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Injective edge-coloring of claw-free graphs with maximum degree 4

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Abstract

An injective k-edge-coloring of a graph G is a mapping ϕ: $E(G) \to \{1, 2, ..., k\}$, such that $\phi(e) \neq \phi(e')$ if edges e and e' are at distance two, or are in a triangle. The smallest integer k such that G has an injective k-edge-coloring is called the injective chromatic index of G, denoted by $\chi_i'(G)$. A graph is called claw-free if it has no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. In this paper, we show that $\chi_i'(G) \leq 13$ for every claw-free graph G with $\Delta(G) \leq 4$, where $\Delta(G)$ is the maximum degree of G.

Keywords: maximum degree, claw-free, injective edge-coloring
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1. Introduction

Only simple and finite graphs are considered in this paper. We use $V(G)$, $E(G)$ and $\Delta(G)$ to denote the vertex set, edge set and maximum degree of a graph G, respectively. For a vertex $v \in V(G)$, $N(v)$ is the set of vertices adjacent to v, and $d(v) = |N(v)|$ is the *degree* of v. Similarly, we can define $N(e)$, the set of edges adjacent to e. A vertex of degree k (at least k, or at most k) is called a *k-vertex* (a *k^+-vertex*, or a *k^--vertex*, respectively). For a vertex subset S of $V(G)$, we use $G[S]$ to denote the subgraph of G that is induced by S. Let n, m be two integers. A complete

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bipartite graph with one part having n vertices and the other part m vertices is denoted by $K_{n,m}$. A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$.

An injective k-edge-coloring of a graph G is a mapping ϕ: $E(G) \to \{1, 2, ..., k\}$, such that $\phi(e) \neq \phi(e')$ if edges e and e' are at distance two, or are in a triangle. The smallest integer k such that G has an injective k-edge-coloring is called the *injective chromatic index* of G, denoted by $\chi_i'(G)$. The concept of injective edge-coloring was proposed in 2015 by Cardose et al. [2] to slove the Packet Radio Network problem and they proved that it is NP-hard to compute the injective chromatic index for any graph. Moreover, Ferdjallah et al.[5] showed that $\chi_i'(G) \leq 2(\Delta(G) - 1)^2$ for any graph G with $\Delta(G) \geq 3$; and $\chi_i'(G) \leq 30$ for any planar graph G. In particular, they proposed the following conjecture.

Conjecture 1. *For every subcubic graph G, $\chi_i'(G) \leq 6$.*

In 2022, Miao et al. [10] posed the following conjecture.

Conjecture 2. *For every simple graph G with maximum degree Δ, $\chi_i'(G) \leq \Delta(\Delta - 1)$.*

Several authors have attacked this upper bound on the injective chromatic index for graphs with small maximum degree. Towards Conjecture 1, Kostochka et al. [8] confirmed that $\chi_i'(G) \leq 7$ for subcubic graphs and proved that the upper bound 7 can be improved to 6 for subcubic planar graphs.

For graphs with maximum degree 4, we summarize the upper bounds of injective chromatic index for graphs with maximum average degree restrictions.

Theorem 1.1. *Let G be a graph with $\Delta(G) = 4$. We say the graph G is a (m, k)-graph if $\mathrm{mad}(G) < m$ and $\chi_i'(G) \leq k$.*

(1) *G is a (m, k)-graph for $m = \frac{7}{3}$ and $k = 5$* [7]*;*
(2) *G is a (m, k)-graph for $(m, k) \in \{(\frac{5}{2}, 6), (\frac{13}{5}, 7), (\frac{36}{13}, 8)\}$* [6]*;*
(3) *G is a (m, k)-graph for $(m, k) \in \{(\frac{14}{5}, 9), (3, 10), (\frac{19}{6}, 11)\}$* [10]*;*
(4) *G is a (m, k)-graph for $m = \frac{33}{10}$ and $k = 12$* [9]*;*
(5) *G is a (m, k)-graph for $(m, k) \in \{(\frac{10}{3}, 13), (\frac{18}{5}, 14), (\frac{15}{4}, 15)\}$* [1].

