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A note on the edge Roman domination in trees

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Abstract

A subset X of edges of a graph G is called an *edge dominating set* of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G. An *edge Roman dominating function* of a graph G is a function $f : E(G) \to \{0, 1, 2\}$ such that every edge e with $f(e) = 0$ is adjacent to some edge e' with $f(e') = 2$. The weight of an edge Roman dominating function f is the value $w(f) = \sum_{e\in E(G)} f(e)$. The edge Roman domination number of G, denoted by $\gamma'_R(G)$, is the minimum weight of an edge Roman dominating function of G. In this paper, we characterize trees with edge Roman domination number twice the edge domination number.

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1. Introduction

For notation and graph theory terminology in general we follow [5]. Let $G = (V, E)$ be a simple graph. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v, denoted by $\deg(v)$, is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. An edge incident to a leaf is called a *pendant edge*. A *strong support vertex* is a vertex that is adjacent to at least two leaves. A tree T is a *double star* if it

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contains exactly two vertices that are not leaves. For $a, b \geq 2$, a double star whose support vertices have degree a and b is denoted by $S(a, b)$. If T is a rooted tree, we for each vertex v, we denote by T_v the sub-rooted tree rooted at v. The height of a rooted tree is the maximum distance from the root to a leaf.

A subset X of E is called an *edge dominating set* of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G. We refer to an edge dominating set with minimum cardinality as a $\gamma'(G)$-set. The concept of edge domination was introduced by Mitchell and Hedetniemi [7]. A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function*, or just RDF, if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G (see [3, 6]).

Roushini Leely Pushpam *et al.* [8] initiated the study of the edge version of Roman domination. An *edge Roman dominating function* (or just ERDF) of a graph G is a function $f : E(G) \to \{0, 1, 2\}$ such that every edge e with $f(e) = 0$ is adjacent to some edge e' with $f(e') = 2$. The weight of an edge Roman dominating function f is the value $w(f) = \sum_{e \in E(G)} f(e)$. The edge Roman domination number of G, denoted by $\gamma'_R(G)$, is the minimum weight of an edge Roman dominating function of G. We refer to an ERDF with minimum weight as a $\gamma'_R(G)$-function. If f is a $\gamma'_R(G)$-function, then we simply write $f = (E_0, E_1, E_2)$, where $E_i = \{e \in E(G) : f(e) = i\}$, $i = 0, 1, 2$. It is easy to see that $\gamma'_R(G) \leq 2\gamma'(G)$ for any graph G. The concept of edge Roman domination is further studied by several authors, (see for example [1, 2, 4]).

In this paper we give a constructive characterization for trees whose edge Roman domination number is twice the edge domination number. We use the following.

Theorem 1.1 ([4]). *For a graph G, $\gamma'_R(G) = 2\gamma'(G)$ if and only if there is a $\gamma'_R(G)$-function f with $E_1 = \emptyset$.*

2. Main result

A support vertex v of a tree is called a *special support vertex* if no $\gamma'_R(T)$-function assigns 2 to a pendant edge at v. Let $\mathcal{F}_1$ be the class of all rooted trees, such that the root has degree at least two, any leaf is within distance two from the root, and any child of the root is either a leaf or a strong support vertex.

Now we present a constructive characterization of trees T with $\gamma'_R(T) = 2\gamma'(T)$. For this purpose, we define a family of trees as follows. Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence $T_1, T_2, \cdots, T_j$ $(j \geq 2)$ such that T_1 is a star $K_{1,r}$ for $r \geq 2$, or a double-star, and if $j \geq 2$, T_{i+1} can be obtained recursively from T_i for $1 \leq i \leq j - 1$ by one of the following operations.

Operation $\mathcal{O}_1$. Assume that $w \in V(T_i)$. Then T_{i+1} is obtained from T_i by joining w to the root of a tree of $\mathcal{F}_1$.

Operation $\mathcal{O}_2$. Assume that $w \in V(T_i)$. Then T_{i+1} is obtained from T_i by joining w to a leaf of a star of order at least four.

Operation $\mathcal{O}_3$. Assume that $w \in V(T_i)$ is a special support vertex or a leaf. Then T_{i+1} is obtained from T_i by joining w to a leaf of a path P_3, or joining w to a center of $S(a,2)$ whose degree is a.

