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# Spectral properties of mixed radial Moore graphs

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**Abstract**

The pioneering study of Moore graphs by Hoffman and Singleton exploits the properties of the eigenvalues of the adjacency matrix of graphs. Radial Moore graphs are extremal graphs related to the Moore ones, where the distance-preserving spanning tree is preserved for some vertices. In this paper, we present some equations involving the adjacency matrix of a mixed radial Moore graph. These equations are used as a basis to create some spectral measures that lead us to rank mixed radial Moore graphs and show which graph is the closest graph to being a mixed Moore graph.

*Keywords:* mixed graph, radial Moore graph, degree/diameter problem, distance matrix, spectrum.
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## 1. Introduction

The degree/diameter problem for mixed graphs asks for the largest possible number of vertices $n(r, z, k)$ in a mixed graph with maximum undirected degree $r$, maximum directed out-degree $z$, and diameter $k$. A natural upper bound for $n(r, z, k)$ is derived by counting the number of vertices at every distance from any given vertex $v$ in a mixed graph with a given maximum undirected degree $r$, maximum directed out-degree $z$, and diameter $k$. This bound is known as the *Moore*

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*bound* for mixed graphs (see Buset, El Amiri, Erskine, Miller, and Pérez-Rosés [2] and Dalfó, Fiol, and López [6]):

$$M(r,z,k) = \frac{1}{r+2z-2} \begin{pmatrix} 1 & 1 \end{pmatrix} \left( \begin{pmatrix} r-1 & r \\ z & z \end{pmatrix}^{k+1} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} r \\ 2-r \end{pmatrix} \qquad (1)$$

We will rediscover Expression (1) in Section 4 (notice the formula given in [6] contains a little mistake in the last column matrix). The mixed Moore bound is a generalization of both the undirected Moore bound and the directed Moore bound. *Mixed Moore graphs* are those with order attaining (1), which means that between any pair of vertices, there is a unique shortest path of length not greater than the diameter. Mixed Moore graphs are totally regular with degree $(r, z)$ (see Bosák [1]), and the distance-preserving spanning tree, also known as the Moore tree, hanging at any vertex $v$ does not depend on the chosen vertex. It is well known that mixed Moore graphs are very rare. This has led to studying mixed graphs 'close' to the Moore ones (see, for instance, [5, 11]). One way to do this is by allowing the existence of vertices with eccentricity that is just one unit more than the value they should have. Hence, a totally $(r, z)$-regular graph of (out-)radius $k$, diameter $k + 1$, and order $M(r, z, k)$ is an $(r, z, k)$-mixed radial Moore graph.

The vertex set of any $(r, z, k)$-mixed radial Moore can be partitioned in two sets, where one set contains all central vertices (those vertices with out-eccentricity $k$, that is, their corresponding distance-preserving spanning tree is a Moore tree) and the other set has all non-central vertices (with out-eccentricity $k + 1$).

Mixed radial Moore graphs can be seen as an approximation of mixed Moore graphs in which the diameter constraint has been relaxed a bit. This relaxation may produce a significant number of graphs that we want to rank in order to see which one is the 'closest' graph to being a Moore graph. The number of central vertices could be a first attempt to measure this closeness, but, as we show later, this measure is not fine since there are many mixed radial Moore graphs with the same number of central vertices.

Two different ranking measures were introduced in Capdevila, Conde, Exoo, Gimbert, and López [3] for the undirected case. One ranking measure is related to the girth since Moore (undirected) graphs attain the minimum order allowed by the girth. However, this measure is no longer useful for mixed graphs, in which short directed cycles appear in mixed Moore graphs. The other ranking measure is based on the 'status' of a vertex in a Moore graph.

*Organization of the paper*

This paper is organized as follows. In Section 2, the reader can find a background on mixed Moore graphs and their algebraic properties. In Section 3, we describe the structural properties of mixed radial Moore graphs. Then, in Section 4, we define a measure related to the spectra of a mixed Moore graph and another one with equations involving the adjacency matrix of mixed Moore graphs. Section 5 is devoted to the ranking of mixed radial Moore graphs for some cases in which their enumeration is known, showing the optimal graphs in each case.


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## 2. Preliminaries

A *mixed* (or *partially directed*) graph $G$ with vertex set $V$ may contain a set $E$ of (undirected) *edges* as well as a set $A$ of directed edges (also known as *arcs*). From this point of view, a *graph* [respectively, *directed graph* or *digraph*] has all its edges undirected [respectively, directed]. The set of vertices that are adjacent from [to] a given vertex $v$ is denoted by $\Gamma^+(v)$ [$\Gamma^-(v)$]. The *undirected degree* of a vertex $v$, denoted by $d(v)$, is the number of edges incident to $v$. The *out-degree* [respectively, *in-degree*] of vertex $v$, denoted by $d^+(v)$ [respectively, $d^-(v)$], is the number of arcs emanating from [respectively, to] $v$. If $d^+(v) = d^-(v) = z$ and $d(v) = r$, for all $v \in V$, then $G$ is said to be *totally regular* of degrees $(r, z)$ (or simply $(r, z)$-*regular*). A *walk* of length $\ell \geq 0$ from $u$ to $v$ is a sequence of $\ell + 1$ vertices, $u_0u_1 \dots u_{\ell-1}u_\ell$, such that $u = u_0$, $v = u_\ell$ and each pair $u_{i-1}u_i$, for $i = 1, \dots, \ell$, is either an edge or an arc of $G$. A *directed walk* is a walk containing only arcs. An *undirected walk* is a walk containing only edges. A walk with different vertices is called a *path*. The length of a shortest path from $u$ to $v$ is the *distance* from $u$ to $v$, and it is denoted by $\mathrm{dist}(u, v)$. Note that $\mathrm{dist}(u, v)$ may be different from $\mathrm{dist}(v, u)$ when the shortest paths between $u$ and $v$ involve arcs.

The *out-eccentricity* of a vertex $u$, denoted by $\mathrm{ecc}^+(u)$, is the maximum distance from $u$ to any vertex in $G$. Analogously, the in-eccentricity of $u$ is $\mathrm{ecc}^-(u) = \max\{\mathrm{dist}(v, u) : v \in V\}$. The maximum distance between any pair of vertices is the *diameter* $k = k(G)$ of $G$, that is, $k(G) = \max\{\mathrm{ecc}^+(u) : u \in V\}$. The *out-radius* and *in-radius* of $G$ are $r^+(G) = \min\{\mathrm{ecc}^+(u) : u \in V\}$ and $r^-(G) = \min\{\mathrm{ecc}^-(u) : u \in V\}$, respectively. According to Knor [9], a *central vertex* is a vertex with minimum radius $r(G) = \max\{r^+(G), r^-(G)\}$. In the case of mixed graphs, we also use the concepts of *in-central* and *out-central* vertices, which are defined as the vertices having minimum in-radius and out-radius, respectively. The *converse* of a digraph $G$, denoted by $\overline{G}$, is the digraph obtained by reversing the orientation of all arcs in $G$. (Of course, if $G$ is a mixed graph, the edges of $\overline{G}$ remain unchanged.) Notice that $k(G) = k(\overline{G})$, $r^+(G) = r^-(\overline{G})$, and $r^-(G) = r^+(\overline{G})$.

