Electronic Journal of Graph Theory and Applications **14** (2) (2026), 471–483



Electronic Journal of
Graph Theory and Applications

# Whitney's theorem for line graphs of multi-graphs

Hans Cuypers

*Department of Mathematics and Computer Science*
*Eindhoven University of Technology*
*P.O. Box 513, 5600 MB, Eindhoven*
*The Netherlands*

f.g.m.t.cuypers@tue.nl

**Abstract**

Whitney's Theorem states that every connected graph, different from $K_3$ or $K_{1,3}$, is uniquely determined by its line graph. The graphs $K_{1,3}$ and $K_3$ have isomorphic line graphs. We provide an analogue of Whitney's Theorem for line graphs of multi-graphs. There are two obvious choices for the definition of a line graph of a multi-graph. A 1-line graph of a multi-graph is the graph with as vertices the edges of the multi-graph, and two edges adjacent if and only if there is a unique vertex which is on both edges. The $\geq 1$-line graph of a multi-graph is the graph on the edges of the multi-graph, where two edges are adjacent if and only if there is at least one vertex which is on both edges. We extend Whitney's Theorem to both classes of line graphs of multi-graphs, and show that most multi-graphs are uniquely determined by their line graph. Moreover, we present an algorithm to determine for a given graph $\Gamma$, if possible, a multi-graph with $\Gamma$ as line graph.

*Keywords:* line graphs, multi-graphs, Whitney's theorem
Mathematics Subject Classification : 05C60, 05C85, 05C76
DOI: 10.5614/ejgta.2026.14.2.14

## 1. Introduction

Let $\Delta = (V, E)$ be an ordinary graph with vertex set $V$ and edge set $E$, i.e. a graph without loops or multiple edges. So, we can consider each edge $e \in E$ as a set of two distinct vertices. Then its line graph $L(\Delta)$ is the graph with vertex set $E$ and two vertices $e, f \in E$ adjacent if and

Received: 04 April 2025, Revised: 1 October 2026, Accepted: 19 October 2026.



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only if they intersect in a single vertex of $\Delta$. The graph $\Delta$ is called a *root graph* for its line graph $L(\Delta)$. Of course, adding isolated vertices to $\Delta$ will not change its line graph. So, we assume a root graph to have no isolated vertices.

An *isomorphism* $\phi$ between two graphs $\Delta = (V, E)$ and $\Delta' = (V', E')$ is a bijection $\phi : V \to V'$, such that the map on $E$ induced by $\phi$ and also denoted by $\phi$, i.e. $\phi(\{v, w\}) = \{\phi(v), \phi(w)\}$ for all $\{v, w\} \in E$, is a bijection from $E$ to $E'$.

Clearly, if $\phi$ is an isomorphism between two graphs $\Delta = (V, E)$ and $\Delta' = (V', E')$, then $\phi$ induces also an isomorphism between the two line graphs $L(\Delta)$ and $L(\Delta')$.

Whitney's Theorem is concerned with the reverse problem:

> Does an isomorphism between line graphs of graphs $\Delta$ and $\Delta'$ also imply the existence of an isomorphism between $\Delta$ and $\Delta'$?
>
> In other words, is the root graph of a line graph unique?

As each isomorphism between the graphs $\Delta$ and $\Delta'$ induces an isomorphism of the line graphs $L(\Delta)$ and $L(\Delta')$, we can add the question:

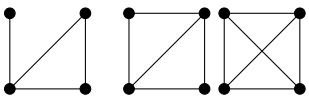
> Is each isomorphism $\phi$ between $L(\Delta)$ and $L(\Delta')$ induced by an isomorphism $\psi$ between $\Delta$ and $\Delta'$? And if so, is the isomorphism $\psi$ then uniquely determined?

The answers to these questions are stated in the following theorem:

**Theorem 1.1** (Whitney's Theorem, [15])**.** *Suppose $\Delta$ and $\Delta'$ are connected graphs and $\phi$ an isomorphism between the line graph $L(\Delta)$ and $L(\Delta')$.*

*Then there is a unique isomorphism $\psi$ between $\Delta$ and $\Delta'$, inducing $\phi$, unless we have one of the following:*

1. *$\Delta$ and $\Delta'$ are (up to permutation) $K_{1,3}$ and $K_3$ and not isomorphic;*
2. *$\Delta$ and $\Delta'$ are isomorphic to $K_2$ and admit two isomorphisms inducing $\phi$; ,*
3. *$\Delta$ is isomorphic to $\Delta'$ and to one of the three graphs*



*In this case, there is an automorphism $\sigma$ of $L(\Delta)$ such that one of $\phi$ or $\phi \circ \sigma$ is induced by a unique isomorphism $\psi$ between $\Delta$ and $\Delta'$.*

An elementary and short proof of this result can be found in [9] and [6].