For claw-free graphs, Dong et al. [4] confirmed that the injective chromatic index of any claw-free subcubic graph is less than or equal to 6 and the upper bound 6 is tight in 2023. Cui and Han [3] proved that $\chi_i'(G) \leq 5$ for every connected claw-free subcubic graph G that is not isomorphic to K_4 and $\overline{C_6}$ in 2024.

In this paper, we consider the injective chromatic index of claw-free graphs with maximum degree at most 4.

Theorem 1.2. *Let G be a claw-free graph with $\Delta(G) \leq 4$. Then $\chi_i'(G) \leq 13$.*

Suppose that G has a partial injective edge-coloring ϕ with the color set C. For each edge e' and e in G, we say that edge e' *sees* the edge e if they are at distance two or are in a triangle. For $e = uv \in E(G)$, we denote the set of the colors of the edges that see e as $F_\phi(e)$ and denote the set of available colors of e as $S_\phi(e)$. Obviously, $S_\phi(e) = C - F_\phi(e)$ and $|F_\phi(e)| \leq 3(d(u) + d(v) - 2)$.

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We simply write $S_\phi(e)$ as $S(e)$ if there is no confusion. For a vertex $v \in V(G)$, we denote the set of the colors of the edges incident with v as $C_\phi(v)$.

For all figures in this paper, a vertex is represented by a solid point when all of its incident edges are drawn; otherwise it is represented by a hollow point. We will use the labels as shown in the figures.

2. Proof of Theorem 1.2

Assume that G is a counterexample of Theorem 1.2 such that $|V(G)|$ is as small as possible. Recall that $\Delta(G) \leq 4$. Then G is a connected claw-free graph.

Remark 2.1. Let $v \in V(G)$ and $uv \in E(G)$. Suppose that u is not adjacent to any other vertices in $N(v) \setminus \{u\}$. Since G is claw-free, we have $xy \in E(G)$ for any two vertices $x, y \in N(u) \setminus \{v\}$. So vu sees at most 6 edges at the vertex u.

Lemma 2.1. $\delta(G) = 4$.

Proof. Suppose to the contrary that G contains a 3^--vertex v. Let $d(v) = k \leq 3$ and $N(v) = \{v_1, v_2, \ldots, v_k\}$. By the minimality of G, $G' = G - v$ has an injective 13-edge-coloring ϕ.

Case 1. $k = 1$.

Since $|S(vv_1)| \geq 13 - 3(d(v_1) - 1) \geq 4$, we can extend ϕ to G, a contradiction.

Case 2. $k = 2$.

First suppose that $v_1v_2 \in E(G)$. Then $|S(vv_1)| \geq 13 - 3(d(v_1) - 2) - (d(v_2) - 1) \geq 4$ and $|S(vv_2)| \geq 13 - 3(d(v_2) - 2) - (d(v_1) - 1) \geq 4$, we can extend ϕ to G, a contradiction.

Next suppose that $v_1v_2 \notin E(G)$. Then vv_1 sees at most 6 edges at the vertex v_1 by Remark 2.1. So $|S(vv_1)| \geq 13 - (6 + (d(v_2) - 1)) \geq 4$. By symmetry, $|S(vv_2)| \geq 4$. We can extend ϕ to G, a contradiction.

Case 3. $k = 3$.

Set $q = |E(G[\{v_1, v_2, v_3\}])|$. Then $1 \leq q \leq 3$ by G is claw-free.

Subcase 3.1. $q = 1$, say $v_1v_2 \in E(G)$.