For results concerning the diameter and order (Moore bounds), see, for instance, Nguyen and Miller [10], and Buset, El Amiri, Erskine, Miller, and Pérez-Rosés [2].

The *adjacency matrix* $\boldsymbol{A}$ of $G$, with rows and columns indexed by the vertices of $G$, has entries $(\boldsymbol{A})_{uv} = 1$ if $(u, v) \in A$ and $(\boldsymbol{A})_{uv} = 0$ otherwise. An edge $\{u, v\}$ can be seen as a *digon* or pair of opposite arcs $(u, v)$ and $(v, u)$. The *characteristic polynomial* $\Phi_G(x)$ of $G$ is simply the characteristic polynomial of its adjacency matrix. Notice that $\Phi_G(x) \in \mathbb{Q}[x]$. Mixed graphs can be seen as a generalization of both undirected and directed graphs or, according to the previous comment, as digraphs having some digons. In this case, we identify an $(r, z)$-regular mixed graph with a $d$-regular digraph with $d = r + z$. Then, the following result of Hoffman and McAndrew [7] is a basic tool for studying algebraic properties.

**Theorem 2.1** ([7])**.** *Let $G$ be a digraph on $n$ vertices and with adjacency matrix $\boldsymbol{A}$.*

$(i)$ *There exists a polynomial $HM(x)$, that we call the* Hoffman-McAndrew polynomial*, such that*

$$HM(\boldsymbol{A}) = \boldsymbol{J}, \tag{2}$$

*where $\boldsymbol{J}$ is the all-1 matrix, if and only if $G$ is strongly connected and $d$-regular.*



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$(ii)$ *The unique polynomial of least degree satisfying* (2) *is* $n\sigma(x)/\sigma(d)$*, where* $m(x) = (x - d)\sigma(x)$ *is the minimal polynomial of* $\boldsymbol{A}$*.*

$(iii)$ *If* $HM(x)$ *is the polynomial of least degree satisfying* (2)*, then the degree* $d$ *of* $G$ *is the greatest real root of* $HM(x) = n$*.*

From this result, we have that the eigenvalues of an strongly connected $(r, z)$-regular mixed graph $G$ (or $d$-regular digraph with $d = r + z$) are $d$ and some of the roots of its Hoffman-McAndrew polynomial. That is, the characteristic polynomial $\phi_G(x)$ contains the factor $(x - d)^1$ and some of the factors of $HM(x)$. Note that if $HM(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ then it is a factor of $\phi_G(x)$. Moreover, $HM(d) = n$.

## 3. Spectral properties of mixed radial Moore graphs

The spectrum of a Moore graph $G$ is well known, and it can be deduced from the fact that its adjacency matrix $\boldsymbol{A}$ satisfies some polynomial equation of the type $p(\boldsymbol{A}) = \boldsymbol{J}$. For instance, if $\boldsymbol{A}$ is the adjacency matrix of a mixed Moore graph of diameter $k = 2$ and undirected degree $r$, due to the uniqueness of trails of length at most two between any pair of vertices, $\boldsymbol{A}$ satisfies $(1 - r)\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 = \boldsymbol{J}$, where $\boldsymbol{I}$ and $\boldsymbol{J}$ denote the identity and the all-ones matrix, respectively. Hence, $\boldsymbol{A}$ satisfies the equation $F_2^r(\boldsymbol{A}) = \boldsymbol{J}$, where

$$F_2^r(x) = x^2 + x + (1 - r). \tag{3}$$

This is precisely the Hoffman-McAndrew polynomial. $\boldsymbol{J}$ has eigenvalues $0$ (with multiplicity $n-1$) and $n$ (with multiplicity $1$), that is, the characteristic polynomial of $\boldsymbol{J}$ is $(x - n)^1 x^{n-1}$. Therefore, as a consequence of $F_2^r(\boldsymbol{A}) = \boldsymbol{J}$, the characteristic polynomial $\Phi_A(x)$ of $\boldsymbol{A}$ is either

- $\Phi_A(x) = (x - d)^1 F_2^r(x)^{\frac{n-1}{2}}$ if $F_2^r(x)$ is irreducible in $\mathbb{Q}[x]$, or
- $\Phi_A(x) = (x - d)^1 (x - \alpha)^a (x - \beta)^b$ if $F_2^r(x)$ is reducible in $\mathbb{Q}[x]$, where $a + b = n - 1$.

Note that the roots of $F_2^r(x)$ are $\alpha = \frac{-1+c}{2}$ and $\beta = \frac{-1-c}{2}$, where $c^2 = 4r - 3$. From the equation $\operatorname{tr} \boldsymbol{A} = 0$, one can deduce well-known properties for the existence of mixed Moore graphs of diameter $2$. Although these extremal graphs do not exist for many pairs $(r, z)$, we may keep their spectral characteristics in order to approximate Moore graphs with graphs that share part of these spectral values. In this case, the eigenvalues of a (mixed) Moore graph of diameter $2$ are $d$ and the zeroes of $F_2^r(x)$. Notice that both $\alpha$ and $\beta$ must have multiplicity greater than or equal to one, since otherwise, from $\operatorname{tr} \boldsymbol{A} = 0$ we would obtain $0 = d + 0\alpha + (n-1)\beta$, that is, $2d = (d + d^2 - r)(1 + c)$. Now, since $d \geq r$, we would obtain $d \geq d^2$, which is impossible. This means that no root of $F_2^r(x)$ may vanish from the spectra of the Moore graph of diameter $2$.

Next, we prove that these three eigenvalues also appear in every totally regular mixed graph of order equal to the Moore bound and radius $2$, for certain values of $r$.