Operation $\mathcal{O}_4$. Assume that $w \in V(T_i)$ is a vertex that has a neighbor u of degree at least two such that any vertex of $N(u) - \{w\}$ is a leaf. Then T_{i+1} is obtained from T_i by joining w to a leaf of a path P_3, or joining w to a center of $S(a,2)$ whose degree is a.

Operation $\mathcal{O}_5$. Assume that $w \in V(T_i)$ is a vertex such that (1) a component of $T - w$ is a path $P_3 : xyz$, where $x \in N_{T_i}(w)$, or (2) a component of $T - w$ is a double-star $S(a,2)$, where w is adjacent to a vertex of maximum degree $S(a,2)$. Then T_{i+1} is obtained from T_i by joining w to a leaf of a path P_3, or joining w to a center of $S(a,2)$ whose degree is a.

Lemma 2.1. *If $\gamma'_R(T_i) = 2\gamma'(T_i)$ and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_1$, the $\gamma'_R(T_{i+1}) = 2\gamma'(T_{i+1})$.*

Proof. Let $\gamma'_R(T_i) = 2\gamma'(T_i)$, and $w \in V(T_i)$. Assume that T_{i+1} is obtained by joining w to the root x of a tree $T \in \mathcal{F}_1$. Let $y_1, \ldots, y_k$ be the children of x which are strong support vertex. Clearly adding xy_i ($i = 1, 2, \ldots, k$) to any $\gamma'(T_i)$-set yields an edge dominating set for T_{i+1}, and so $\gamma'(T_{i+1}) \leq \gamma'(T_i) + k$. Furthermore, any $\gamma'_R(T_i)$-function can be extended to an ERDF for T_{i+1} by assigning 2 to xy_i ($i = 1, 2, \ldots, k$), and 0 to wx and each other edge of T_{i+1}. Thus $\gamma'_R(T_{i+1}) \leq \gamma'_R(T_i) + 2k$. Let $f = (E_0, E_1, E_2)$ be a $\gamma'_R(T_{i+1})$-function such that $|E_2|$ is maximum. Clearly we may assume that $f(xy_i) = 2$ ($i = 1, 2, \ldots, k$). If $f(wx) = 2$, then we replace $f(wx)$ by 0, and one edge of T_i at w by 2. Thus we may assume that $f(xw) = 0$. Then $f|_{V(T_i)}$ is an ERDF for T_i, implying that $\gamma'_R(T_i) \leq \gamma'_R(T_{i+1}) - 2k$. Thus $\gamma'_R(T_{i+1}) = \gamma'_R(T_i) + 2k$. Now,

$$\gamma'(T_i) = \frac{\gamma'_R(T_i)}{2} = \frac{\gamma'_R(T_{i+1}) - 2k}{2} \leq \frac{2\gamma'(T_{i+1}) - 2k}{2} = \gamma'(T_{i+1}) - k,$$

and thus $\gamma'(T_{i+1}) \geq \gamma'(T_i) + k$. Thus $\gamma'(T_{i+1}) = \gamma'(T_i) + k$. Now $\gamma'_R(T_{i+1}) = \gamma'_R(T_i) + 2k = 2\gamma'(T_i) + 2k = 2\gamma'(T_{i+1})$. □

Lemma 2.2. *If $\gamma'_R(T_i) = 2\gamma'(T_i)$ and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_2$, the $\gamma'_R(T_{i+1}) = 2\gamma'(T_{i+1})$.*

