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Some algebraic invariants of ideals associated with graph families

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Abstract

In this article, we study some algebraic invariants of monomial ideals associated with graphs. Specifically, we compute the depth and Stanley depth of edge ideals arising from sunlet graphs, cycle-related graphs, and wheel-related graphs. In addition, we show that the path ideal of length two in centipede-related graphs satisfies both the persistence and strong persistence properties.

Keywords: depth, Stanley depth, persistence property, strong persistence property, edge ideals
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1. Introduction

Throughout this paper, $\mathcal{A}$ denotes a commutative Noetherian ring over a field K. For any graph G, we write $I(G)$ to denote its edge ideal. In particular, we use the notations $I(S'_{2n})$, $I(C'_n)$, $I(W'_n)$, $I(\mathcal{Z}_{2n})$, and $I(L_n)$ to represent the edge ideals of the sunlet graph, cycle-related graph, wheel-related graph, centipede graph, and line graph, respectively. Moreover, $P_2(G)$ and $P_2(G')$ denote the path ideals of length 2 associated with the graphs G and G', respectively.

The *edge ideal* $I(G)$ of a simple graph G is the monomial ideal generated by $\{x_ix_j \mid \{x_i, x_j\} \in E(G)\}$. The edge ideal was first introduced by Villarreal [22]. The *path ideal of length* 2 of G,

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is the monomial ideal generated by all monomials corresponding to paths of length 2 in G, i.e, $P_2(G) = (x_ix_jx_k : x_i - x_j - x_k$ is a path of length 2 in $G)$. The *depth* of a finitely generated $\mathcal{A}$-module M, is the length of a maximal M-regular sequence in $\mathcal{A}$. The Stanley depth is a combinatorial invariant defined using Stanley decompositions. A Stanley decomposition of a $\mathbb{Z}^n$-graded $\mathcal{A}$-module M is a representation

$$M = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

where each u_i is a homogeneous element of M and $Z_i \subseteq \{x_1, \ldots, x_n\}$. The Stanley depth of this decomposition is $\min |Z_i|$, and the Stanley depth of M, is the maximum Stanley depth taken over all such decompositions.

The study of depth and Stanley depth of monomial ideals has attracted considerable attention. Brodmann [6] proved that $depth(\mathcal{A}/I^n)$ is constant when n is sufficiently large. Herzog et al. [13] proposed a method for computing the Stanley depth of quotients of monomial ideals, which was later refined into an effective algorithm by Rinaldo [20] and implemented in the computer algebra system CoCoA [8]. Cimpoeaș [7] investigated the path ideal associated with a line graph and computed the Stanley depth of its quotient ring, observing that it coincides with its depth; in particular, the Stanley inequality holds in this case. Zahid et al.[25] studied the depth and Stanley depth of powers of edge ideals associated with caterpillar and lobster trees, and established sharp lower bounds for these invariants.

A prime ideal $p \subset \mathcal{A}$ is an associated prime of an ideal I if there exists an element v in $\mathcal{A}$ such that $p = (I : v)$, where $(I : v) = \{r \in \mathcal{A} : rv \in I\}$. The set of associated primes of I, denoted by $Ass(\mathcal{A}/I)$. An ideal $I \subseteq \mathcal{A}$ is said to exhibit the *persistence property* if $Ass(\mathcal{A}/I^k) \subseteq Ass(\mathcal{A}/I^{k+1})$ for all $k \geq 1$. It holds the strong persistence property if $(I^{k+1} : I) = I^k$. Brodmann [5] showed that $Ass(\mathcal{A}/I^n)$ is stabilizes for large n. Persistence and strong persistence properties have been widely studied in the literature; see, [19], [23], [12], [16], and [14]. Herzog et al.[11] gave results on the persistence and stability of powers of ideals. They also proved that a strong persistence property implies a persistence property. Nasernejad [17] proved that if an ideal in a ring $\mathcal{A}$ admits a strongly superficial element of degree one, then it satisfies the persistence property. Furthermore, Rajaee et al. [21] established that an ideal has a superficial element if and only if it satisfies the strong persistence property. Al-Ayyoub et al. [2], [1] investigated the normality and strong persistence property of cover ideals associated with theta graphs and Jahangir graphs, respectively. They proved that these cover ideals are normal squarefree monomial ideals and studied their associated primes. A. Bretto et al. [4], show that any square-free monomial ideal in $K[x_1, x_2, x_3, x_4, x_5]$ has the strong persistence property.

In [3], Bibi et al. computed the depth of the edge ideals of centipede and centipede-related graphs. Motivated by these results, we determine the depth of the edge ideal of the sunlet graph (see Theorem 2.1). Furthermore, [3] also discusses the persistence and strong persistence properties of centipede-related graphs. We extend this study by examining the strong persistence property of the path ideal of length two in a generalized form of centipede-related graphs, see Theorem 3.1 and 3.2. All calculations used to obtain these results are performed using the computer algebra system SINGULAR [10] and Macaulay2 [9].

This paper is organized as follows. The first section focuses on computing the depth and

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Stanley depth of edge ideals associated with sunlet, wheel-related, and cycle-related graphs. In the second part, we extend the study of the strong persistence property of the path ideal of length two in a generalized family of centipede-related graphs. (see Theorems 3.1 and 3.2).

2. Depth and Stanley depth of Edge Ideals

2.1. Sunlet Graph

A sunlet graph is obtained from a cycle by attaching one pendant edge to every vertex of the cycle. We denote this graph by S'_{2n}, where n is the number of vertices in the underlying cycle. The vertex set of S'_{2n} is $\{x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{2n}\}$. The first n vertices form a cycle, while each vertex x_i for $1 \leq i \leq n$ is joined to a new pendant vertex x_{n+i}. Accordingly, the edge set will be $\{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{x_i x_{n+i} : 1 \leq i \leq n\}$. The following graph is a sunlet graph.