Then $v_2v_3 \notin E(G)$ and $v_1v_3 \notin E(G)$. Then vv_3 sees at most 6 edges at the vertex v_3 by Remark 2.1. Hence $|S(vv_3)| \geq 13 - (6 + (d(v_1) + d(v_2) - 3)) \geq 2$. Next we can show that $|S(vv_1)| \geq 2$. In fact, if $d(v_1) = 3$ or $d(v_2) = 3$, then $|S(vv_1)| \geq 13 - 3(d(v_1) - 2) + (d(v_2) - 1) + 3) \geq 2$. Now we can suppose that $d(v_1) = d(v_2) = 4$. Then $xy \in E(G)$ by $G[\{v, x, y\}]$ is not isomorphic to $K_{1,3}$, where $x, y \in N(v_1) \setminus \{v, v_2\}$. So $|S(vv_1)| \geq 13 - (5 + (d(v_2) - 1) + (d(v_3) - 1)) \geq 2$. Hence, we show that $|S(vv_1)| \geq 2$. By symmetry, $|S(vv_2)| \geq 2$. Then ϕ can be extended to be an injective 13-edge-coloring of G, a contradiction.

Subcase 3.2. $q = 2$, say $v_1v_2 \in E(G)$ and $v_2v_3 \in E(G)$.

Then $|S(vv_1)| \geq 13 - (3(d(v_1) - 2) + (d(v_2) - 1) + (d(v_3) - 2)) \geq 2$, and $|S(vv_2)| \geq 13 - (3 + (d(v_1) - 1) + (d(v_3) - 1)) \geq 4$. By symmetry, $|S(vv_3)| \geq 2$. So we can extend ϕ to G, a contradiction.

Subcase 3.3. $q = 3$, say $v_1v_2 \in E(G)$, $v_2v_3 \in E(G)$ and $v_1v_3 \in E(G)$.

Then $|S(vv_i)| \geq 13 - (3 + 5) = 5$ for each $i \in \{1, 2, 3\}$, we can extend ϕ to G, a contradiction. □

Lemma 2.2. *G does not contain K_4 as a subgraph.*

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Proof. Suppose that G contains K_4 as a subgraph. Set $V(K_4) = \{v_1, v_2, v_3, v_4\}$. Let u_i be the neighbor of v_i not in $V(K_4)$ for each $i \in \{1, 2, 3, 4\}$.

Suppose that $u_1 = u_2$. By the minimality of G, $G' = G - v_1$ has an injective 13-edge-coloring ϕ. Since $|S(v_1u_1)| \geq 13-3(d(u_1)-2)-(d(v_2)+d(v_3)+d(v_4)-6) = 1$, $|S(v_1v_2)| \geq 13-(d(u_1)-1) - (d(v_3) + d(v_4) - 3) = 5$, and $|S(v_1v_3)| \geq 13 - (3 + (d(v_4) + d(v_2) - 3) + (d(u_1) - 2)) = 3$. By symmetry, $|S(v_1v_4)| \geq 3$. So we can extend ϕ to G, a contradiction.

So we may assume that any two of u_1, u_2, u_3, u_4 are not coincide. By the minimality of G, $G' = G - \{v_1, v_2, v_3, v_4\}$ has an injective 13-edge-coloring ϕ. Then v_iu_i sees at most 6 edges at the vertex u_i by Remark 2.1 for each $i \in \{1, 2, 3, 4\}$. So $|S(u_iv_i)| \geq 13 - 6 = 7$. Since $|S(v_iv_j)| \geq 13 - (d(u_i) + d(u_j) - 2) = 7$ for each pair $i, j \in \{1, 2, 3, 4\}$, we can extend ϕ to G by coloring $v_1v_4, v_1v_2, v_2v_3, v_3v_4, v_1v_3, v_2v_4, v_4u_4, v_2u_2, v_3u_3$ and v_1u_1 in order, a contradiction. □

Lemma 2.3. *Any* 4*-vertex in* G *is incident with at most two 3-cycles.*

Proof. Suppose to the contrary that there exists 4-vertex v incident with three 3-cycles in G. Let $N(v) = \{v_1, v_2, v_3, v_4\}$. By Lemma 2.2 and G is claw-free, we may assume that $v_1v_2 \in E(G)$, $v_2v_3 \in E(G)$ and $v_1v_4 \in E(G)$. Set $N(v_1) = \{v, v_2, v_4, u_1\}$ and $N(v_2) = \{v, v_1, v_3, u_2\}$. By Lemma 2.2, $u_1 \neq v_3$ and $u_2 \neq v_4$. By the minimality of G, $G' = G - v$ has an injective 13-edge-coloring ϕ. Since G is claw-free, $u_1 = u_2$, or $u_1v_4 \in E(G)$ and $u_2v_3 \in E(G)$.