**Proposition 3.1.** *Let* $G'$ *be a totally regular mixed graph of degrees* $(r, z)$*, order equal to the Moore bound* $M(r, z, 2)$ *and radius* $2$*, such that* $4r - 3$ *is not a square, having either one central vertex or two in-central or out-central vertices. Then, the three eigenvalues* $d, \alpha$*, and* $\beta$ *of a mixed Moore graph of diameter* $2$ *are also eigenvalues of* $G'$*.*



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*Proof.* Let $\boldsymbol{A}'$ be the adjacency matrix of a totally regular mixed graph $G'$ of degrees $(r, z)$, order the Moore bound $M(r, z, 2)$ and radius 2, having one central vertex. First of all, the total degree $d$ is in fact an eigenvalue of $\boldsymbol{A}'$, being the all-1 vector $\boldsymbol{j}$ its eigenvector due to the total regularity of $G'$. The evaluation of $\boldsymbol{A}'$ in the polynomial $F_2^r(x)$ no longer produces the matrix $\boldsymbol{J}$ (unless every vertex of $G'$ would be central, which is the trivial case where $G'$ is a mixed Moore graph of diameter 2). Instead, $F_2^r(\boldsymbol{A}')$ produces a matrix containing at least one row and column of ones corresponding to a central vertex of $G'$. Without loss of generality, let us assume that the first row and column of $\boldsymbol{A}'$ are indexed by a central vertex so that $F_2^r(\boldsymbol{A}')$ is

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & b_{n2} & \dots & b_{nn} \end{pmatrix}.$$

Recall that $d$ is an eigenvalue of $\boldsymbol{A}'$, being the all-1 vector $\boldsymbol{j}$ its eigenvector. Therefore, $\boldsymbol{j}$ is also an eigenvector of $F_2^r(\boldsymbol{A}')$ with eigenvalue $F_2^r(d) = (1 - r) + d + d^2$, which is precisely the Moore bound $n$, as expected. Now from $F_2^r(\boldsymbol{A}')\boldsymbol{j} = n\boldsymbol{j}$ we get that

$$\sum_{j=2}^{n} b_{ij} = n - 1 \quad \text{for} \quad i = 2, \dots, n. \tag{4}$$

From this, we see that the vector $\boldsymbol{u} = (n - 1, -1, \dots, -1)^\top$ is an eigenvector of $F_2^r(\boldsymbol{A}')$ with eigenvalue 0. Moreover, we reach the same conclusion if $G'$ contains at least two in- or out-central vertices because, then, $F_2^r(\boldsymbol{A}')$ contains two different columns or rows of 1's and, hence, the matrix $F_2^r(\boldsymbol{A}')$ has some null eigenvalue.

At this point, we have seen that the characteristic polynomial of $F_2^r(\boldsymbol{A}')$ is $(x - n)^1 x^a q(x)$ for $a \geq 1$, where $q(x)$ is a polynomial of degree $n - a - 1$. We recall that $F_2^r(x) = x^2 + x + (1 - r)$ decomposes into two rational factors if and only if its discriminant $4r - 3$ is a square. So, similarly to what happens in a mixed Moore graph of diameter 2, the characteristic polynomial of $\boldsymbol{A}'$ has the factor $(x - d)^1$ (corresponding to $(x - n)^1$ in $F_2^r(\boldsymbol{A}')$) and either one of the following factors (corresponding to $x^a$ in $F_2^r(\boldsymbol{A}')$):

1. $(x - \alpha)^{a_1}(x - \beta)^{a_2}$, where $a_1 + a_2 = a$, if $\alpha$ and $\beta$ are both rational, that is, $4r - 3 = c^2$ for some positive integer $c$; or
2. $(F_2^r(x))^{a/2}$, otherwise.

By assumption $4r - 3$ is not a square, so $\boldsymbol{A}'$ falls into the second case, where this irreducible factor of degree 2 encodes the two remaining eigenvalues $\alpha$ and $\beta$ that we wanted to show belong to the set of eigenvalues of $\boldsymbol{A}'$, so the proof is concluded. □

It would be nice to know if the statement remains true in the first case, where $F_2^r(x)$ splits into two rational terms $(x-\alpha)^{a_1}(x-\beta)^{a_2}$, where $a_1+a_2 = a \geq 1$, inside the characteristic polynomial of $\boldsymbol{A}'$. In this case, at least one factor must be in the characteristic polynomial of $\boldsymbol{A}'$ (since $a \geq 1$), but we could find no proof that both factors must be there. For the smallest cases $r = 1, z = 1$


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Figure 1. The unique mixed Moore graph $G$ on 6 vertices (left) and a mixed radial Moore graph $H$ (right). Both share the same spectrum $\{2^{[1]}, 0^{[3]}, -1^{[2]}\}$ and their corresponding Hoffman-McAndrew polynomial is $HM(x) = x(x+1)$, which is $F_2^r(x)$ for the special case $r = 1$.

and $r = 3, z = 0$, where an easy computation provides all totally regular graphs of radius 2 (two mixed graphs in the first case and four undirected graphs in the second one), it can be checked that the proposition remains true.

The adjacency matrix $\boldsymbol{A}$ of an undirected Moore graph of degree $r$ and diameter $k$ satisfies $F_k^r(\boldsymbol{A}) = \boldsymbol{J}$, where the polynomial $F_k^r(x)$ defined recursively as $F_0^r(x) = 1$, $F_1^r(x) = x + 1$ and $F_k^r(x) = xF_{k-1}^r(x) - (r-1)F_{k-2}^r(x)$. In the cubic case of diameter 3, the corresponding polynomial $F_3^3(x) = x^3 + x^2 - 4x - 2$ has three real roots $\gamma_1, \gamma_2$, and $\gamma_3$, and it is irreducible in $\mathbb{Q}[x]$. Applying the same argument as the one given in the previous proof, it is easy to see that $F_3^3(x)$ also appears as a factor of the characteristic polynomial of every cubic regular graph of radius 3 containing one central vertex, proving that the three real roots of $F_3^3(x)$ are also part of the spectra of these graphs.

**Corollary 3.1.** *Let $G'$ be a cubic graph of order equal to the Moore bound $M(3, 0, 3)$ and radius 3 that has one central vertex. Then, the four eigenvalues $d, \gamma_1, \gamma_2$, and $\gamma_3$ of a (hypothetical) Moore graph of diameter 3 are also eigenvalues of $G'$.*

Among the vast number of regular graphs with given radius $k$, we will keep in those with small diameter: We recall that an $(r, z)-$regular graph of (out-)radius $k$, diameter $k + 1$ and order $M(r, z, k)$ is an $(r, z, k)-$mixed radial Moore graph. As a consequence of the previous results, we have the following consequence.

**Corollary 3.2.** *Let $(r, z, k)$ be either $(3, 0, 3)$ or $(r, z, 2)$ (where $4r - 3$ is not a square). Then, the set of the different eigenvalues of any $(r, z, k)$-mixed Moore graph is contained in the set of eigenvalues of every $(r, z, k)$-mixed radial Moore graph.*

A natural question arises at this point: *Which are the eigenvalues of mixed radial Moore graphs that are not eigenvalues of mixed Moore graphs?* After analyzing the particular case $(r, z, k) = (2, 1, 2)$, described in Ceresuela, López, and Chemisana [4], there are many eigenvalues different than the ones that a hypothetical Moore graph should have, although some real eigenvalues appear quite frequently in many graphs (like $-1, 0$, and 1), other complex eigenvalues belong to the spectra of radial Moore graphs (see Figure 2).