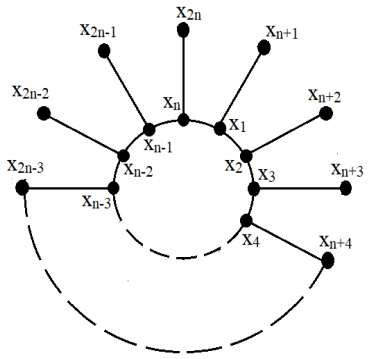


Figure 1. Graph S'_{2n}

Theorem 2.1. *Let* $S'_{2n}, n \geq 3$ *be a sunlet graph with* $2n$ *vertices, then* $\operatorname{depth}\big(\mathcal{A}/I(S'_{2n})\big) = n$.

Proof. We consider the following short exact sequence.

$$0 \to \mathcal{A}/\big(I(S'_{2n}) : x_n\big) \to^{x_n} \mathcal{A}/I(S'_{2n}) \to \mathcal{A}/\big(I(S'_{2n}), x_n\big) \to 0$$

where

$$\begin{aligned} I(S'_{2n}) = (&x_1x_2,\ x_2x_3,\ x_3x_4,\ \ldots,\ x_{n-3}x_{n-2},\ x_{n-2}x_{n-1},\ x_{n-1}x_n,\ x_nx_1,\\ &x_1x_{n+1},\ x_2x_{n+2},\ x_3x_{n+3},\ \ldots,\ x_{n-2}x_{2n-2},\ x_{n-1}x_{2n-1},\ x_nx_{2n}) \end{aligned}$$

we have

$$\begin{aligned} \big(I(S'_{2n}) : x_n\big) &= (x_1x_2,\ x_2x_3,\ x_3x_4,\ \ldots,\ x_{n-3}x_{n-2},\ x_{n-2}x_{n-1},\ x_{n-1},\ x_1,\\ &\quad x_1x_{n+1},\ x_2x_{n+2},\ x_3x_{n+3},\ \ldots,\ x_{n-2}x_{2n-2},\ x_{n-1}x_{2n-1},\ x_{2n}).\\ &= (x_2x_3,\ x_3x_4,\ \ldots,\ x_{n-3}x_{n-2},\ x_{n-1},\ x_1,\\ &\quad x_2x_{n+2},\ x_3x_{n+3},\ \ldots,\ x_{n-3}x_{2n-3},\ x_{n-2}x_{2n-2},\ x_{2n}). \end{aligned}$$

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$$\begin{aligned}\operatorname{depth}\big(\mathcal{A}/(I(S'_{2n}):x_n)\big) =& \operatorname{depth}\big(K[x_1,x_2,x_3,\ldots,x_{n-1},x_n,x_{n+1},\ldots,x_{2n-2},x_{2n-1},x_{2n}]/(x_2x_3,\\ & x_3x_4,\ \ldots,\ x_{n-3}x_{n-2},\ x_{n-1},\ x_1,\ x_2x_{n+2},\ x_3x_{n+3},\ \ldots,\ x_{n-3}x_{2n-3},\\ & x_{n-2}x_{2n-2},\ x_{2n})\big)\\ =& \operatorname{depth}\big(K[x_2,x_3,\ldots,x_{n-2},x_n,x_{n+1},\ldots,x_{2n-2},x_{2n-1}]/(x_2x_3,\ \ldots,\\ & x_{n-4}x_{n-3},\ x_{n-3}x_{n-2},\ x_2x_{n+2},\ x_3x_{n+3},\ \ldots,\ x_{n-3}x_{2n-3},\ x_{n-2}x_{2n-2})\big)\\ =& \operatorname{depth}\big(\mathcal{A}'/I(\mathcal{Z}_{2n-6})\big)+3\\ =& \frac{2n-6}{2}+3=n\end{aligned}$$

where $I(\mathcal{Z}_{2n-6})$ is the edge ideal of the centipede graph on the ring $\mathcal{A}'$ with variables $2n-6$. By using [3, Theorem 2.1] $\operatorname{depth}\big(\mathcal{A}'/I(\mathcal{Z}_{2n-6})\big)=\frac{2n-6}{2}$.

$$(I(S'_{2n}),\ x_n) \ = \ (x_1x_2,\ x_2x_3,\ \ldots,\ x_{n-2}x_{n-1},\ x_1x_{n+1},\ x_2x_{n+2},\ \ldots,\ x_{n-2}x_{2n-2},\ x_{n-1}x_{2n-1},\ x_n).$$

$$\begin{aligned}\operatorname{depth}(\mathcal{A}/(I(S'_{2n}),\ x_n)) =& \operatorname{depth}\big(K[x_1,x_2,x_3,\ldots,x_{n-1},x_n,x_{n+1},\ldots,x_{2n-1},x_{2n}]/(x_1x_2,\\ & x_2x_3,\ \ldots,\ x_{n-2}x_{n-1},\ x_1x_{n+1},\ x_2x_{n+2},\ \ldots,\ x_{n-2}x_{2n-2},\ x_{n-1}x_{2n-1},\ x_n)\big)\\ =& \operatorname{depth}\big(K[x_1,x_2,x_3,\ldots,x_{n-1},x_{n+1},\ldots,x_{2n-1},x_{2n}]/(x_1x_2,\\ & x_2x_3,\ \ldots,\ x_{n-2}x_{n-1},\ x_1x_{n+1},\ x_2x_{n+2},\ \ldots,\ x_{n-2}x_{2n-2},\ x_{n-1}x_{2n-1})\big)\\ =& \operatorname{depth}\big(\mathcal{A}''/I(\mathcal{Z}_{2n-2})\big)+1\\ =& \frac{2n-2}{2}+1=n\end{aligned}$$

where $I(\mathcal{Z}_{2n-2})$ denotes the edge ideal of the centipede graph defined over the ring $\mathcal{A}''$ with $2n-2$ variables. By applying [3, Theorem 2.1], we obtain $\operatorname{depth}(\mathcal{A}''/I(\mathcal{Z}_{2n-2}))=\frac{2n-2}{2}$. Furthermore, an application of [24, Lemma 3.1.4] yields

$$\operatorname{depth}\big(\mathcal{A}/I(S'_{2n})\big)=n.$$

□

Lemma 2.1. *Let $\mathcal{Z}_n$ be the centipede graph of n vertices where $n\geq 2$, then*