Proof. Let $\gamma'_R(T_i) = 2\gamma'(T_i)$, and $w \in V(T_i)$. Assume that T_{i+1} is obtained by joining w to a leaf x of a star of order at least four. Let y be the center of the added star and $x, y_1, \ldots, y_l$ ($l \geq 2$) be the leaves of the added star. Clearly adding xy to any $\gamma'(T_i)$-set yields an edge dominating set for T_{i+1}, and so $\gamma'(T_{i+1}) \leq \gamma'(T_i) + 1$. Furthermore, any $\gamma'_R(T_i)$-function can be extended to an ERDF for T_{i+1} by assigning 2 to xy and 0 to wx and yy_i ($i = 1, \ldots, l$). Thus $\gamma'_R(T_{i+1}) \leq \gamma'_R(T_i) + 2$. Let f be a $\gamma'_R(T_{i+1})$-function. Clearly we may assume that $f(xy) = 2$. If $f(wx) = 2$, then may assume that $f(e) = 0$ for every edge of T_i at w. Then we replace $f(wx)$ by 0, and one edge of T_i incident with w by 2. Thus we may assume that $f(xw) = 0$. Then $f|_{V(T_i)}$ is an ERDF for T_i, implying that $\gamma'_R(T_i) \leq \gamma'_R(T_{i+1}) - 2$. Thus $\gamma'_R(T_{i+1}) = \gamma'_R(T_i) + 2$. Now, $\gamma'(T_i) = \frac{\gamma'_R(T_i)}{2} = \frac{\gamma'_R(T_{i+1})-2}{2} \leq \frac{2\gamma'(T_{i+1})-2}{2} = \gamma'(T_{i+1}) - 1$, and this implies that $\gamma'(T_{i+1}) = \gamma'(T_i) + 1$. Now $\gamma'_R(T_{i+1}) = \gamma'_R(T_i) + 2 = 2\gamma'(T_i) + 2 = 2\gamma'(T_{i+1})$. □

Lemma 2.3. *If $\gamma'_R(T_i) = 2\gamma'(T_i)$ and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_3$, the $\gamma'_R(T_{i+1}) = 2\gamma'(T_{i+1})$.*

Proof. Let $\gamma'_R(T_i) = 2\gamma'(T_i)$. Assume that w is a special support vertex of T_i, and assume that T_{i+1} is obtained by joining w to the leaf x of a path xyz. Clearly adding xy to any $\gamma'(T_i)$-set yields an edge dominating set for T_{i+1}, and so $\gamma'(T_{i+1}) \leq \gamma'(T_i) + 1$. Furthermore, any $\gamma'_R(T_i)$-function can be extended to an ERDF for T_{i+1} by assigning 2 to xy, and 0 to wx and yz. Thus $\gamma'_R(T_{i+1}) \leq \gamma'_R(T_i) + 2$. Clearly $\gamma'_R(T_{i+1}) \geq \gamma'_R(T_i) + 1$. Suppose that $\gamma'_R(T_{i+1}) = \gamma'_R(T_i) + 1$. Let $f = (E_0, E_1, E_2)$ be a $\gamma'_R(T_{i+1})$-function such that $|E_2|$ is maximum and $f(yz) \neq 2$. If $f(xy) = 2$, then $f|_{V(T_i)}$ is an ERDF for T_i, a contradiction. Thus $f(xy) \neq 2$. Then $f(xw) = 2$, and so $f(yz) = 1$. Let w_1 be a leaf of T_i adjacent to w. Then clearly $f(ww_1) = 0$. Now replacing $f(ww_1)$ by 2 yields a $\gamma'_R(T_i)$-function contradicting the speciality of w. Thus $\gamma'_R(T_{i+1}) = \gamma'_R(T_i) + 2$. Now $\gamma'(T_i) = \frac{\gamma'_R(T_i)}{2} = \frac{\gamma'_R(T_{i+1})-2}{2} \leq \frac{2\gamma'(T_{i+1})-2}{2} = \gamma' T_{i+1} - 1$, and thus $\gamma'(T_{i+1}) \geq \gamma'(T_i) + 1$. Thus $\gamma'(T_{i+1}) = \gamma'(T_i) + 1$. Now $\gamma'_R(T_{i+1}) = \gamma'_R(T_i) + 2 = 2\gamma'(T_i) + 2 = 2\gamma'(T_{i+1})$. If w is a leaf, or T_{i+1} is obtained by joining w to a center of a double star $S(a, 2)$ whose degree is a, then similarly $\gamma'_R(T_{i+1}) = 2\gamma'(T_{i+1})$. □

Lemma 2.4. *If $\gamma'_R(T_i) = 2\gamma'(T_i)$ and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_4$, the $\gamma'_R(T_{i+1}) = 2\gamma'(T_{i+1})$.*