Suppose that $u_1 = u_2$. First suppose that $v_3v_4 \notin E(G)$. We erase the color of v_1v_4. Then vv_i sees at most 7 edges at the vertex v for each $i \in \{1, 2\}$, and vv_i sees at most 6 edges at the vertex v for each $i \in \{3, 4\}$. Since $|S(vv_3)| \geq 13 - (2 \times 3 + 6) = 1$, $|S(vv_4)| \geq 13 - (2 \times 3 + 6) = 1$, $|S(v_1v_4)| \geq 13 - (2 \times 3 + 4) = 3$, $|S(vv_1)| \geq 13 - (7 + 2) = 4$ and $|S(vv_2)| \geq 13 - (7 + 2) = 4$, we can extend ϕ to G, a contradiction. Next suppose that $v_3v_4 \in E(G)$. Then vv_i sees at most 7 edges at the vertex v for each $i \in \{1, 2, 3, 4\}$. Since $|S(vv_3)| \geq 13 - (7 + 3) = 3$, $|S(vv_4)| \geq 13 - (7 + 3) = 3$, $|S(vv_1)| \geq 13 - (7 + 2) = 4$ and $|S(vv_2)| \geq 13 - (7 + 2) = 4$, we can extend ϕ to G, a contradiction.

Suppose that $u_1v_4 \in E(G)$ and $u_2v_3 \in E(G)$. By the above case, we can deduce that $v_3v_4 \notin E(G)$. Then vv_i sees at most 7 edges at the vertex v for each $i \in \{3, 4\}$, and vv_i sees at most 8 edges at the vertex v for each $i \in \{1, 2\}$. Since $|S(vv_3)| \geq 13 - (7 + 3 + 2) = 1$, $|S(vv_4)| \geq 13 - (7 + 3 + 2) = 1$, $|S(vv_1)| \geq 13 - (8 + 2) = 3$ and $|S(vv_2)| \geq 13 - (8 + 2) = 3$, we can extend ϕ to G, a contradiction. □

By Lemma 2.2 and 2.3, the following lemma holds trivially.

Lemma 2.4. *Each* 4*-vertex in* G *is incident with exactly two edge-disjoint triangles.*

Lemma 2.5. *There is no* 4*-cycles in* G*.*

Proof. Suppose to the contrary that there exists a 4-cycle $xyuvx$. By Lemma 2.4, each vertex in $\{x, y, u, v\}$ is incident with exactly two edge-disjoint triangles, as shown in Figure 1. Let $N(y_1) = \{y, u, y_1', y_1''\}$ and $N(u_1) = \{u, v, u_1', u_1''\}$. By Lemma 2.4, $y_1'y_1'' \in E(G)$ and $u_1'u_1'' \in E(G)$. By the minimality of G, $G' = G - \{y, y_1, u, v, u_1\}$ has an injective 13-edge-coloring ϕ.

Case 1. $(N(y_1) \cap N(u_1)) \setminus \{u\} \neq \emptyset$, say $y_1'' = u_1'$.