$$\operatorname{sdepth}\big(\mathcal{A}/I(\mathcal{Z}_n)\big)\geq\frac{n}{2}.$$

Proof. Consider the following short exact sequence

$$0\to\mathcal{A}/\big(I(\mathcal{Z}_n):x_{n-1}\big)\to^{x_{n-1}}\mathcal{A}/I(\mathcal{Z}_n)\to\mathcal{A}/\big(I(\mathcal{Z}_n),x_{n-1}\big)\to 0$$

where $I(\mathcal{Z}_n)$ = $(x_1x_2,\ x_1x_3,\ x_3x_4,\ x_3x_5,\ \ldots,\ x_{n-2}x_{n-3},\ x_{n-3}x_{n-1},\ x_{n-1}x_n)$.
We have $\big(I(\mathcal{Z}_n):x_{n-1}\big)$ = $\big(I(\mathcal{Z}_{n-4}),\ x_{n-3},\ x_n\big)$.

$$\begin{aligned}\operatorname{sdepth}\big(\mathcal{A}/(I(\mathcal{Z}_n):x_{n-1})\big) &= \operatorname{sdepth}\big(K[x_1,x_2,\ldots,x_{n-1},x_n]/(I(\mathcal{Z}_{n-4}),\ x_{n-3},\ x_n)\big)\\ &= \operatorname{sdepth}\big(\mathcal{A}'/I(\mathcal{Z}_{n-4})\big)+2=\frac{n-4}{2}+2=\frac{n}{2}\end{aligned}$$

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where $\mathcal{A}'$ is the ring on $n-4$ variables. Thus $\operatorname{sdepth}\big(\mathcal{A}/(I(\mathcal{Z}_n) : x_{n-1})\big) = \frac{n}{2}$. Note that $(I(\mathcal{Z}_n), x_{n-1}) = (I(\mathcal{Z}_{n-2}), x_{n-1})$.

$$\begin{aligned}\operatorname{sdepth}\big(\mathcal{A}/(I(\mathcal{Z}_n), x_{n-1})\big) &= \operatorname{sdepth}\big(K[x_1, x_2, \ldots, x_{n-1}, x_n]/(I(\mathcal{Z}_{n-2}), x_{n-1})\big)\\ &= \operatorname{sdepth}\big(\mathcal{A}''/I(\mathcal{Z}_{n-2})\big) + 1\\ &= \frac{n-2}{2} + 1 = \frac{n}{2}\end{aligned}$$

where $\mathcal{A}''$ is the ring having variables $n-2$. So $\operatorname{sdepth}\big(\mathcal{A}/(I(\mathcal{Z}_n), x_{n-1})\big) = \frac{n}{2}$ and by using [18, Lemma 2.2] we get $\operatorname{sdepth}\big(\mathcal{A}/I(\mathcal{Z}_n)\big) \geq \frac{n}{2}$.

□

Theorem 2.2. *For $n \geq 3$, the Stanley depth of $\mathcal{A}/I(S'_{2n})$ satisfies*

$$\operatorname{sdepth}\big(\mathcal{A}/I(S'_{2n})\big) \geq n.$$

Proof. Consider the following short exact sequence.

$$0 \to \mathcal{A}/\big(I(S'_{2n}) : x_n\big) \to^{x_n} \mathcal{A}/I(S'_{2n}) \to \mathcal{A}/\big(I(S'_{2n}), x_n\big) \to 0$$

$$\begin{aligned}\operatorname{sdepth}\big(\mathcal{A}/(I(S'_{2n}) : x_n)\big) &= \operatorname{sdepth}\big(\mathcal{A}'/I(\mathcal{Z}_{2n-6})\big) + 3\\ &= \frac{2n-6}{2} + 3 = n\end{aligned}$$

where $I(\mathcal{Z}_{2n-6})$ is the edge ideal of centipede graph over the ring $\mathcal{A}'$ on $2n-6$ variables. Thus, by using Lemma 2.1, we obtain $\operatorname{sdepth}\big(A'/I(\mathcal{Z}_{2n-6})\big) = n-3$.

$$\begin{aligned}\operatorname{sdepth}\big(\mathcal{A}/(I(S'_{2n}), x_n)\big) &= \operatorname{sdepth}\big(\mathcal{A}''/I(\mathcal{Z}_{2n-2})\big) + 1\\ &= \frac{2n-2}{2} + 1 = n\end{aligned}$$

where $I(\mathcal{Z}_{2n-2})$ is the edge ideal of the centipede graph over the ring $\mathcal{A}''$ on $2n-2$ variables. Thus, by using Lemma 2.1, we obtain $\operatorname{sdepth}\big(A''/I(\mathcal{Z}_{2n-2})\big) = n-1$. Furthermore, applying [18, Lemma 2.2], we derive the inequality $\operatorname{sdepth}\big(\mathcal{A}/I(S'_{2n})\big) \geq n$. □

2.2. Cycle-related Graph

The Cycle-related graph is formed by attaching one edge to the cycle graph. The vertex set is $\{x_1, x_2, \ldots, x_{n-2}, x_{n-1}, x_n\}$, and the edge set is

$$\{x_i x_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{x_1 x_{n-1}\}.$$

It is denoted as C'_n, where n is the total number of vertices. The following is a cycle-related graph.