In this note we consider the same problem for multi-graphs. Here a *multi-graph* $\Delta = (V, E, \sim)$ consists of a set of vertices $V$ and a set of edges $E$ together with a relation $\sim$ between $V$ and $E$ such that for any element $e \in E$ there are exactly two distinct elements $v, w \in V$ with $v, w \sim e$. If $v \sim e$, then we will say $v$ is on $e$. Moreover, we call two vertices $v, w$ adjacent if and only if there is an edge $e$ with $v, w \sim e$. Of course, an ordinary graph is a multi-graph in which the relation $\sim$ is the relation $\in$.



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There are at least two different obvious choices for the definition of the line graph of a multi-graph. The vertices of the line graph of $\Delta = (V, E, \sim)$ are the elements from $E$. But two elements $e, f \in E$ can be defined to be adjacent if and only if there is a *unique* vertex with $v \sim e$ and $v \sim f$, or, if and only if there is *at least one* vertex $v$ with $v \sim e$ and $v \sim f$. We consider both situations.

The 1-*line graph* $L_1(\Delta)$ of a multi-graph $\Delta = (V, E, \sim)$ is the graph with vertex set $E$, such that two vertices $e, f \in E$ are adjacent if and only if there is a *unique* vertex $v$ of $\Delta$ with $v \sim e$ and $v \sim f$. By $L_{\geq 1}(\Delta)$ we denote the $\geq 1$-*line graph* of $\Delta$, the graph with vertex set $E$ and two edges $e, f$ adjacent if $e \neq f$ and there is *at least one* vertex $v$ of $\Delta$ with $v \sim e$ and $v \sim f$.

Notice that both types of line graphs are ordinary graphs.

If it is clear from the context whether we are considering the 1-line graph or $\geq 1$-line graph, we will just refer to these graphs as line graphs. In both cases the graph $\Delta$ is called a *root graph* for its line graph.

An *isomorphism between two multi-graphs* $\Delta = (V, E, \sim)$ and $\Delta' = (V', E', \sim')$ is a pair $(\phi_V, \phi_E)$ of bijections

$$\phi_V : V \to V', \phi_E : E \to E'$$

such that for all $v \in V$ and $e \in E$ we have

$$v \sim e \Leftrightarrow \phi_V(v) \sim' \phi_E(e).$$

An isomorphism $(\phi_V, \phi_E)$ between multi-graphs $\Delta = (V, E, \sim)$ and $\Delta' = (V', E', \sim')$ induces an isomorphism, the isomorphism $\phi_E$, between the graphs $L_1(\Delta)$ and $L_1(\Delta')$, as well as between $L_{\geq 1}(\Delta)$ and $L_{\geq 1}(\Delta')$.

Again we can ask the following questions:

> Does an isomorphism $\phi$ between line graphs of multi-graphs $\Delta$ and $\Delta'$ also imply the existence of an isomorphism between $\Delta$ and $\Delta'$ inducing $\phi$?
>
> In other words, is the root graph of a line graph of a multi-graph unique?

The answers to these questions are given in the following theorems. Indeed, Theorem 1.2 describes the solution to the problem for 1-line graphs, while Theorem 1.4 does this for $\geq 1$-line graphs. We also discuss the implications of Theorem 1.2 on *generalized line graphs*, a class of graphs which can be considered to be a subclass of 1-line graphs. This result is Theorem 1.3.

We start with 1-line graphs of multi-graphs.

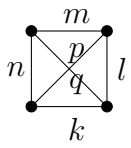
**Theorem 1.2.** *Suppose $\Delta$ and $\Delta'$ are connected multi-graphs and $\phi$ an isomorphism between the line graphs $L_1(\Delta)$ and $L_1(\Delta')$.*

*Then there is a unique isomorphism between $\Delta$ and $\Delta'$ inducing $\phi$, unless we have one of the following:*

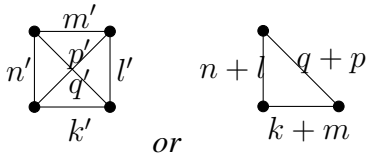
1. *If $\Delta$ and $\Delta'$ have only two vertices, then $\Delta'$ and $\Delta$ are isomorphic, and there are two isomorphisms $\psi$ between $\Delta$ and $\Delta'$ inducing $\phi$.*
2. *If (up to switching $\Delta$ and $\Delta'$) the multi-graph $\Delta'$ has four vertices and is the graph*



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*for some cardinal numbers $k, \ldots, q$, then $\Delta$ is one of the following graphs*




*for cardinal numbers $k', \ldots, q'$ with $k + m = k' + m'$, $n' + l' = n + l$ and $p' + q' = p + q$.*

Here an edge on two vertices labeled with a cardinal $x$ represents $x$ edges between the vertices.