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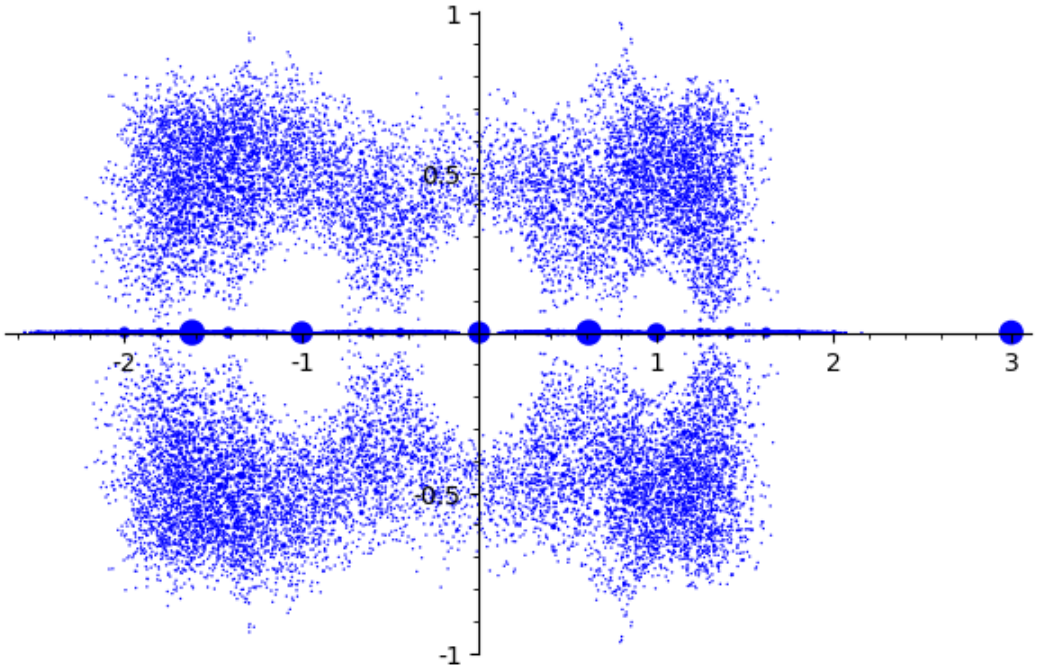



Figure 2. Distribution of the eigenvalues of all 9486 mixed radial Moore graphs corresponding to the case $(r, z, k) = (2, 1, 2)$. Each blue point depicted in $\mathbb{C}$ is an eigenvalue of a mixed graph, and its size is proportional to the number of occurrences in the whole set of mixed radial Moore graphs. The biggest dots correspond to the eigenvalues of a putative $(2, 1, 2)$-mixed Moore graph, and they are located at $\{3, \frac{1}{2}(-1 \pm \sqrt{5})\}$ in the real line.

## 4. Two spectral measures to rank mixed radial Moore graphs

The results in the previous section provide a basis for defining a new closeness measure to rank mixed radial Moore graphs. Since the set of eigenvalues of a $(r, z, 2)$-mixed radial Moore graph contains the eigenvalues that a hypothetical Moore graph should have, our purpose is to study how far the remaining eigenvalues are from the Moore ones. To this end, given the eigenvalues set $\{\lambda_0 = r + z, \lambda_1, \lambda_2, \cdots, \lambda_{n-1}\}$ of a $(r, z, 2)$-mixed radial Moore graph $G$, we define its *spectral closeness Moore vector* as:

$$\hat{\mathbf{s}}(G) = (F_2^r(\lambda_1), \ldots, F_2^r(\lambda_{n-1}))$$

where $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n-1}|$ and $F_2^r(x) = x^2 + x + (1 - r)$ is the Hoffman-McAndrew polynomial of a mixed Moore graph of diameter 2. Notice that the eigenvalues of $G$ are complex numbers in general, so $\hat{\mathbf{s}}(G) \in \mathbb{C}^{n-1}$. By Corollary 3.2, the vector $\hat{\mathbf{s}}(G)$ has at least two null entries and, clearly, it becomes the all-zero vector if $G$ is an $(r, z, 2)$-mixed Moore graph. The converse is not necessarily true since there are mixed graphs cospectral with mixed Moore graphs, as the one depicted in Figure 1.

Let us denote by $\mathcal{RM}(r, z, 2)$ the set of all nonisomorphic $(r, z, 2)$-mixed radial Moore graphs. Our purpose is to rank each mixed graph of this 'population' in terms of how close it is to being



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a mixed Moore graph. Given two mixed graphs $G_1, G_2 \in \mathcal{RM}(r, z, 2)$, we define $G_1$ and $G_2$ to be *spectral distance equivalent*, $G_1 \sim G_2$, if their corresponding spectral closeness Moore vectors $\hat{\mathbf{s}}(G_1)$ and $\hat{\mathbf{s}}(G_2)$ have the same distance to $0 \in \mathbb{C}^{n-1}$, which is the place where the spectral closeness Moore vector of a Moore graph is. This distance can be computed using

$$\hat{N}(G) = \|\hat{\mathbf{s}}(G)\| = \sqrt{\sum_{i=1}^{n-1} |F_2^r(\lambda_i)|^2}$$

for any $G \in \mathcal{RM}(r, z, 2)$. In the quotient set of $\mathcal{RM}(r, z, k)$ by $\sim$, we say that $G_1$ is closer than $G_2$ to be a mixed Moore graph if $\hat{N}(G_1) < \hat{N}(G_2)$, in which case it is denoted by $G_1 < G_2$. We call it the *spectral polynomial measure*.