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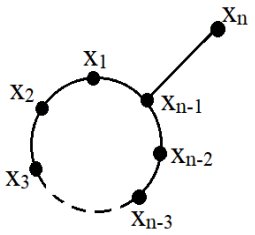


Figure 2. Graph C'_n

Theorem 2.3. *Let C'_n, $n \geq 3$ be a cycle-related graph, then*

$$\operatorname{depth}\left(\mathcal{A}/I(C'_n)\right) = \left\lceil \frac{n-1}{3} \right\rceil.$$

Proof. Consider the following short exact sequence.

$$0 \to \mathcal{A}/\big(I(C'_n) : x_{n-1}\big) \to^{x_{n-1}} \mathcal{A}/I(C'_n) \to \mathcal{A}/\big(I(C'_n), x_{n-1}\big) \to 0$$

where $I(C'_n) = (x_1x_2, x_2x_3, \ldots, x_{n-3}x_{n-2}, x_{n-2}x_{n-1}, x_{n-1}x_1, x_{n-1}x_n)$. Then $\big(I(C'_n) : x_{n-1}\big) = \big(x_1,\ I(L_{n-4}),\ x_{n-2},\ x_n\big)$, here $I(L_{n-4})$ is the edge ideal of the line graph having $n-4$ vertices.

$$\begin{aligned}\operatorname{depth}\left(\mathcal{A}/(I(C'_n) : x_{n-1})\right) &= \operatorname{depth}\left(K[x_1, x_2, \ldots, x_n]/(x_1,\ I(L_{n-4}),\ x_{n-2},\ x_n)\right)\\ &= \operatorname{depth}\left(K[x_{n-1}]/I(L_{n-4})\right)\\ &= \operatorname{depth}\left(K/[I(L_{n-4})]\right) + 1.\end{aligned}$$

By using [15, Lemma 2.8], the $\operatorname{depth}\left(\mathcal{A}/(I(C'_n) : x_{n-1})\right) = \lceil \frac{n-4}{3} \rceil + 1 = \lceil \frac{n-1}{3} \rceil$. Now $\big(I(C'_n), x_{n-1}\big) = \big(I(L_{n-2}), x_{n-1}\big)$, here $I(L_{n-2})$ is the edge ideal of the line graph having $n-2$ vertices.

$$\begin{aligned}\operatorname{depth}\left(\mathcal{A}/(I(C'_n), x_{n-1})\right) &= \operatorname{depth}\left(K[x_1, x_2, \ldots, x_n]/(L_{n-2},\ x_{n-1})\right)\\ &= \operatorname{depth}\left(K/[I(L_{n-2})]\right) + 1.\end{aligned}$$

By applying [15, Lemma 2.8], we obtain $\operatorname{depth}\left(\mathcal{A}/(I(C'_n), x_{n-1})\right) = \lceil \frac{n-2}{3} \rceil + 1$. Furthermore, it follows from [24, Lemma 3.1.4] that

$$\operatorname{depth}\left(\mathcal{A}/I(C'_n)\right) = \left\lceil \frac{n-1}{3} \right\rceil.$$

□

Theorem 2.4. *For $n \geq 3$, the Stanley depth of $\mathcal{A}/I(C'_n)$ satisfies*

$$\operatorname{sdepth}\left(\mathcal{A}/I(C'_n)\right) \geq \lceil \frac{n-1}{3} \rceil.$$

Proof. Consider the following short exact sequence.

$$0 \to \mathcal{A}/\big(I(C'_n) : x_{n-1}\big) \to \mathcal{A}/I(C'_n) \to \mathcal{A}/\big(I(C'_n), x_{n-1}\big) \to 0.$$

As $\operatorname{sdepth}(\mathcal{A}/(I(C'_n) : x_{n-1})) = \lceil \frac{n-4}{3} \rceil + 1 = \lceil \frac{n-1}{3} \rceil$, and $\operatorname{sdepth}(\mathcal{A}/(I(C'_n), x_{n-1})) = \lceil \frac{n-2}{3} \rceil + 1$. By using [18, Lemma 2.2], we get $\operatorname{sdepth}\left(\mathcal{A}/I(C'_n)\right) \geq \lceil \frac{n-1}{3} \rceil$. □

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2.3. Wheel-related graph

A wheel-related graph is a graph obtained by adding an edge at any vertex, except a common vertex, to a wheel graph. It is denoted as W'_n.

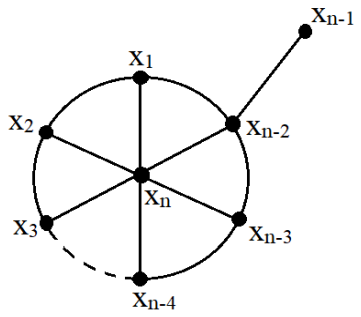


Figure 3. Graph W'_n

Theorem 2.5. *Let W'_n be a wheel-related graph then*

$$\operatorname{depth}\left(\mathcal{A}/I(W'_n)\right) = \begin{cases} 1 & \text{if } n = 4, 5, \\ 2 & \text{if } n \geq 6. \end{cases}$$

Proof. Consider the following short exact sequence.

$$0 \to \mathcal{A}/\big(I(W'_n) : x_n\big) \to \mathcal{A}/I(W'_n) \to \mathcal{A}/\big(I(W'_n), x_n\big) \to 0$$

For $n = 4$, we have $I(W'_4) = (x_1x_2, x_2x_3, x_1x_4, x_2x_4)$. So $\operatorname{depth}(\mathcal{A}/I(W'_4) : x_4) = 1$, and By using [15, Lemma 2.8] we get, $\operatorname{depth}\left(\mathcal{A}/I(W'_4), x_4\right) = \operatorname{depth}\left(\mathcal{A}[x_1, x_2, x_3, x_4]/(x_1x_2, x_2x_3, x_4)\right) = \lceil \frac{3}{3} \rceil = 1$. By applying [24, Lemma 3.1.4], $\operatorname{depth}(\mathcal{A}/I(W'_4)) = 1$.
For $n = 5$, we have $I(W'_5) = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_3x_1,\ x_1x_5,\ x_2x_5,\ x_3x_5)$. So $\operatorname{depth}\left(\mathcal{A}/I(W'_5) : x_5\right) = 1$, and By applying Theorem 2.3 we get, $\operatorname{depth}\left(\mathcal{A}/I(W'_5), x_5\right) = \operatorname{depth}\left(\mathcal{A}[x_1, x_2, x_3, x_4, x_5]/(x_1x_2, x_2x_3, x_3x_4, x_3x_1, x_5)\right) = \lceil \frac{3}{3} \rceil = 1$. By using [24, Lemma 3.1.4], $\operatorname{depth}(\mathcal{A}/I(W'_5)) = 1$.
Now we will prove it for $n \geq 6$. $I(W'_n) = (x_1x_2,\ x_2x_3,\ \ldots,\ x_{n-4}x_{n-3},\ x_{n-3}x_{n-2},\ x_{n-2}x_{n-1},\ x_{n-2}x_1,\ x_1x_n,\ x_2x_n,\ \ldots,\ x_nx_{n-2})$. Then $(I(W'_n) : x_n)) = (x_1, x_2, x_3, \ldots, , x_{n-2})$.