As in Whitney's Theorem 1.1 the exceptional cases in this theorem are the cases where the graph $\Delta$ has four vertices. These cases correspond to the graph $K_{1,3}$ in Whitney's Theorem and can be reduced to a graph on 3 points, related to $K_3$.

The above theorem therefore implies the following:

**Corollary 1.1.** *Let $\Delta$ be a connected multi-graph, then there is up to isomorphism a unique connected multi-graph $\Delta'$ not on 4 vertices such that $L_1(\Delta)$ and $L_1(\Delta')$ are isomorphic.*

A class of 1-line graphs of multi-graphs that received a lot of attention is the class of so-called generalized line graphs, as they are graphs with eigenvalues $\geq -2$. See for example [3] or [2], and various references in [2].

A *generalized line graph* $\Gamma$ is a graph which can be constructed from a line graph of an ordinary graph $\Delta = (V, E)$ and for each vertex $v \in V$ a cocktail party graph $\Delta_v$ (a possibly empty ordinary graph in which each vertex is adjacent to precisely all but one other vertices). The graph $\Gamma$ is the union of the line graph $L(\Delta)$ and all the graphs $\Delta_v$ with $v \in V$ to which the following edges are added: a vertex $e$ of $L(\Delta)$ is adjacent to all vertices of $\Delta_v$, where $v$ is a vertex on $e$.

An equivalent definition is the following.

A graph $\Gamma$ is a generalized line graph if and only if it is the 1-line graph of a multi-graph $\Delta$ such that:

1. Two vertices of $\Delta$ are on at most two common edges.
2. If $v, w$ are two vertices of $\Delta$ which are on two common edges, then one of the vertices is on no other edges.

See [5, Proposition 6.2].

Using this definition of a generalized line graph we can apply Theorem 1.2 and easily deduce the analogue of Whitney's Theorem for generalized line graphs as in [4] and [6, Theorem 2.3.4]. This provides a new elementary proof for the following result.

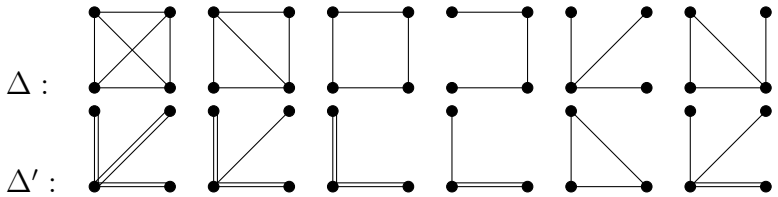
**Theorem 1.3.** *Let $\Gamma$ be a connected generalized line graph. Then there is a unique connected multi-graph $\Delta$ satisfying:*



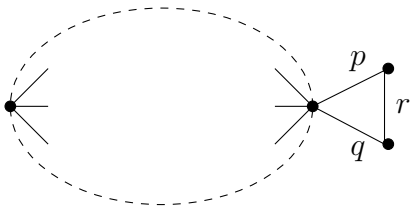
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1. *Two vertices are on at most two common edges;*
2. *If $v, w$ are two vertices which are on two common edges, then one of the vertices is on no other edges;*

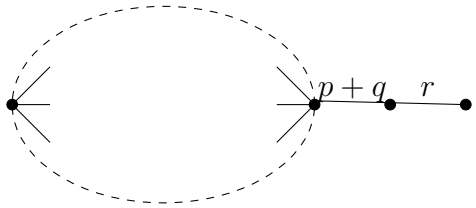
*such that $\Gamma$ is isomorphic to $L_1(\Delta)$, or we have $\Gamma$ isomorphic to $L_1(\Delta)$ as well as $L_1(\Delta')$ where $\Delta$ and $\Delta'$ are one of the following multi-graphs satisfying (i) and (ii):*




