_j|S) \neq m(v_{n+1-j}|S)$.

Similarly, $m(u_{j,k_j}|S) \neq m(u_{l,k_l}|S)$ and $m(u_{j,1}|S) \neq m(u_{l,1}|S)$.

Next, consider $1 \leq s, t \leq n$. Since distance 1 occurs in $m(v_s|S)$, but not in $m(u_{t,k_t}|S)$, we have $m(v_s|S) \neq m(u_{t,k_t}|S)$. Distance 2 occurs in $m(v_s|S)$, but not in $m(u_{t,1}|S)$, and so $m(v_s|S) \neq m(u_{t,1}|S)$. Finally, distance 1 occurs in $m(u_{s,1}|S)$, but not in $m(u_{t,k_t}|S)$, which means $m(u_{s,1}|S) \neq m(u_{t,k_t}|S)$. □

In Theorem 3.4, we provide an example of a graph G that is not multiset distance irregular, together with a collection $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$, for which $\dim_{ms}(G \odot \mathcal{H})$ equals the lower bound in Theorem 3.3. However, when the k_i are not pairwise distinct, it remains open which other graphs have this property. This leads to the following question: Characterize all connected graphs G of order n and all collections $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$, for which $dim_{ms}(G \odot \mathcal{H}) = \sum_{i=1}^{n}(k_i - 2)$.

We conclude this subsection by asking a general question related to the outer multiset dimension of the extended corona product with stars.

Problem 4. *Let G be a connected graph and $\mathcal{H} = \{ S_{k_i} \mid k_i \geq 3,\ 1 \leq i \leq n \}$, determine $dim_{ms}(G \odot \mathcal{H})$.*

4. Concluding Remarks

The results of this paper suggest three main directions for further research. First, close the gap between the lower and upper bounds of the outer multiset dimension of corona-type products, particularly by determining which bounds are sharp and characterizing the graphs that attain or come close to them. The second is to investigate how the outer multiset dimension depends on the choice of the graphs H and the collection of graphs $\mathcal{H}$. And lastly, we aim to extend the study to other graph products, such as the Cartesian and strong products.

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