$$\operatorname{depth}\left(\mathcal{A}/(I(W'_n) : x_n)\right) \;=\; \operatorname{depth}\left(K[x_1, x_2, \ldots, x_n]/(x_1, x_2, \ldots, x_{n-2})\right) = 2.$$

Similarly, $(I(W'_n), x_n)) = (x_1x_2,\ x_2x_3,\ \ldots,\ x_{n-4}x_{n-3},\ x_{n-3}x_{n-2},\ x_{n-2}x_{n-1},\ x_{n-2}x_1,\ x_1x_n,\ x_2x_n,\ \ldots,\ x_nx_{n-2},\ x_n)$. This implies

$$(I(W'_n), x_n)) = (I(C'_{n-1}), x_n)$$

where $I(C'_{n-1})$ is the edge ideal of the cycle-related graph in the ring $\mathcal{A} = K[x_1, x_2, \ldots, x_{n-1}]$. Thus by Theorem 2.3, we get

$$\operatorname{depth}\left(\mathcal{A}/(I(W'_n), x_n)\right) = \left\lceil \frac{n-2}{3} \right\rceil.$$

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Hence by using [24, Lemma 3.1.4], we get $\operatorname{depth}\left(\mathcal{A}/(I(W_n')\right) = 2$. □

Theorem 2.6. *Let W_n' be wheel related graph, then*

$$\operatorname{sdepth}\left(\mathcal{A}/I(W_n')\right) \geq \begin{cases} 1, & \text{if } n = 4, 5, \\ 2, & \text{if } n \geq 6. \end{cases}$$

Proof. Consider a short exact sequence.

$$0 \to \mathcal{A}/\big(I(W_n') : x_n\big) \to \mathcal{A}/I(W_n') \to \mathcal{A}/\big(I(W_n'), x_n\big) \to 0$$

then clearly, by using [18, Lemma 2.2], we obtain the inequalities. □

3. Strong Persistence Property of Path Ideal of Length 2 in Centipede Related Graph

A centipede graph is a graph on the vertex set $\{u_1, u_2, \ldots, u_n\}$ and the edge set is $\{\{u_i, u_{i+1}\} : i = 1, 3, 5, 7, \ldots, n-1\} \cup \{\{u_j, u_{j+2}\} : j = 1, 3, 5, 7, \ldots, n-3\}$. The following graph is a centipede graph.

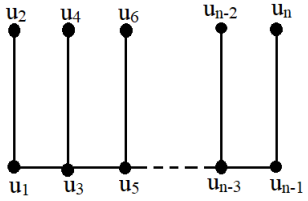


Figure 4. Centipede Graph

Theorem 3.1. *Let $\mathcal{Z}_{2n}$, with $n \geq 2$, be a centipede graph. Define a graph G_n with vertex set $V(\mathcal{Z}_{2n}) \cup \{2n+1, 2n+2, \ldots, 2n+m\}$, and the edge set is $I(\mathcal{Z}_{2n}) \cup \{\{3, 2n+1\}, \{5, 2n+2\}, \{7, 2n+3\}, \ldots, \{2n-1, 2n+m\}\}$, where $m \in \mathbb{N}$. We call G_n a* centipede-related graph. *Then the path ideal of length two $P_2(G_n)$, satisfies the strong persistence property.*

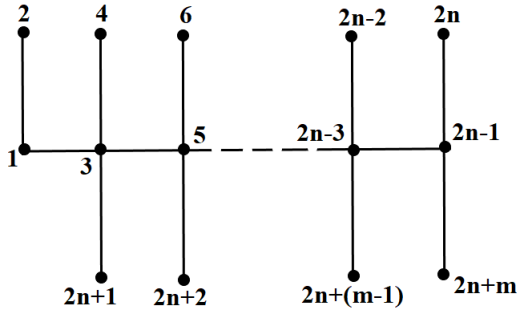


Figure 5. Graph G_n

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Proof. Let $H = P_2(G_n)$. Then the generating set of H is

$$\begin{aligned}\mathcal{G}(H) =&\{x_{2k-1}x_{2k}x_{2k+1},\ x_{2k-1}x_{2k+1}x_{2k+2} :\ k = 1, 2, \ldots, n-1\}\\ &\cup \{x_{2r-1}x_{2r+1}x_{2r+3} :\ r = 1, 2, \ldots, n-2\}\\ &\cup \{x_{2t-1}x_{2t+1}x_{2n+t},\ x_{2t+2}x_{2t+1}x_{2n+t} :\ t = 1, 2, \ldots, n-1\}\\ &\cup \{x_{2t+1}x_{2t+3}x_{2n+t} :\ t = 1, 2, \ldots, n-2\}.\end{aligned}$$

It is clear that the ideal H has exactly unique minimal generators $6n-8$, $n \in \mathbb{N}$. Suppose that $\mathcal{G}(H) = \{v_1, v_2, \ldots, v_k\}$ with $k = 6n-8$ such that $v_1 = x_1x_2x_3,\ v_2 = x_1x_3x_{2n+1},\ v_3 = x_3x_4x_5,\ v_4 = x_3x_5x_7,\ v_5 = x_3x_5x_6,\ v_6 = x_3x_4x_{2n+1},\ v_7 = x_3x_5x_{2n+2},\ v_8 = x_3x_5x_{2n+1}$ and so on. We claim that

$$(H^{m+1} : v_1) = H^m \quad \forall \quad m \in \mathbb{N}.$$

To prove the claim, pick any $m \in \mathbb{N}$. Then

$$\begin{aligned}(H^{m+1} : v_1) &= (H^mH : v_1)\\ &= (H^m(v_1, v_2, v_3, \ldots, v_k) : v_1)\\ &= \sum_{i=1}^{k}(H^mv_i : v_1)\\ &= H^m + \sum_{i=2}^{k}(H^mv_i : v_1).\end{aligned}$$