**Example.** *Two $(2, 1, 2)$-mixed radial Moore graphs $G_1$ and $G_2$ are shown in Figure 3. The Hoffman-McAndrew polynomials of $G_1$ and $G_2$ are $HM_1(x) = \frac{1}{6}x(x-1)(x^2+x-1)$ and $HM_2(x) = \frac{1}{8}(x^2-1)(x^2+x-1)$, respectively. More precisely, their spectrum is $\{3^{[1]}, 1^{[1]}, 0^{[1]}, \lambda_1^{[4]}, \lambda_2^{[4]}\}$ and $\{3^{[1]}, 1^{[2]}, (-1)^{[2]}, \lambda_1^{[3]}, \lambda_2^{[3]}\}$, respectively, where $\lambda_1 = \frac{-1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{-1-\sqrt{5}}{2}$. The set of eigenvalues $\{3, \lambda_1, \lambda_2\}$ is the set that a mixed Moore graph should have in this case. Hence,*

$$\hat{\mathbf{s}}(G_1) = (F_2^r(1)^{[1]}, F_2^r(\lambda_1)^{[4]}, F_2^r(0)^{[1]}, F_2^r(\lambda_2)^{[4]}),$$
$$\hat{\mathbf{s}}(G_2) = (F_2^r(1)^{[2]}, F_2^r(\lambda_1)^{[3]}, F_2^r(-1)^{[2]}, F_2^r(\lambda_2)^{[3]}).$$

*As a consequence, $\hat{N}(G_1) = \sqrt{2}$ and $\hat{N}(G_2) = 2$, so $G_1 < G_2$, that is, $G_1$ is closer than $G_2$ to being a Moore graph according to the spectral polynomial measure.*

We can do the same with the case $(3, 0, 3)$, for which Corollary 3.2 assures that every cubic radial Moore graph has the four eigenvalues that the corresponding (hypothetical) Moore graph has. Even for the general undirected case $(r, 0, k)$, we could apply this closeness measure since there is a polynomial $F_k^r(x)$ satisfying $F_k^r(\boldsymbol{A}) = \boldsymbol{J}$ (although there is no guarantee that they share more than one eigenvalue). Nevertheless, we can not do this in general for the mixed case since the matrix equations involving the relationship between $\boldsymbol{A}$ and $\boldsymbol{J}$ require the decomposition of $\boldsymbol{A}$ in two binary matrices $\boldsymbol{R}$ and $\boldsymbol{Z}$, whose entries encode the edges and the arcs of the mixed graph, respectively, as we show next.

In general, given a mixed graph $G$ with diameter $k$, let us consider the distance matrices of $G$, denoted by $\boldsymbol{R}_i$ and $\boldsymbol{Z}_i$, whose entries are defined as follows:

$$(\boldsymbol{R}_i)_{uv} = \begin{cases} 1 & \text{if } \mathrm{dist}(u, v) = i, \\ & \text{and the shortest path from } u \text{ to } v \text{ ends with an edge } v' \sim v. \\ 0 & \text{otherwise,} \end{cases}$$

$$(\boldsymbol{Z}_i)_{uv} = \begin{cases} 1 & \text{if } \mathrm{dist}(u, v) = i, \\ & \text{and the shortest path from } u \text{ to } v \text{ ends with an arc } u' \to v. \\ 0 & \text{otherwise.} \end{cases}$$



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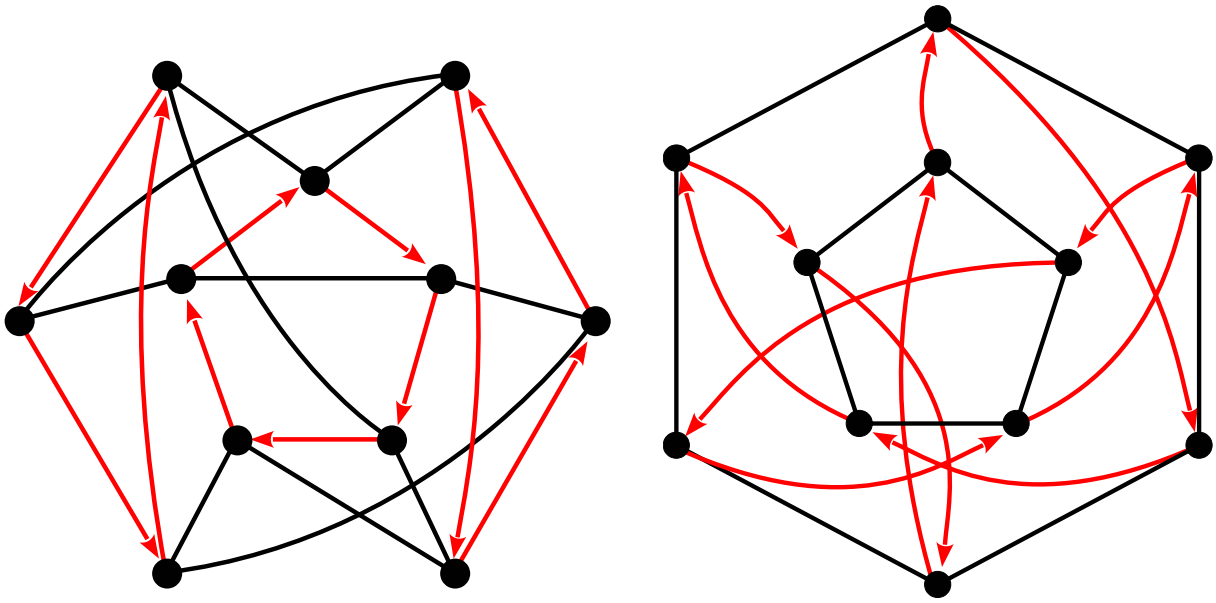


Figure 3. Two $(2,1,2)$-mixed radial Moore graphs $G_1$ (left) and $G_2$ (right). Precisely, $G_1$ achieves the minimum value for this measure, as we show in Section 5.

For convenience, we define $\boldsymbol{R}_0 = \boldsymbol{O}$ (the all-0 matrix) and $\boldsymbol{Z}_0 = \boldsymbol{I}$. Moreover, notice that the adjacency matrix of $G$ is $\boldsymbol{A} = \boldsymbol{R}_1 + \boldsymbol{Z}_1 = \boldsymbol{R} + \boldsymbol{Z}$ and, in general, the $i$-distance matrix of $G$ is $\boldsymbol{A}_i = \boldsymbol{R}_i + \boldsymbol{Z}_i$ for $i = 0, 1, \ldots, k$.

**Lemma 4.1.** *Let $G$ be an $(r,z,k)$-mixed Moore graph. Then, the distance matrices $\boldsymbol{R}_i$ and $\boldsymbol{Z}_i$ satisfy the recurrences*

$$\boldsymbol{R}_i = \boldsymbol{A}_{i-1}\boldsymbol{R} - (r-1)\boldsymbol{R}_{i-2} - r\boldsymbol{Z}_{i-2}, \tag{5}$$
$$\boldsymbol{Z}_i = \boldsymbol{A}_{i-1}\boldsymbol{Z}, \tag{6}$$

*starting from $\boldsymbol{R}_0 = \boldsymbol{O}$ and $\boldsymbol{Z}_0 = \boldsymbol{I}$, and $\boldsymbol{A}_{i-1} = \boldsymbol{R}_{i-1} + \boldsymbol{Z}_{i-1}$.*

*Proof.* Use induction from $\boldsymbol{R}_1 = \boldsymbol{R}$ and $\boldsymbol{Z}_1 = \boldsymbol{Z}$, considering that there is a unique shortest path between any pair of vertices. □

These recurrences may be defined using polynomials. Note that since $\boldsymbol{R}$ and $\boldsymbol{Z}$ do not necessarily commute, the corresponding polynomials neither commute.