We now focus on $\geq 1$-line graphs.
The $\geq 1$-line graphs of the graphs



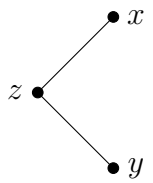

and




with $p, q$ and $r$ cardinals and the same multi-graph inside the ellipse are isomorphic. Avoiding the first of such graphs with $p, q > 0$ yields the uniqueness of a root graph:

**Theorem 1.4.** *Suppose $\Delta$ and $\Delta'$ are connected multi-graphs and $\phi$ an isomorphism between the line graph $L_{\geq 1}(\Delta)$ and $L_{\geq 1}(\Delta')$.*

*Then there is a unique isomorphism between $\Delta$ and $\Delta'$, inducing $\phi$, unless $\Delta$ or $\Delta'$ contains a subgraph $\Delta_0$*






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*such that the vertices $x$ and $y$ have no other neighbors outside $\Delta_0$.*

*If $\Delta$ does contain such subgraph $\Delta_0$ then there is, up to isomorphism, a unique connected multi-graph $\Delta''$ not containing such subgraph $\Delta_0$, with $L_{\geq 1}(\Delta'')$ isomorphic to $L_{\geq 1}(\Delta)$.*

Notice that $\Delta_0$ is not necessarily an induced subgraph, we allow multiple edges on $x, z$ or $y, z$ or $x, y$.

After finishing the research for this note, we found that results closely related to Theorem 1.4 have also been obtained by Zverovich [16] and Hemminger [8]. Both Zverovich and Hemminger also consider line graphs of multi-graphs. Zverovich [16] investigates $\geq 1$-line graphs which contain subgraphs of type $\Delta_0$ and shows that one may remove these subgraphs without changing the $\leq 1$-line graph, while Hemminger [8] studies the line graphs of pseudographs, which are graphs with multiple edges as well as loops and hence include multi-graphs. Both authors obtain results close to Theorem 1.4 using methods different from our approach.

We present a uniform proof for Theorem 1.2, Theorem 1.3 and Theorem 1.4 in the next section. This proof can be transformed into an algorithm that on input of an ordinary graph $\Gamma = (V, E)$ decides whether $\Gamma$ is a line graph of a multi-graph and also determines in the case that $\Gamma$ is indeed a line graph a root graph $\Delta$ for $\Gamma$.

The algorithm has complexity $O(|V| + |E|)$. It is discussed in Section 3.

The algorithm of Section 3 can be modified to find a root graph for a generalized line graph. See also [14] for a different algorithm to find such root graph of a generalized line graph.

## 2. Line graphs

In this section we prove Theorem 1.2, Theorem 1.3 and Theorem 1.4. We start with some definitions.

Let $\Gamma = (V, E)$ be an ordinary graph. On the vertex set of $\Gamma$ we define two equivalence relations $\equiv$ and $\bowtie$ by the following: two vertices $v, w$ of $\Gamma$ are in relation $\equiv$ if and only if they are equal or they have the same set of neighbors (and hence are themselves not adjacent), while two vertices $v, w$ of $\Gamma$ are in relation $\bowtie$ if and only if they are equal or they are adjacent and each other vertex adjacent to one of them is also adjacent to the other. For $v$ a vertex of $\Gamma$, denote by $[v]$ the $\equiv$-class of $v$ and by $\langle v \rangle$ the $\bowtie$-class of $v$. Two $\equiv$-equivalent vertices are also called *false twins*, while $\bowtie$-equivalent vertices are called *true twins*.

True or false twins do occur in line graphs of multi-graphs. Indeed, let $x$ and $y$ be edges of a multi-graph $\Delta$ on the same vertices. If $\Gamma$ is the 1-line graph of $\Delta$, then these two edges $x$ and $y$ of $\Delta$ are non-adjacent in $\Gamma$. However, any edge of $\Delta$ which is inside $\Gamma$ adjacent to one of $x$ or $y$ is also adjacent to the other. So $x$ and $y$ are false twins in $\Gamma$. However, if $\Gamma$ is the $\geq 1$-line graph of the graph $\Delta$, then $x$ and $y$ are adjacent. Moreover, any edge $z$ of $\Delta$ meeting $x$ or $y$ is in $\Gamma$ adjacent to both $x$ and $y$. So, $x$ and $y$ are true twins in $\Gamma$.