We have $|S(yu)| \geq 13 - (d(x_1) + d(x) - 4) = 9$. By symmetry, $|S(uy_1)| \geq 9$, $|S(uv)| \geq 9$

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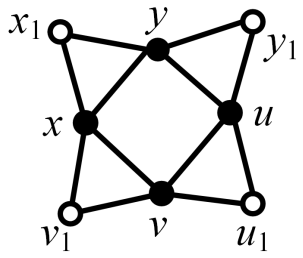


Figure 1. There exists a 4-cycle $xyuvx$ in G.

and $|S(uu_1)| \geq 9$. Since $|S(yu)| + |S(uv)| \geq 18 > 13$, we have $|S(yu) \cap S(uv)| \geq 1$. We color yu and uv with a color $\alpha \in S(yu) \cap S(uv)$, and denote this new coloring as ϕ'. Now $|S_{\phi'}(uy_1)| \geq 9 - 1 = 8$ and $|S_{\phi'}(uu_1)| \geq 9 - 1 = 8$. Similarly, we can color uy_1 and uu_1 with a color $\beta \in S_{\phi'}(uy_1) \cap S_{\phi'}(uu_1)$ by $|S_{\phi'}(uy_1)| + |S_{\phi'}(uu_1)| \geq 16 > 13$. Denote this new coloring as ϕ''. Then $|S_{\phi''}(yx_1)| \geq 13 - ((d(x) - 2) + 5 + 2) = 4$, $|S_{\phi''}(yx)| \geq 13 - ((d(x_1) - 1) + (d(v_1) - 2) + 2) = 6$ and $|S_{\phi''}(yy_1)| \geq 13 - ((d(x_1) + d(x) - 4) + (d(y_1') + d(y_1'') - 4) + 2) = 3$. By symmetry, $|S_{\phi''}(y_1y_1')| \geq 4$, $|S_{\phi''}(y_1y_1'')| \geq 6$, $|S_{\phi''}(vv_1)| \geq 4$, $|S_{\phi''}(vx)| \geq 6$, $|S_{\phi''}(u_1u_1'')| \geq 4$, $|S_{\phi''}(u_1y_1'')| \geq 6$ and $|S_{\phi''}(vu_1)| \geq 3$. Hence we can extend ϕ'' to G by coloring $y_1y_1', y_1y_1'', u_1y_1'', u_1u_1'', yx_1, vv_1, yx, xv, yy_1$ and vu_1 in order, a contradiction.

Case 2. $(N(y_1) \cap N(u_1)) \setminus \{u\} = \emptyset$.

We have $|S(yu)| \geq 13 - (d(x_1) + d(x) - 4) = 9$, $|S(uy_1)| \geq 13 - (d(y_1') + d(y_1'') - 3) = 8$. By symmetry, $|S(uv)| \geq 9$ and $|S(uu_1)| \geq 8$. Since $|S(yu)| + |S(uv)| \geq 18 > 13$, we have $|S(yu) \cap S(uv)| \geq 1$. We color yu and uv with a color $\alpha \in S(yu) \cap S(uv)$, and denote this new coloring as ϕ'. Now $|S_{\phi'}(uy_1)| \geq 8 - 1 = 7$ and $|S_{\phi'}(uu_1)| \geq 8 - 1 = 7$. Similarly, we can color uy_1 and uu_1 with a color $\beta \in S_{\phi'}(uy_1) \cap S_{\phi'}(uu_1)$ by $|S_{\phi'}(uy_1)| + |S_{\phi'}(uu_1)| \geq 14 > 13$. Denote this new coloring as ϕ''. Then $|S_{\phi''}(y_1y_1')| \geq 13 - ((d(y_1'') - 1) + 5 + 2) = 3$, $|S_{\phi''}(yx_1)| \geq 13 - ((d(x) - 2) + 5 + 2) = 4$, $|S_{\phi''}(yx)| \geq 13 - ((d(x_1) - 1) + (d(v_1) - 2) + 2) = 6$ and $|S_{\phi''}(yy_1)| \geq 13 - ((d(x_1) - 1) + (d(x) - 3) + (d(y_1') + d(y_1'') - 3) + 2) = 2$. By symmetry, $|S_{\phi''}(y_1y_1'')| \geq 3$, $|S_{\phi''}(u_1u_1')| \geq 3$, $|S_{\phi''}(u_1u_1'')| \geq 3$, $|S_{\phi''}(vv_1)| \geq 4$, $|S_{\phi''}(xv)| \geq 6$ and $|S_{\phi''}(vu_1)| \geq 2$. Hence we can extend ϕ'' to G by coloring $y_1y_1', y_1y_1'', u_1u_1', u_1u_1'', yx_1, vv_1, yx, xv, yy_1$ and vu_1 in order, a contradiction. □