**Corollary 4.1.** *Let $p_i$ and $q_i$ be noncommutative polynomials in the two variables $x, y$ satisfying the recurrences:*

$$p_i = (p_{i-1} + q_{i-1})x - (r-1)p_{i-2} - rq_{i-2},$$
$$q_i = (p_{i-1} + q_{i-1})y,$$

*with $p_0 = 0$ and $q_0 = 1$. Then, the distance matrices $\boldsymbol{R}_i$ and $\boldsymbol{Z}_i$ of an $(r,z,k)$-mixed Moore graph are the noncommutative polynomials $p_i$ and $q_i$ when evaluated at $x = \boldsymbol{R}$ and $y = \boldsymbol{Z}$. That is,*

$$\boldsymbol{R}_i = p_i(\boldsymbol{R}, \boldsymbol{Z}), \qquad \boldsymbol{Z}_i = q_i(\boldsymbol{R}, \boldsymbol{Z}).$$



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*Hence, the distance matrix $\boldsymbol{A}_i$ is a noncommutative polynomial in $\boldsymbol{R}, \boldsymbol{Z}$: $\boldsymbol{A}_i = (p_i + q_i)(\boldsymbol{R}, \boldsymbol{Z})$.*

Thus, with $s = x + y$, the first noncommutative polynomials are:

$$\begin{array}{ll} p_0(x,y) = 0, & q_0(x,y) = 1, \\ p_1(x,y) = x, & q_1(x,y) = y, \\ p_2(x,y) = sx - r, & q_2(x,y) = sy, \\ p_3(x,y) = [s^2 - r + 1]x - sr, & q_3(x,y) = [s^2 - r]y, \\ p_4(x,y) = [s^3 - 2rs + s + x]x - rs^2 + r(r-1), & q_4(x,y) = [s^3 - 2rs + x]y. \end{array}$$

**Lemma 4.2.** *The noncommutative polynomials $p_i$ and $q_i$, when evaluated at $(x, y) = (r, z)$, satisfy the recurrences*

$$\begin{aligned} p_i(r,z) &= (r-1)p_{i-1}(r,z) + rq_{i-1}(r,z), \\ q_i(r,z) &= zp_{i-1}(r,z) + zq_{i-1}(r,z), \end{aligned}$$

*or, in matrix form,*

$$\begin{pmatrix} p_i(r,z) \\ q_i(r,z) \end{pmatrix} = \begin{pmatrix} r-1 & r \\ z & z \end{pmatrix} \begin{pmatrix} p_{i-1}(r,z) \\ q_{i-1}(r,z) \end{pmatrix},$$

*starting from $p_0(r, z) = 0$ and $q_0(r, z) = 1$.*

*Proof.* Use induction with Corollary 4.1. □

Since $G$ is an $(r, z)$-regular mixed graph, given a vertex $u$ and for $i = 0, 1, \ldots, k$, the maximum number of vertices at distance $i$ from $u$ is

$$N_i = (\boldsymbol{R}_i + \boldsymbol{Z}_i)\mathbf{1} = (p_i(\boldsymbol{R}, \boldsymbol{Z}) + q_i(\boldsymbol{R}, \boldsymbol{Z}))\mathbf{1} = p_i(r, z) + q_i(r, z).$$

With $R_i = p_i(z, r)$ and $Z_i = q_i(r, z)$, and using Lemma 4.2, we can reobtain the Moore bound (1), as was done by some of the authors in [6],

$$\begin{aligned} M(r,z,k) = \sum_{i=0}^{k} N_i &= \begin{pmatrix} 1 & 1 \end{pmatrix} \sum_{i=0}^{k} \boldsymbol{M}^i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{r+2z-2} \begin{pmatrix} 1 & 1 \end{pmatrix} (\boldsymbol{M}^{k+1} - \boldsymbol{I}) \begin{pmatrix} r \\ 2-r \end{pmatrix}, \end{aligned} \tag{7}$$

where $\boldsymbol{M} = \begin{pmatrix} r-1 & r \\ z & z \end{pmatrix}$, and $r + 2z \neq 2$.

**Proposition 4.1.** *Let $\boldsymbol{A} = \boldsymbol{R} + \boldsymbol{Z}$ be the adjacency matrix of an $(r, z, k)$-mixed Moore graph $G$, where $k \in \{3, 4\}$. The following equations must hold:*

$(i)$ *If $k = 3$, then*

$$\boldsymbol{A}^3 + \boldsymbol{A}^2 + (1-2r)\boldsymbol{A} + \boldsymbol{R} + (1-r)\boldsymbol{I} = \boldsymbol{J}. \tag{8}$$



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$(ii)$ *If* $k = 4$*, then*

$$\boldsymbol{A}^4+\boldsymbol{A}^3+(1-3r)\boldsymbol{A}^2+(1-2r)\boldsymbol{A}+\boldsymbol{R}\boldsymbol{A}+\boldsymbol{A}\boldsymbol{R}+\boldsymbol{R}+(1-r)^2\boldsymbol{I}=\boldsymbol{J}. \tag{9}$$

*Proof.* By using the noncommutative polynomials $p_i$ and $q_i$ of Corollary 4.1, and for $k \in \{2, 3, 4\}$, let us consider the sum polynomials $F_k(x, y) = \sum_{i=0}^{k}(p_i + q_i)(x, y)$, where $s = x + y$. Namely,

$$\begin{aligned} F_3(x, y) &= s^3 + s^2 + (1 - 2r)s + x + (1 - r),\\ F_4(x, y) &= s^4 + s^3 + (1 - 3r)s^2 + (1 - 2r + x)s + (s + 1)x + (1 - r)^2. \end{aligned}$$

Then, since $\boldsymbol{A}_i = (p_i + q_i)(\boldsymbol{R}, \boldsymbol{Z})$ for $k \geq 0$, such polynomials satisfy $F_k(\boldsymbol{R}, \boldsymbol{Z}) = \boldsymbol{J}$, as claimed (recall that $xy \neq yx$ must be taken into account since the variables of the polynomials are matrices). □

Notice that the equations given in Proposition 4.1 require the matrix $\boldsymbol{R}$ in order to give the relationship between $\boldsymbol{A}$ and $\boldsymbol{J}$. This means that the spectral closeness measure defined at the beginning of this section is no longer useful in general for diameters $k \geq 3$. This fact encourages us to define a new measure based on the matrix polynomial $F_k(\boldsymbol{R}, \boldsymbol{Z})$ to compute the closeness to a mixed Moore graph. So, given any $(r, z, k)$-mixed radial Moore graph $G$ with adjacency matrix $\boldsymbol{A} = \boldsymbol{R} + \boldsymbol{Z}$, we can compute the value of $F_k(\boldsymbol{R}, \boldsymbol{Z})$ and rate how close it is to $\boldsymbol{J}$. To this end, good results are obtained when we use the *spectral norm* of a matrix $\boldsymbol{B}$, defined as

$$\|\boldsymbol{B}\|_2 = \sqrt{\lambda_{\max}(\boldsymbol{B}^\top \boldsymbol{B})}.$$