To complete the evidence, we show that

$$(H^mv_i : v_1) \subseteq H^m \quad \forall \quad i = 2, 3, 4, \ldots, k.$$

To do this, pick any arbitrary element $u \in (H^mv_i :\ v_1)$ for some $2 \leq i \leq k$. This implies $uv_1 \in H^mv_i$ and so there exist a generator of H^m such as $v_1^{b_1}v_2^{b_2}v_3^{b_3}\ldots v_k^{b_k}$ with $b_1 + b_2 + b_3 + \ldots + b_k = m$ such that $v_iv_1^{b_1}v_2^{b_2}v_3^{b_3}\ldots v_k^{b_k}$ divides uv_1. There exist a monomial $w \in \mathcal{A}$ such that $uv_1 = wv_iv_1^{b_1}v_2^{b_2}v_3^{b_3}\ldots v_k^{b_k}$. If $b_1 \geq 1$ then $u = wv_iv_1^{b_1-1}v_2^{b_2}v_3^{b_3}\ldots v_k^{b_k}$ as $v_iv_1^{b_1-1}v_2^{b_2}v_3^{b_3}\ldots v_k^{b_k} \in H^m$, consequently, $u \in H^m$. If $b_1 = 0$ then $b_2 + b_3 + \ldots + b_k = m$. It follows that $ux_1x_3 = w^{'}v_iv_2^{b_2}v_3^{b_3}\ldots v_k^{b_k}$ such that $w = x_2w^{'}$ for any monomial $w^{'} \in \mathcal{A}$, since $x_2 \notin v_j$ for all $j = 2, 3, 4, \ldots, k$.

As a result, we obtain the following cases.

Case 1. If $v_i = v_2 = x_1x_3x_4$ or $v_i = v_3 = x_1x_3x_5$ or $v_i = v_4 = x_1x_3x_{2n+1}$. Then $u = w^{'}x_4v_2^{b_2}v_3^{b_3}\ldots v_k^{b_k}$ or $u = w^{'}x_5v_2^{b_2}v_3^{b_3}\ldots v_k^{b_k}$ or $u = w^{'}x_{2n+1}v_2^{b_2}v_3^{b_3}\ldots v_k^{b_k}$. This implies $u \in H^m$.

Case 2. If $v_i = v_5 = x_3x_4x_5$ then $ux_1x_3 = w^{'}x_3x_4x_5v_2^{b_2}v_3^{b_3}v_4^{b_4}\ldots v_k^{b_k}$, this implies $ux_1 = w^{'}x_4x_5v_2^{b_2}v_3^{b_3}v_4^{b_4}\ldots v_k^{b_k}$. Therefore, we take the following subcases.

Subcase 2.1. If $b_2 \geq 1$ or $b_3 \geq 1$ or $b_4 \geq 1$ then

$$\begin{aligned}&\Rightarrow u = w^{'}x_4v_5v_2^{b_2-1}v_3^{b_3}v_4^{b_4}\ldots v_k^{b_k}\\ \text{or}\quad &u = w^{'}x_5v_5v_2^{b_2}v_3^{b_3-1}v_4^{b_4}v_5^{b_5}v_6^{b_6}\ldots v_k^{b_k}\\ \text{or}\quad &u = w^{'}x_5v_8v_2^{b_2-1}v_3^{b_3}v_4^{b_4-1}v_5^{b_5}v_6^{b_6}\ldots v_k^{b_k}.\end{aligned}$$

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This implies $u \in H^m$.

Subcase 2.2. If $b_2 = 0$, $b_3 = 0$ and $b_4 = 0$, then $ux_1 = w^{'}x_4x_5v_5^{b_5}v_6^{b_6}\dots v_k^{b_k}$. As $x_1 \notin v_j \ \forall \ j = 5, 6, 7, \dots, k$, it follows that $u = w^{''}x_4x_5v_5^{b_5}v_6^{b_6}\dots v_k^{b_k}$ such that $w^{'} = x_1w^{''}$ for any monomial $w^{''} \in \mathcal{A}$. This implies $u \in H^m$.

Case 3. If $v_i = v_r$ with $r \geq 6$, then $ux_1x_3 = w^{'}v_rv_2^{b_2}v_3^{b_3}\dots v_k^{b_k}$. Therefore, we take the following subcases:

Subcase 3.1. If $b_2 \geq 1$ or $b_3 \geq 1$ or $b_4 \geq 1$, then the proof follows the same steps as Subcase 2.1.

Subcase 3.2. If $b_2 = 0$ and $b_3 = 0$ and $b_4 = 0$ then $ux_1x_3 = w^{'}v_rv_5^{b_5}v_6^{b_6}v_7^{b_7}\dots v_k^{b_k}$.
As $x_1 \notin v_j \ \forall \ j = 5, 6, \dots, k$, this implies $ux_3 = w^{''}v_rv_5^{b_5}v_6^{b_6}v_7^{b_7}\dots v_k^{b_k}$ such that $w^{'} = x_1w^{''}$ for any monomial $w^{''} \in \mathcal{A}$.
If $b_5 \geq 1$ or $b_6 \geq 1$ or $b_7 \geq 1$ or $b_8 \geq 1$ or $b_9 \geq 1$ or $b_{10} \geq 1$, then

$$\begin{aligned}
&\Rightarrow u = w^{''}x_4x_5v_rv_5^{b_5-1}v_5^{b_5}v_6^{b_6}v_7^{b_7}\dots v_k^{b_k}\\
\text{or}\quad &u = w^{''}x_5x_7v_rv_5^{b_5-1}v_6^{b_6-1}v_7^{b_7}v_8^{b_8}\dots v_k^{b_k}\\
\text{or}\quad &u = w^{''}x_5x_6v_rv_5^{b_5}v_6^{b}v_7^{b_7-1}v_8^{b_8}\dots v_k^{b_k}\\
\text{or}\quad &u = w^{''}x_4x_{2n+1}v_rv_5^{b_5}v_6^{b}v_7^{b_7}v_8^{b_8-1}v_9^{b_9}v_{10}^{b_{10}}\dots v_k^{b_k}\\
\text{or}\quad &u = w^{''}x_5x_{2n+2}v_rv_5^{b_5}v_6^{b}v_7^{b_7}v_8^{b_8}v_9^{b_9-1}v_{10}^{b_{10}}v_{11}^{b_{11}}v_{12}^{b_{12}}\dots v_k^{b_k}\\
\text{or}\quad &u = w^{''}x_5x_{2n+1}v_rv_5^{b_5}v_6^{b}v_7^{b_7}v_8^{b_8}v_9^{b_9}v_{10}^{b_{10}-1}v_{11}^{b_{11}}v_{12}^{b_{12}}\dots v_k^{b_k}
\end{aligned}$$