By $\overline{\Gamma}$ we denote the graph with vertex set the $\equiv$-classes of vertices of $\Gamma$, and two such classes $[v]$ and $[w]$ adjacent if and only if $v$ and $w$ are adjacent in $\Gamma$. The graph on the $\bowtie$-classes, where two such classes $\langle v \rangle$ and $\langle w \rangle$ are adjacent if and only if $v$ and $w$ are adjacent in $\Gamma$, is denoted by $\widehat{\Gamma}$.

If an $\equiv$- or $\bowtie$-class has size one, we often identify it with the element it contains. In this way we justify a statement like $\overline{\Gamma} = \Gamma$, or $\widehat{\Gamma} = \Gamma$, in case all equivalence classes have size one.



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**Lemma 2.1.** *Let $\Gamma$ be a graph. Then $\overline{\overline{\Gamma}} = \overline{\Gamma}$ and $\widehat{\widehat{\Gamma}} = \widehat{\Gamma}$.*

*Proof.* Consider two vertices $u, v$ in distinct $\equiv$-classes $[u]$ and $[v]$ of $\Gamma$ with $[u]$ and $[v]$ nonadjacent in $\overline{\Gamma}$. As $u$ and $v$ are not in relation $\equiv$, there is, up to permutation of $u$ and $v$, a vertex $w$ of $\Gamma$ adjacent to $u$ and not to $v$. But then $[w]$ is adjacent to $[u]$ and not to $[v]$. This implies that in $\overline{\Gamma}$ the $\equiv$-classes have size 1.

Similarly, we find in $\widehat{\Gamma}$ all $\bowtie$-classes to have size 1. □

We now focus on proving Theorem 1.2 and Theorem 1.4 and will consider multi-graphs and their line graphs. The next lemma implies that we can restrict our attention to connected graphs.

**Lemma 2.2.** 1. *Let $\Gamma$ be a connected graph, which is the line graph of a multi-graph $\Delta$ without isolated vertices. Then $\Delta$ is also connected.*

2. *Suppose $\Delta$ is a connected multi-graph, then its line graph is connected, unless $\Delta$ is a graph on two vertices with at least two edges on these vertices and $\Gamma$ its 1-line graph.*

*Proof.* Suppose $\Gamma$ is a connected line graph of the multi-graph $\Delta$, which is not connected.

Let $v$ and $w$ be two vertices of $\Delta$ on an edge $e$ and $u$ a vertex of $\Delta$ which is in a different connected component than $v$ and $w$.

Then as $\Gamma$ is connected, we find for every edge $f$ on $u$ a path inside $\Gamma$ from $f$ to $e$. This would result in a path from $u$ to $v$, contradicting that they are in different connected components of $\Delta$.

So, there is no edge on $u$ and $u$ is an isolated vertex in $\Delta$. This proves (i).

Now assume that $\Delta$ is a connected multi-graph on at least 3 vertices. Let $e, f$ be two edges in $\Delta$. If $e, f$ are not on the same vertices, then there is a path from a vertex $v$ of $e$ but not on $f$ to a vertex $w$ of $f$ not on $e$. This path gives rise to a path in $\Gamma$ from $e$ to $f$.

So, if $e, f$ are vertices of disjunct connected components of $\Gamma$, then they are on the same vertices of $\Delta$. This clearly implies $\Delta$ to have only two vertices and all edges to be on these two vertices. Moreover, $\Gamma$ has to be the 1-line graph to be disconnected. We have proven (ii). □

From now on we will only be concerned with connected graphs, unless stated explicitly that they are not connected.

Let $\Gamma$ be a connected graph which is the 1-line (or $\geq$ 1-line) graph of a multi-graph $\Delta = (V_\Delta, E_\Delta, \sim_\Delta)$. Then for each $\equiv$-class (or $\bowtie$-class) of $\Gamma$ we pick a single vertex $v$ which is an edge of $\Delta$. Let $[\Delta]$ (respectively $\langle\Delta\rangle$) be the subgraph of $\Delta$ with edges the various $v$ just chosen, and as vertices those elements of $\Delta$ which are on such an edge.

Notice that $[\Delta]$ and $\langle\Delta\rangle$ are ordinary graphs, as any two of their vertices are on at most one edge, and uniquely defined up to isomorphism by the graph $\Delta$.

**Proposition 2.1.** 1. *If $\Gamma$ is the 1-line graph of a connected multi-graph $\Delta$, then the map from the edge set of $[\Delta]$ to the vertex set of $\overline{\Gamma}$ mapping an edge $e$ to $[e]$ induces an isomorphism between $L([\Delta])$ and $\overline{\Gamma}$.*

2. *If $\Gamma$ is the $\geq$ 1-line graph of a connected multi-graph $\Delta$, then the map from the edge set of $\langle\Delta\rangle$ to the vertex set of $\widehat{\Gamma}$ mapping an edge $e$ to $\langle e\rangle$ induces an isomorphism between $L([\Delta])$ and $\widehat{\Gamma}$.*



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*Proof.* We prove the first statement.