The spectral norm is widely used to measure the distance between matrices. Given two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, the quantity $\|\boldsymbol{A} - \boldsymbol{B}\|_2$ measures the largest possible difference between the actions of $\boldsymbol{A}$ and $\boldsymbol{B}$ on unit vectors (see Horn and Johnson [8]). Then, given a radial $(r, z, k)$-mixed Moore graph $G$ with adjacency matrix $\boldsymbol{A} = \boldsymbol{R} + \boldsymbol{Z}$, we define

$$\tilde{N}(G) = \|F_k(\boldsymbol{R}, \boldsymbol{Z}) - \boldsymbol{J}\|_2.$$

Accordingly, if $G$ is a mixed Moore graph, then $\tilde{N}(G) = 0$. We define $G_1$ and $G_2$ to be *spectral matrix equivalent*, $G_1 \asymp G_2$, if and only if $\tilde{N}(G_1) = \tilde{N}(G_2)$. Moreover, in the quotient set $\mathcal{RM}(r, z, k)/\asymp$, we say that $G_1$ is closer than $G_2$ to be a mixed Moore graph if $\tilde{N}(G_1) < \tilde{N}(G_2)$, in which case we denote $G_1 \prec G_2$. This measure is fine enough to discern cases when other approaches fail. For instance, for the two graphs of Figure 3, we get

$$\tilde{N}(G_1) = \sqrt{10} \approx 3.1623 \qquad \text{and} \qquad \tilde{N}(G_2) \approx 2.6554,$$

so we infer that $G_2 \prec G_1$, that is, $G_2$ is closer than $G_1$ to becoming a mixed Moore graph according to this *spectral matrix measure*.



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| $(r,z,k)$ | $\lvert\mathcal{RM}(r,z,k)\rvert$ | $\lvert\mathcal{RM}(r,z,k)/\sim\rvert$ | $\lvert\mathcal{RM}(r,z,k)/\asymp\rvert$ |
|---|---|---|---|
| $(3,0,2)$ | 4 | 4 | 4 |
| $(0,2,2)$ | 6 | 6 | 6 |
| $(3,0,3)$ | 1062 | 79 | 854 |
| $(2,1,2)$ | 9486 | 3750 | 7535 |
| $(4,0,2)$ | 282224 | 58 | 257821 |

Table 1. Total population of (mixed) radial Moore graphs for different cases $(r,z,k)$ and their corresponding number of classes according to the spectral polynomial measure and the spectral matrix measure.

## 5. Top spectral ranked graphs

The enumeration of mixed (out-)radial Moore graphs is known for $(r,z,k) = (2,1,2)$ (see Ceresuela, López, and Chemisana [4]). Besides, for the undirected case, the total population of radial Moore graphs was previously computed in Capdevila, Conde, Exoo, Gimbert, and López [3] for the cubic case $r = 3$ and radius $k \in \{2,3\}$ and the case $(4,0,2)$. In addition, we have performed the ranking for the (out-)radial Moore digraphs corresponding to the case $(0,2,2)$, where the digraphs have order 7, which is small enough to perform a simple computation to get them all (there are only 6 of them). All the computations with the new defined measures have been done with the open source software SageMath and all the data as well as the implemented measures to rank the graphs are available online[1]. Table 1 shows the number of classes defined by the equivalence relations $\sim$ and $\asymp$ defined in $\mathcal{RM}(r,z,k)$. Both measures define a total ordering in the smaller cases $(3,0,2)$ and $(0,2,2)$. More generally, the spectral matrix measure is finer than the spectral polynomial measure in the sense that it produces many more classes, especially for undirected graphs.

For every family $\mathcal{RM}(r,z,k)$, we computed the rankings for both the spectral polynomial measure and spectral matrix measure. In Table 2, one can find the top-ranked graphs according to these measures, where we denote by $\hat{G}_{(r,z,k)}$ [respectively, $\tilde{G}_{(r,z,k)}$] a graph with minimum value of $\hat{N}(G)$ [respectively, $\tilde{N}(G)$] for every graph in $\mathcal{RM}(r,z,k)$. Some graphs may share the same minimum value for one of the two spectral norms; in that case, we choose those better positioned in the other ranking. Whenever a graph has both minimum norm values, then we simply denote it by $G_{(r,z,k)}$, and we add a diamond if the corresponding graph is unique. A graphical representation of these graphs is given in Figures 4 and 5.

The values of $\alpha_i$ (for $i = 1,2,3$) are the roots of the polynomials

$$\begin{aligned}
p_1(x) = {} & x^7 - 42x^6 + 691x^5 - 5565x^4 + 22267x^3 - 39870x^2 + 27297x \\
& -2916; \\
p_2(x) = {} & x^7 - 30x^6 + 333x^5 - 1694x^4 + 3922x^3 - 3484x^2 + 1149x - 96; \\
p_3(x) = {} & x^{10} - 28x^9 + 328x^8 - 2084x^7 + 7780x^6 - 17278x^5 + 22108x^4 \\
& -15260x^3 + 5400x^2 - 884x + 49.
\end{aligned}$$

[1] https://udlcat-my.sharepoint.com/:f:/g/personal/nacho_lopez_udl_cat/IgAKLJ8-kvs1QYZ72FZ3ebF9AbZnoaleyDPEvy6jsakzgUk