$$\Rightarrow u \in H^m.$$

If $b_i = 0 \ \forall \ i = 5, 6, 7, 8, 9, 10$. Since $x_3 \notin v_j \ \forall \ j = 11, 12, 13, 14, \dots, k$, it follows that $u = w_1v_{11}^{b_{11}}v_{12}^{b_{12}}v_{13}^{b_{13}}\dots v_k^{b_k}$ such that $w^{''}v_r = x_3w_1$ for some monomial $w_1 \in \mathcal{A}$.
Consequently, $u \in H^m$. This shows that v_1 is the strongly superficial element of degree one of H. Hence, by [17, Proposition 2.9] ideal H has the persistence property.
Let $h = x_1x_2x_3$. Since $h \in H = P_2(G_n)$, similarly we can drives

$$H^{m+1} : h = H^m \quad \forall \quad m \in \mathbb{N}.$$

This implies that $h\mathcal{A}$ is the superficial ideal of H. Hence, the ideal $P_2(G_n)$ holds a strong persistence property. □

Example 3.1. *For $n = 4$, we have $6n - 8 = 6(4) - 8 = 16$, so the path ideal $H = P_2(G_4)$ has exactly 16 distinct minimal generators. The graph G_4 consists of the centipede graph $\mathcal{Z}_8$ together with the additional edges $\{3, 9\}, \{5, 10\}, \{7, 11\}$ as shown in the following figure.*

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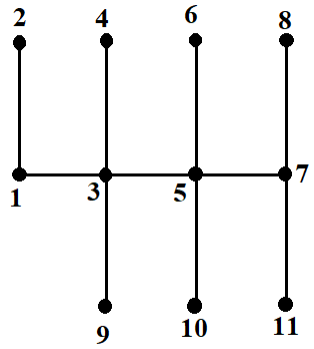


Figure 6. Graph G_4

Hence,

$$\begin{aligned}\mathcal{G}(H) =&\langle x_1x_2x_3,\ x_1x_3x_4,\ x_3x_4x_5,\ x_3x_5x_6,\ x_5x_6x_7,\ x_5x_7x_8,\ x_1x_3x_5,\ x_3x_5x_7,\\ &x_1x_3x_9,\ x_3x_5x_9,\ x_3x_5x_{10},\ x_5x_7x_{10},\ x_5x_7x_{11},\ x_3x_4x_9,\ x_5x_6x_{10},\ x_7x_8x_{11}\rangle.\end{aligned}$$

Therefore, $H = P_2(G_4)$ has exactly 16 *distinct minimal generators.*

Remark 3.1. If an additional edge is attached to the vertex x_1 of the graph G_n in the Figure 5, then the monomial $x_1x_2x_3$ is not a strongly superficial element of degree one. Consequently, the ideal $(x_1x_2x_3)\mathcal{A}$ is not a superficial ideal of H.

This can be seen from the following example.

Example 3.2. *Consider the following graph. Here, an additional edge $\{1, 5\}$ is attached at the vertex* 1.

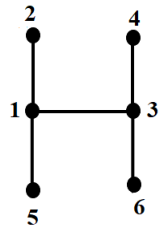


Figure 7. Graph G_2

Let $H = P_2(G_2)$. Then the generating set of H is $\langle v_1v_2v_3,\ v_1v_3v_4,\ v_1v_3v_5,\ v_1v_3v_6,\ v_1v_2v_5,\ v_3v_4v_6\rangle$ and $(H^2 : v_1v_2v_3) = (v_4v_5v_6,\ v_1v_5v_6,\ v_3v_4v_6,\ v_1v_3v_6,\ v_1v_5^2,\ v_1v_4v_5,\ v_1v_3v_5,\ v_1v_2v_5,\ v_1v_3v_4,\ v_1v_2v_3)$. This shows that $H \subseteq (H^2 : v_1v_2v_3)$ however $(H^2 : v_1v_2v_3) \nsubseteq H$.
Hence,

$$(H^n : v_1v_2v_3) \neq H^{n-1} \quad \forall \quad n \in \mathbb{N}.$$

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This implies $v_1v_2v_3$ is not a strongly superficial element of degree one and thus $v_1v_2v_3\mathcal{A}$ is not a superficial ideal of H.

Theorem 3.2. *Let $\mathcal{Z}_{2n}$ $(n \geq 2)$, be a centipede graph. Define a graph G'_n obtained by adding m new vertices $\{2n+1, 2n+2, \ldots, 2n+m\}$ to $\mathcal{Z}_n$. Then the vertex set of G'_n is $V(\mathcal{Z}_{2n}) \cup \{2n+1, 2n+2, \ldots, 2n+m\}$ and edge set of G'_n is $I(\mathcal{Z}_{2n}) \cup \{\{2n-1, 2n+1\}, \{3, 2n+2\}, \{5, 2n+3\}, \ldots, \{2n-3, 2n+(m-1)\}, \{2n-1, 2n+m\}\}$. Then the path ideal of length two of G'_n, denoted by $P_2(G'_n)$, satisfies the strong persistence property.*