Clearly the edges $e$ of $[\Delta]$ are in bijective correspondence with the vertices $[e]$ of $\overline{\Gamma}$.

If two such edges $e$ and $f$ of $[\Delta]$ meet in a vertex of $[\Delta]$, then $e$ and $f$ have a unique intersection point in $\Delta$ and hence are adjacent in $\Gamma$. But then $[e]$ and $[f]$ are adjacent in $\overline{\Gamma}$.

On the other hand, if $e$ and $f$ are edges of $\Delta$ such that $[e]$ and $[f]$ are different but adjacent vertices in $\overline{\Gamma}$, then $e$ and $f$ are adjacent vertices of $\Gamma$ and, as edges of $\Delta$, meet in a single vertex of $\Delta$. But then they also meet in a single vertex of $[\Delta]$.

We conclude that $L([\Delta])$ is isomorphic to $\overline{\Gamma}$.

The proof of the second statement is similar. □

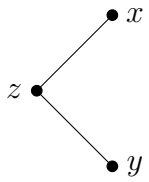
For a multi-graph $\Delta = (V, E, \sim)$ we denote by $\underline{\Delta}$ the ordinary graph obtained from $\Delta$ by replacing all multiple edges by single ones. So, $\underline{\Delta} = (V, \{\{x, y\} \mid x \neq y \in V \text{ with } x \text{ adjacent to } y\})$.

If two edges of $\Delta$ are on the same vertices, then in its line graphs $L_1(\Delta)$ and $L_{\geq 1}(\Delta)$ they are in relation $\equiv$ and $\bowtie$, respectively.

As we can consider $[\Delta]$, $\langle\Delta\rangle$ and $\underline{\Delta}$ as graphs on (a subset of) $V$, we can view $[\Delta]$ and $\langle\Delta\rangle$ to be induced subgraphs of $\underline{\Delta}$.

**Proposition 2.2.** *Let $\Delta$ be a connected multi-graph.*

1. *$[\Delta] = \underline{\Delta}$ if and only if $\Delta$ does not contain precisely $4$ vertices or $\Delta$ contains precisely $4$ vertices and $\underline{\Delta}$ is isomorphic to $K_{1,3}$.*
2. *$\langle\Delta\rangle = \underline{\Delta}$ if and only if $\Delta$ contains no (not necessarily induced) subgraph $\Delta_0$*

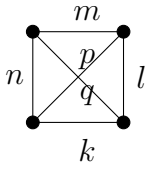



*such that the vertices $x$ and $y$ have no other neighbors outside $\{x, y, z\}$.*

*Proof.* If two edges of $\Delta$ are on the same two vertices if and only if they are in relation $\equiv$ in $L_1(\Delta)$, then clearly $[\Delta]$ and $\underline{\Delta}$ are the same.

So, suppose there are two edges in $\Delta$, say $e \neq f$ that do not meet each other, such that each edge meeting one of $e$ or $f$ in a unique vertex meets the other also in a unique vertex. Then, by connectedness of $\Delta$, every edge of $\Delta$ meets both $e$ and $f$ in a unique vertex, or is on the same two vertices as $e$ or $f$. But then we find $\Delta$ to be a connected graph on $4$ vertices.

If $\Delta$ is a connected multi-graph on $4$ vertices, then $\Delta$ is the multi-graph






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with $k, l, m$ cardinal numbers at least 1 and $n, p, q$ cardinals at least 0, or with $k, l, q$ to be at least 1 and $m = n = p = 0$.

So, $\underline{\Delta}$ is a connected graph on four vertices. However, $[\Delta]$ contains at most three edges, and hence can only equal $\underline{\Delta}$ if $\underline{\Delta}$ is isomorphic to $K_{1,3}$.

This proves the first part.

Now consider the second statement. If two edges of $\Delta$ are on the same two vertices if and only if they are in relation $\bowtie$ in $L_{\geq 1}(\Delta)$, then $\langle\Delta\rangle = \underline{\Delta}$.