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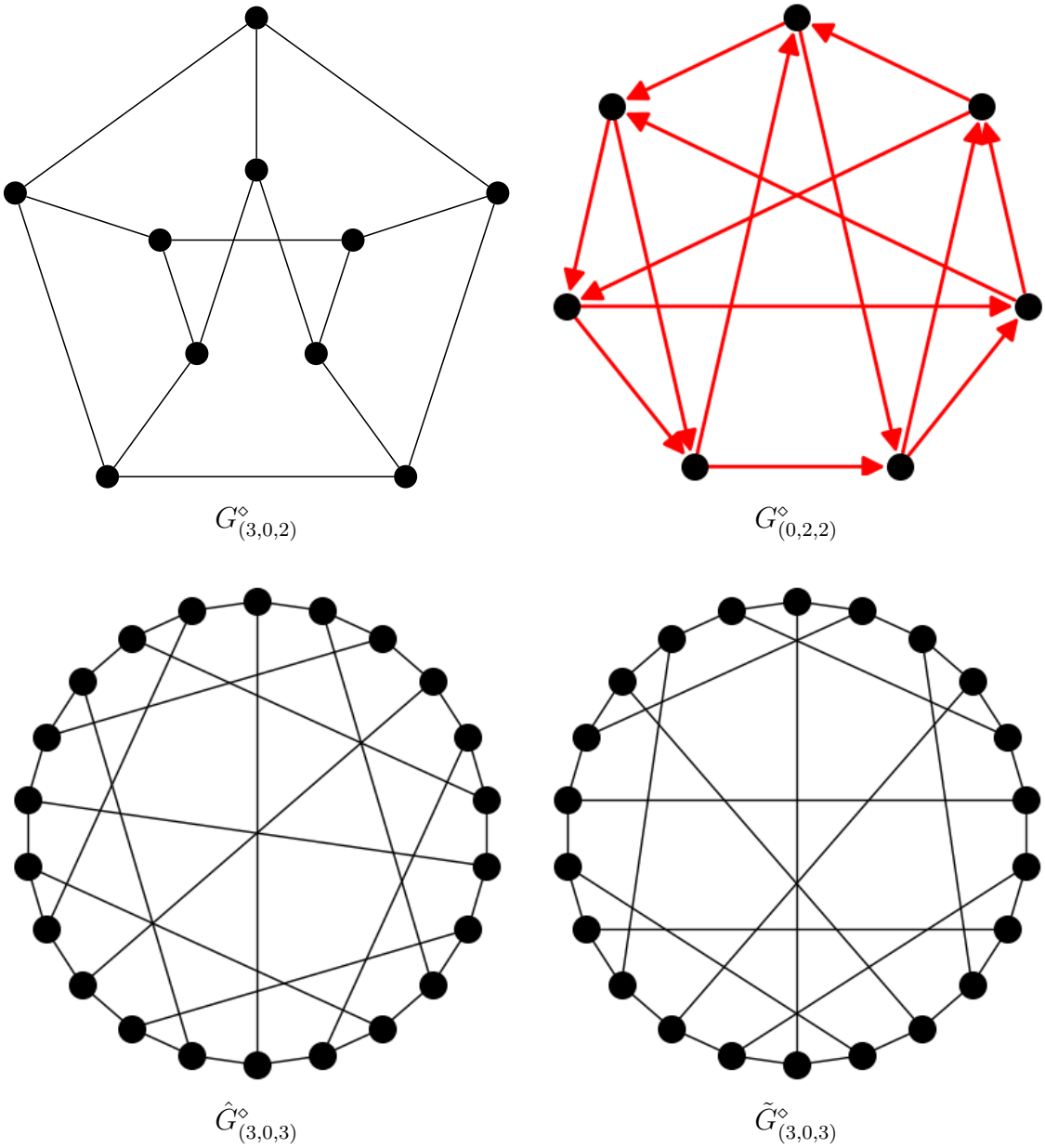



Figure 4. Some of the top-ranked graphs given in Table 2 (a diamond indicates the graph is unique).

They are irreducible factors of the characteristic polynomial of the matrix $\boldsymbol{B}^{\top}\boldsymbol{B}$, where $\boldsymbol{B} = F_k(\boldsymbol{R}, \boldsymbol{Z}) - \boldsymbol{J}$ as defined in Section 4.



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| Graph $G$ | $\hat{N}(G)$ | $\tilde{N}(G)$ | $\mathrm{rank}_{\prec}(G)$ | $\mathrm{rank}_{\succ}(G)$ |
|---|---|---|---|---|
| $G^{\diamond}_{(3,0,2)}$ | 4 | 4 | 1 | 1 |
| $G^{\diamond}_{(0,2,2)}$ | $\sqrt{2}$ | $\sqrt{10}$ | 1 | 1 |
| $\hat{G}^{\diamond}_{(3,0,3)}$ | $4\sqrt{3}$ | 4 | 1 | 12 |
| $\tilde{G}^{\diamond}_{(3,0,3)}$ | $6\sqrt{3}$ | $\sqrt{\alpha_1}$ | 12 | 1 |
| $\hat{G}_{(2,1,2)}$ | $\sqrt{2}$ | $\sqrt{\alpha_2}$ | 1 | 245 |
| $\tilde{G}^{\diamond}_{(2,1,2)}$ | $\beta_1$ | $\sqrt{\alpha_3}$ | 10 | 1 |
| $G_{(4,0,2)}$ | 6 | 3 | 1 | 1 |

Table 2. Top-ranked graphs according to the spectral polynomial measure and spectral matrix measure and their corresponding norm values, where $\alpha_1 \approx 10.5305$, $\alpha_2 \approx 11.1482$, $\alpha_3 \approx 6.4503$ and $\beta_1 \approx 2.5890$. A diamond indicates the graph is unique.

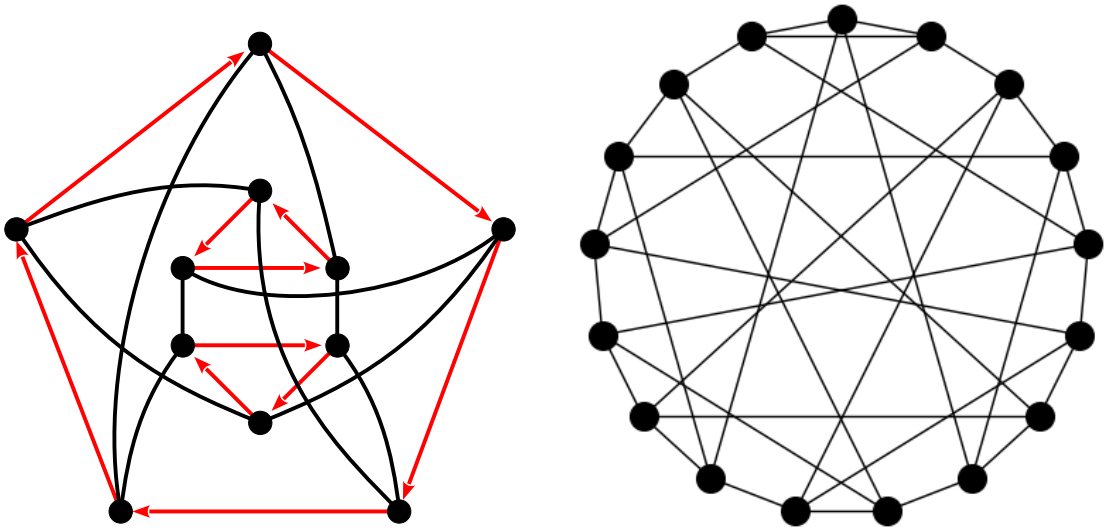



Figure 5. The remaining top-ranked graphs given in Table 2. Graph $\hat{G}_{(2,1,2)}$ has been depicted before in Figure 3 as graph $G_1$.

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