Now we are ready to show Theorem 1.2. Let $v \in V(G)$ with $N(v) = \{u_1, u_2, u_3, u_4\}$. By Lemma 2.4, we may assume that $u_1u_2 \in E(G)$ and $u_3u_4 \in E(G)$. Let $N(u_1) = \{x_1, y_1, v, u_2\}$, $N(u_2) = \{x_2, y_2, v, u_1\}$, $N(u_3) = \{x_3, y_3, v, u_4\}$, and $N(u_4) = \{x_4, y_4, v, u_3\}$. By Lemma 2.4, $x_iy_i \in E(G)$, $u_i \neq x_j$ and $u_i \neq y_j$ for each pair $i, j \in \{1, 2, 3, 4\}$. By Lemma 2.4 and 2.5, $x_i \neq x_j$, $y_i \neq y_j$, $x_i \neq y_j$ for each $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$, as shown in Figure 2. By the minimality of G, $G' = G - \{v, u_1, u_2, u_3, u_4\}$ has an injective 13-edge-coloring ϕ with the color set C.

Since G is claw-free and $\Delta(G) \leq 4$, we have $|S(vu_i)| \geq 13 - 5 = 8$, $|S(u_ix_i)| \geq 13 - 8 = 5$, $|S(u_iy_i)| \geq 13 - 8 = 5$ for each $i \in \{1, 2, 3, 4\}$. Since $|S(vu_1)| + |S(vu_3)| \geq 16 > 13$, we have $|S(vu_1) \cap S(vu_3)| \geq 1$. We color vu_1 and vu_3 with a color $\alpha \in S(vu_1) \cap S(vu_3)$, and denote this new coloring as ϕ'. Now $|S_{\phi'}(u_ix_i)| \geq 5 - 1 = 4$ and $|S_{\phi'}(u_iy_i)| \geq 5 - 1 =$

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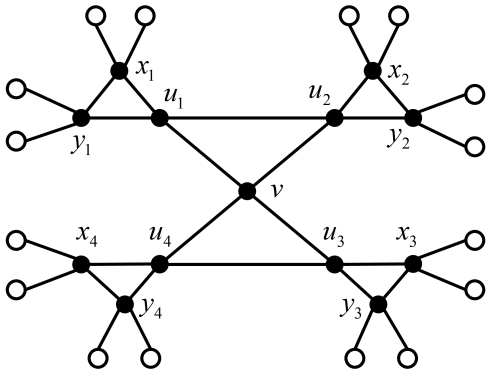


Figure 2. The configuration used in the proof of Theorem 1.2.

4 for each $i \in \{1,2,3,4\}$. We can color $u_1x_1, u_1y_1, u_2x_2, u_2y_2, u_3x_3, u_3y_3, u_4x_4$ and u_4y_4 with $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$ in order, and denote the obtained new coloring as ϕ''. Then $|S_{\phi''}(vu_2)| \geq 8-1-6=1$ and $|S_{\phi''}(vu_4)| \geq 8-1-6=1$. We color vu_2 with β_1, and color vu_4 with β_2. Denote the obtained new coloring as ϕ^*.