So, assume there are two edges $e$ and $f$ of $\Delta$ on a unique common vertex $z$. Let $x$ be the second vertex on $e$ and $y$ the second vertex on $f$. Assume that each edge which meets $e$ also meets $f$ and vice versa. That implies that an edge meeting $e$ or $f$ meets them both in $z$, or it is an edge on both $x$ and $y$. So, the subgraph $\Delta_0$ of $\Delta$ on the vertices $x, y$ and $z$ and the edges $e$ and $f$ is a graph as in the statement of the lemma.

Now suppose $\Delta$ contains three vertices $x, y, z$ such that there are edges $e$ and $f$ on $x$ and $z$ and on $y$ and $z$, respectively, such that $x$ and $y$ have no neighbors outside $\{x, y, z\}$. Then $e$ and $f$ are adjacent vertices $L_{\geq 1}(\Delta)$ and in relation $\bowtie$. Moreover any edge $g$ of $\Delta$ which is a vertex of $\widehat{\Gamma}$ adjacent to $e$ or $f$ is adjacent to both $e$ and $f$. So, we find $e \bowtie f$.

This implies that we find in $\langle\Delta\rangle$ that $z$ is not adjacent to $x$ or not to $y$.

This proves statement (ii). □

We say that a (not necessarily induced) subgraph on three distinct vertices $x, y, z$ of a multi-graph $\Delta$ is of *type* $\Delta_0$, if $x, y$ are adjacent to $z$ but to no vertex outside $\{x, y, z\}$.

A multi-graph $\Delta$ that does not contain any subgraph of type $\Delta_0$ is called $\Delta_0$-*free*.

**Corollary 2.1.** *Let $\Delta$ be a connected multi-graph. Then we have:*

1. $\overline{L_1(\Delta)}$ *is isomorphic to* $L(\underline{\Delta})$*, if $\Delta$ has not precisely 4 vertices, or $\underline{\Delta}$ is isomorphic to* $K_{1,3}$.
2. $\widehat{L_{\geq 1}(\Delta)}$ *is isomorphic to* $L(\underline{\Delta})$*, if and only if $\Delta$ is $\Delta_0$-free.*

*Proof.* Notice that by Proposition 2.1 the graphs $\overline{L_1(\Delta)}$ and $L([\Delta])$ as well as $\overline{L_{\geq 1}(\Delta)}$ and $L(\langle\Delta\rangle)$ are isomorphic. Moreover, by Proposition 2.2 we only need to show that $[\Delta]$ and $\underline{\Delta}$, and $\langle\Delta\rangle$ and $\underline{\Delta}$ are isomorphic if and only if their line graphs are isomorphic.

Clearly $[\Delta]$ and $\underline{\Delta}$, or $\langle\Delta\rangle$ and $\underline{\Delta}$ being isomorphic implies their line graphs to be isomorphic.

So, suppose $[\Delta]$ and $\underline{\Delta}$, or $\langle\Delta\rangle$ and $\underline{\Delta}$ are not isomorphic, but their line graphs are. Whitney's Theorem 1.1 implies these graphs to be isomorphic to $K_{1,3}$ or $K_3$. But this contradicts $[\Delta]$ and $\langle\Delta\rangle$ to be subgraphs of $\underline{\Delta}$. This proves part (i) and (ii). □

Now we can prove Theorem 1.2 and Theorem 1.4 in the generic cases.

**Proposition 2.3.** *Let $\Delta$ and $\Delta'$ be two connected multi-graphs.*

1. *Suppose $\phi$ is an isomorphism between $L_1(\Delta)$ and $L_1(\Delta')$. Assume $\Delta$ and $\Delta'$ do not contain 4 vertices. Then there is a unique isomorphism between $\Delta$ and $\Delta'$ inducing $\phi$.*
2. *Suppose $\phi$ is an isomorphism between $L_{\geq 1}(\Delta)$ and $L_{\geq 1}(\Delta')$. Assume $\Delta$ and $\Delta'$ are $\Delta_0$-free. Then there is a unique isomorphism between $\Delta$ and $\Delta'$ inducing $\phi$.*



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*Proof.* Suppose we are in case (i) or (ii) and let $\phi$ be the given isomorphism in these cases.

By Corollary 2.1 and the assumptions we find $\overline{L_1(\Delta)} = L(\underline{\Delta})$ and $\overline{L_1(\Delta')} = L(\underline{\Delta'})$ to be isomorphic in case (i), and $\widehat{L_1(\Delta)} = L(\underline{\Delta})$ and $\widehat{L_1(\Delta')} = L(\underline{\Delta'})$ to be isomorphic in case (ii). These isomorphisms are induced by $\phi$.