Proof. Let $\gamma'_R(T_i) = 2\gamma'(T_i)$, $w \in V(T_i)$, and $u \in N(w)$ be the vertex such that any vertex of $N(u) - \{w\}$ is a leaf. First assume that T_{i+1} is obtained by joining w to the leaf x of a path xyz. As Lemma 2.3, we have $\gamma'(T_{i+1}) \leq \gamma'(T_i) + 1$ and $\gamma'_R(T_{i+1}) \leq \gamma'_R(T_i) + 2$. Clearly $\gamma'_R(T_{i+1}) \geq \gamma'_R(T_i) + 1$. Let $f = (E_0, E_1, E_2)$ be a $\gamma'_R(T_{i+1})$-function with pendant edges assigned the value 2 as few as possible. By our choice $f(vw) = f(xy) = 2$. Hence $f|_{E(T_i)}$ is an ERDF and $\gamma'(T_i) \leq \gamma'(T_{i+1}) - 2$. Thus $\gamma'_R(T_{i+1}) = \gamma'_R(T_i) + 2$. Now $\gamma'(T_i) = \frac{\gamma'_R(T_i)}{2} = \frac{\gamma'_R(T_{i+1})-2}{2} \leq \frac{2\gamma'(T_{i+1})-2}{2} = \gamma' T_{i+1} - 1$, and thus $\gamma'(T_{i+1}) \geq \gamma'(T_i) + 1$. Thus $\gamma'(T_{i+1}) = \gamma'(T_i) + 1$. Now $\gamma'_R(T_{i+1}) = \gamma'_R(T_i) + 2 = 2\gamma'(T_i) + 2 = 2\gamma'(T_{i+1})$. If T_{i+1} is obtained by joining w to a center of a double star $S(a, 2)$ whose degree is a, then similarly $\gamma'_R(T_{i+1}) = 2\gamma'(T_{i+1})$. □

Similarly the following is verified.

Lemma 2.5. *If $\gamma'_R(T_i) = 2\gamma'(T_i)$ and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_5$, the $\gamma'_R(T_{i+1}) = 2\gamma'(T_{i+1})$.*

We now are ready to state the main result of this paper.

Theorem 2.1. *For a tree T, $\gamma'_R(T) = 2\gamma'(T)$ if and only if $T \in \mathcal{T}$.*

Proof. The sufficiency follows by an induction on the edge Roman domination number and Lemmas 2.1, 2.2, 2.3, 2.4, and 2.5. We need to prove the necessity. We prove by induction on the edge domination number $\gamma'(T)$ of a tree T with $\gamma'_R(T) = 2\gamma'(T)$ that $T \in \mathcal{T}$. If $\gamma'_R(T) = 1$, then since $\gamma'_R(K_2) \neq 2\gamma'(K_2)$, T is a star with at least three vertices, or a double-star, and so $T \in \mathcal{T}$. Suppose the result is true for all trees T' with $\gamma'_R(T') = 2\gamma'(T')$ and $\gamma'(T') < \gamma'(T)$. Since $\gamma'(T) > 1$, we obtain $diam(T) \geq 4$. Among all diametrical paths in T, let $xx_1x_2 \ldots x_d$ be a diametrical path in T such that $\deg(x_{d-1})$ is maximum. We root T at x. By Theorem 1.1 there is a $\gamma'_R(T)$-function $f = (E_0, E_1, E_2)$ with $E_1 = \emptyset$. We may assume that $f(x_{d-1}x_{d-2}) = 2$.

Assume that $d = 4$. Clearly, we may assume that x_1 and x_3 are strong support vertices, and any child of x_1 different from x_2 is a leaf. Thus we may assume that $f(x_1x_2) = f(x_2x_3) = 2$. Since $E_1 = \emptyset$ we obtain that any child of x_2 is a leaf or a strong support vertex. Clearly $T - T_{x_2}$ is a star of order at least three, and so belongs to $\mathcal{T}$. If $\deg(x_2) \geq 3$, then $T_{x_2} \in \mathcal{F}_1$, and so T is obtained from $T - T_{x_2}$ by Operation $\mathcal{O}_1$. Thus $\deg(x_2) = 2$. Then T is obtained from $T - T_{x_2}$ by Operation $\mathcal{O}_2$. We thus assume that $d \geq 5$. We consider the following two cases.

Case 1. $\deg(x_{d-1}) \geq 3$.