Suppose that $\beta_1 = \beta_2$. Since $|S_{\phi^*}(u_1u_2)| \geq 13-(5+5+2)=1$ and $|S_{\phi^*}(u_3u_4)| \geq 13-(5+5+2)=1$, we can extend ϕ^* to G, a contradiction. So we may assume that $\beta_1 \neq \beta_2$, say $\alpha=1, \beta_1=2$ and $\beta_2=3$. If $|S_{\phi^*}(u_1u_2)| \geq 1$ and $|S_{\phi^*}(u_3u_4)| \geq 1$, then we can extend ϕ^* to G, a contradiction. By symmetry, we may assume that $|S_{\phi^*}(u_1u_2)| = 0$. That is $(C_{\phi^*}(x_1) \cup C_{\phi^*}(y_1)) \setminus \{b_1, b_2\} = \{4,5,6,7,8\}$ and $(C_{\phi^*}(x_2) \cup C_{\phi^*}(y_2)) \setminus \{b_3, b_4\} = \{9,10,11,12,13\}$. If we can recolor vu_2 with 3, then turn to the case that $\beta_1 = \beta_2$, a contradiction. Hence $3 \in F_{\phi*}(vu_2)$, say $3 \in \{b_1, b_2, b_5, b_6, b_7, b_8\}$. We can deduce that $3 \in \{b_7, b_8\}$, say $b_7 = 3$, by ϕ^* is the partial injective edge-coloring of G.

- If there exists a color $\gamma \in \{4,5,6,7,8\}$ such that $\gamma \notin F_{\phi*}(vu_2)$, say $\gamma = 4$, then we recolor vu_2 with γ and color u_1u_2 with 2. Denote this new coloring as ϕ_1. Now if $|S_{\phi_1}(u_3u_4)| \geq 1$, then we can extend ϕ_1 to G, a contradiction. Hence $F_{\phi_1}(u_3u_4) = C$. Let $(C_{\phi_1}(x_3) \cup C_{\phi_1}(y_3)) \setminus \{b_5, b_6\} = \{c_1, c_2, \ldots, c_5\}$ and $(C_{\phi_1}(x_4) \cup C_{\phi_1}(y_4)) \setminus \{b_7, b_8\} = \{d_1, d_2, \ldots, d_5\}$. Then $\{c_1, \ldots, c_5, d_1, \ldots, d_5\} = \{2, 5, 6, \ldots, 13\}$. If we can recolor vu_4 with 4, then we can color u_3u_4 with 3 to obtain an injective 13-edge-coloring of G, a contradiction. Hence $4 \in F_{\phi_1}(vu_4)$, say $4 \in \{b_1, b_2, \ldots, b_6\}$. By ϕ_1 is the partial injective edge-coloring of G, we have $4 \in \{b_3, b_4\}$, say $b_3 = 4$. Note that $3 \notin S_{\phi_1}(vu_3)$. If we can recolor vu_3 with a color $\gamma \in S_{\phi_1}(vu_3) \setminus \{1\}$, then we can recolor or color vu_4, u_3u_4 with 1, 3, respectively. The obtained coloring is the injective 13-edge-coloring of G, a contradiction. Hence $F_{\phi_1}(vu_3) \cup \{1\} = C$. That is $\{b_1, b_2, b_4, b_8, 2, c_1, c_2, \ldots, c_5\} = \{2, 5, 6, \ldots, 13\}$. Recall that $\{c_1, c_2, \ldots, c_5, d_1, d_2, \ldots, d_5\} = \{2, 5, 6, \ldots, 13\}$. We can deduce that $\{b_1, b_2, b_4, b_8\} \subseteq \{d_1, d_2, \ldots, d_5\}$. Now $|F_{\phi_1}(vu_4)| \leq 10$, we can recolor vu_4 with a color $\eta \in S_{\phi_1}(vu_4) \setminus \{3\}$ and color u_3u_4 with 3 to obtain the injective 13-edge-coloring of G, a contradiction.

- If $\{4,5,6,7,8\} \subseteq F_{\phi*}(vu_2)$, then $\{b_1, b_2, b_5, b_6, b_8\} = \{4,5,6,7,8\}$. Now we erase the color of vu_1, and recolor or color vu_2, u_1u_2 with 1, 2, respectively. We denote this new coloring

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as ϕ_2. Since $|S_{\phi_2}(u_3u_4)| \geq 13 - 12 = 1$ and $|S_{\phi_2}(vu_1)| \geq 13 - 10 = 3$, we can extend ϕ_2 to G, a contradiction.

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