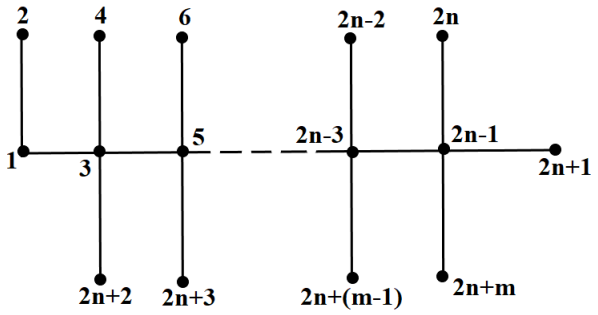


Figure 8. Graph G'_n

Proof. Let $M = P_2(G'_n)$. Then the generating set of M is

$$\begin{aligned}\mathcal{G}(M) =& \{x_{2k-1}x_{2k}x_{2k+1} : \ k = 1, 2 \ldots, n\} \cup \{x_{2k-1}x_{2k+1}x_{2k+2} : \ k = 1, 2, \ldots, n-1\} \\ & \cup \{x_{2t-3}x_{2t-1}x_{2n+t}, \ x_{2t-1}x_{2t+1}x_{2n+t}, \ x_{2t}x_{2t-1}x_{2n+t} : \ t = 2, 3, \ldots, n\} \\ & \cup \{x_{2r-1}x_{2r+1}x_{2r+3} : \ r = 1, 2, \ldots, n-1\}.\end{aligned}$$

It is clear that the ideal M has exactly unique minimal generators $6n-5$, $n \in \mathbb{N}$. Suppose that $\mathcal{G}(M) = \{v_1, v_2, v_3, \ldots, v_k\}$ with $k = 6n-5$ such that $v_1 = x_1x_2x_3$, $v_2 = x_1x_3x_{2n+2}$, $v_3 = x_3x_4x_5$, $v_4 = x_3x_5x_7$, $v_5 = x_3x_5x_6$, $v_6 = x_3x_4x_{2n+2}$, $v_7 = x_3x_5x_{2n+3}$, $v_8 = x_3x_5x_{2n+2}$ and so on. Our objective is to prove that

$$(M^{m+1} : v_1) = M^m \quad \forall \quad m \in \mathbb{N}.$$

Now, same as Theorem 3.1, it can be easily proved that $x_1x_2x_3$ is the strongly superficial element of degree one. Hence, by [17, Proposition 2.9], ideal M has the persistence property.
Consider $x_1x_2x_3\mathcal{A} \in M$. This is the superficial ideal; it follows that $P_2(G'_n)$ satisfies the strong persistence property, as it is proved in Theorem 3.1. □

Example 3.3. *For $n = 4$, we have $6n-5 = 6(4)-5 = 19$, so the path ideal $M = P_2(G'_4)$ has exactly 19 distinct minimal generators. The graph G'_4 consists of the centipede graph $\mathcal{Z}_8$ together with the additional edges $\{3, 10\}, \{5, 11\}, \{7, 12\}$, and $\{7, 9\}$ as shown in the following figure.*

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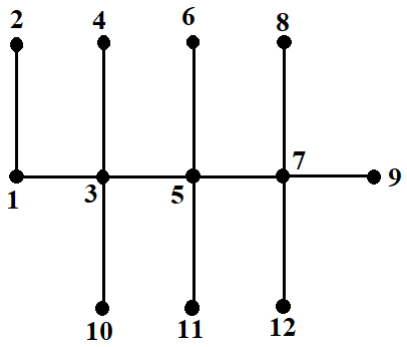


Figure 9. Graph G'_4

$\mathcal{G}(M) = \langle x_1x_2x_3,\ x_1x_3x_4,\ x_3x_4x_5,\ x_3x_5x_6,\ x_5x_6x_7,\ x_5x_7x_8,\ x_7x_8x_9,\ x_1x_3x_{10},\ x_3x_5x_{10},\ x_3x_5$
$x_{11},\ x_5x_7x_{11},\ x_5x_7x_{12},\ x_7x_9x_{12},\ x_1x_3x_5,\ x_3x_5x_7,\ x_5x_7x_9,\ x_3x_4x_{10},\ x_5x_6x_{11},\ x_7x_8x_{12}\rangle$.
Therefore, $M = P_2(G'_4)$ *has exactly* 19 *distinct minimal generators.*

Remark 3.2. Assume that an additional edge is attached to the vertex x_1 of the graph G'_n in Figure 8. In this case, the monomial $x_1x_2x_3$ is not a strongly superficial element of degree one. Consequently, the ideal generated by $x_1x_2x_3\mathcal{A}$ is not superficial with respect to M.

This can also be seen from the following example.

Example 3.4. *Consider the following graph. Here, an additional edge* $\{1, 6\}$ *is attached at the vertex* 1.

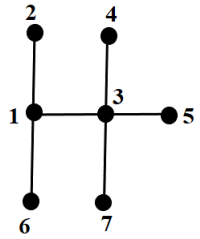


Figure 10. Graph G'_2

Let $M = P_2(G'_2)$. *Then the generating set of* M *is* $\langle v_1v_2v_3,\ v_3v_4v_5,\ v_1v_3v_5,\ v_1v_3v_6,\ v_1v_3v_7,\ v_1v_2v_6,$
$v_3v_4v_7\rangle$ *and* $(M^2 : v_1v_2v_3) = (v_5v_6v_7,\ v_4v_6v_7,\ v_1v_6v_7,\ v_4v_5v_7,\ v_3v_5v_7,\ v_4^2v_7,\ v_3v_4v_7,\ v_1v_4v_7,\ v_1v_3$
$v_7,\ v_1v_6^2,\ v_4v_5v_6,\ v_1v_5v_6,\ v_1v_4v_6,\ v_1v_3v_6,\ v_1v_2v_6,\ v_4^2v_5,\ v_3v_4v_5,\ v_1v_4v_5,\ v_1v_3v_5,\ v_1v_2v_4,\ v_1v_2v_3)$.
This shows that $M \subseteq (M^2 : v_1v_2v_3)$ *however* $(M^2 : v_1v_2v_3) \nsubseteq M$.
Hence

$$(M^n : v_1v_2v_3) \neq M^{n-1} \quad \forall \quad n \in \mathbb{N}.$$

This implies $v_1v_2v_3$ *is not strongly superficial element of degree one and thus* $v_1v_2v_3\mathcal{A}$ *is not a superficial ideal of* M.

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