Assume that $\deg(x_{d-2}) \geq 3$. Since $E_1 = \emptyset$ we obtain that any child of x_{d-2} is a leaf or a strong support vertex. Let $T_1 = T - T_{x_{d-2}}$, and assume that x_{d-2} has precisely k children that are strong support vertices. Then we may assume that $f(x_{d-2}u) = 2$ for each child u of x_{d-2} with $\deg(u) \geq 3$. If $f(x_{d-3}x_{d-2}) = 2$, then we change $f(x_{d-3}x_{d-2})$ to 0, and assign 2 to one of edges of T_1 incident with x_{d-3}. Thus we may assume that $f(x_{d-3}x_{d-2}) = 0$. Then $f|_{V(T_1)}$ is an ERDF for T_1 implying that $\gamma'_R(T_1) \leq \gamma'_R(T) - 2k$. Similarly $\gamma'(T_1) \leq \gamma'(T) - k$. On the other hand any $\gamma'_R(T_1)$-function can be extended to an ERDF for T by assigning 2 to the $x_{d-2}u$ for each child u of x_{d-2} with $\deg(u) \geq 3$, and 0 to $x_{d-3}x_{d-2}$ and any other edge of $T_{x_{d-2}}$. So $\gamma'_R(T) \leq \gamma'_R(T_1) + 2k$, and thus $\gamma'_R(T) = \gamma'_R(T_1) + 2k$. Similarly we obtain $\gamma'(T) = \gamma'(T_1) + k$. Then $\gamma'_R(T_1) = \gamma'_R(T) - 2k = 2\gamma'(T) - 2k = 2\gamma'(T_1)$. By the inductive hypothesis $T_1 \in \mathcal{T}$. It is also clear that $T_{x_{d-2}} \in \mathcal{F}_1$. Thus $T \in \mathcal{T}$, and is obtained from T_1 by Operation $\mathcal{O}_1$.

We next assume that $\deg(x_{d-2}) = 2$. Clearly we may assume that $f(x_{d-3}x_{d-2}) = 0$. Let $T_2 = T - T_{x_{d-2}}$. As before, we can see that $\gamma'_R(T) = \gamma'_R(T_2) + 2$, and $\gamma'(T) = \gamma'(T_2) + 1$, and so we obtain that $\gamma'_R(T_2) = 2\gamma'(T_2)$. By the inductive hypothesis $T_2 \in \mathcal{T}$. Thus T is obtained from T_2 by Operation $\mathcal{O}_2$.

Case 2. $\deg(x_{d-1}) = 2$.

Then each child of x_{d-2} is a leaf or a support vertex of degree two. Assume that x_{d-2} has a child $u \neq x_{d-1}$ with $\deg(u) = 2$, and u_1 is the child of u. Then clearly we may assume that $f(x_{d-2}u) = 0$. But then $f(uu_1) = 1$, a contradiction. We deduce that x_{d-1} is the unique child of x_{d-2} that is not a leaf. Assume that $\deg(x_{d-3}) \geq 3$. Let $T_3 = T - T_{x_{d-2}}$. As before, we can see that $\gamma'_R(T) = \gamma'_R(T_3) + 2$, and $\gamma'(T) = \gamma'(T_3) + 1$, and so we obtain that $\gamma'_R(T_3) = 2\gamma'(T_3)$. By the inductive hypothesis $T_3 \in \mathcal{T}$. Assume that x_{d-3} is a support vertex. If there is a $\gamma'_R(T_3)$-function such that assigns 2 to a pendant edge e incident with x_{d-3}, then we replace $f(e)$ by 0, $f(x_{d-3}x_{d-2})$ by 2, $f(x_{d-1}x_d)$ by 1, and assign 0 to any other edge of $T_{x_{d-2}}$ to obtain an ERDF for T of weight less than $\gamma'_R(T)$, a contradiction. Thus x_{d-3} is a special support vertex of T_3. Consequently, $T \in \mathcal{T}$ and is obtained from T_3 by Operation $\mathcal{O}_3$. Thus we may assume that x_{d-3} is not a support vertex. Assume that x_{d-3} has a child u such that any child of u is a leaf. Then T is obtained from T_3 by Operation $\mathcal{O}_4$. Thus x_{d-3} has no child u such that any child of u is a leaf. Since $\deg(x_{d-3}) \geq 3$, we obtain that a component of $T - x_{d-3}$ is a double-star $S(a, 2)$, where x_{d-3} is adjacent to a vertex of maximum degree $S(a, 2)$. We conclude that T is obtained from T_3 by Operation $\mathcal{O}_5$. Thus $\deg(x_{d-3}) = 2$. Then x_{d-3} is a leaf of T_3, and T is obtained from T_3 by Operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$ and the proof is complete. □

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