By $\overline{\phi}$ we denote the isomorphism between $\overline{L_1(\Delta)}$ and $\overline{L_1(\Delta')}$ in case (i) and by $\widehat{\phi}$ the isomorphism between $\widehat{L_1(\Delta)}$ and $\widehat{L_1(\Delta')}$ in case (ii).

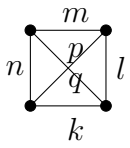
As the graphs $\underline{\Delta}$ and $\underline{\Delta'}$ are not, up to permutation, isomorphic to $K_3$ and $K_{1,3}$ (indeed, $K_{1,3}$ is not $\Delta_0$-free), we can by Whitney's Theorem 1.1 find a unique isomorphism $\overline{\phi}$ from $\underline{\Delta}$ to $\underline{\Delta'}$, which induces $\overline{\phi}$ between $\overline{L_1(\Delta)} = L(\underline{\Delta})$ and $\overline{L_1(\Delta')} = L(\underline{\Delta'})$, and a unique isomorphism $\widehat{\phi}$ from $\underline{\Delta}$ to $\underline{\Delta'}$, which induces $\widehat{\phi}$ between $\widehat{L_{\geq 1}(\Delta)} = L(\underline{\Delta})$ and $\widehat{L_{\geq 1}(\Delta')} = L(\underline{\Delta'})$.

By the uniqueness of $\overline{\phi}$ and $\widehat{\phi}$ we find that the pairs $(\overline{\phi}, \phi)$ and $(\widehat{\phi}, \phi)$ are the unique isomorphisms between $\Delta$ and $\Delta'$ we are looking for in case (i) and (ii), respectively. □

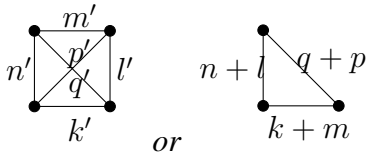
It remains to consider the cases excluded in the above Proposition 2.3. These are handled by the following result.

**Proposition 2.4.**

1. *Suppose $\Delta$ is the connected multi-graph*




*for some cardinal numbers $k, \ldots, q$. If $\Delta'$ is a connected multi-graph with $L_1(\Delta')$ isomorphic to $L_1(\Delta)$, then $\Delta'$ is one of the following multi-graphs:*

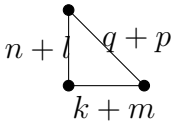



*for cardinal numbers $k', \ldots, q'$ with $k + m = k' + m'$, $n' + l' = n + l$ and $p' + q' = p + q$.*

2. *If $\Delta$ is a connected multi-graph which is not $\Delta_0$-free then there is, up to isomorphism, a unique $\Delta_0$-free multi-graph $\Delta'$ with $L_{\geq 1}(\Delta')$ and $L_{\geq 1}(\Delta)$ isomorphic.*

*Proof.* We first consider case (i).

Let $\Delta''$ be the multi-graph




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Then $L_1(\Delta'')$ is isomorphic to $L_1(\Delta)$ and by Proposition 2.3 we find $\Delta''$ to be the unique such graph which is not on $4$ vertices.

The graphs $\Delta'$ on $4$ vertices with $L_1(\Delta')$ isomorphic to $L_1(\Delta)$ and hence $L_1(\Delta'')$ are then precisely the graphs mentioned in statement (i).

Now consider (ii). So, suppose $\Delta$ is not $\Delta_0$-free.

Then consider the graph $\langle\Delta\rangle$. By construction this graph is $\Delta_0$-free.

Indeed, in this graph we take a unique element in each equivalence class of the relation $\bowtie$, which implies that we can not have a $\Delta_0$ subgraph on vertices $x, y$ and $z$ as the edges on $\{x, z\}$ and $\{y, z\}$ are $\bowtie$-equivalent.

Now take for $\Delta'$ the graph with vertex set the set of vertices of $\langle\Delta\rangle$ and for each edge $e$ of $\Delta$ define $e'$ to be an edge of $\Delta'$ which is on the two vertices $v$ and $w$ which are on $\langle e\rangle$. By construction we find $\Delta'$ to be $\Delta_0$-free and $L_{\geq 1}(\Delta)$ and $L_{\geq 1}(\Delta')$ to be isomorphic. Again Proposition 2.3 implies $\Delta'$, which is $\Delta_0$-free, to be the unique such graph. □

Theorem 1.2 and Theorem 1.4 follow from Proposition 2.3 and Proposition 2.4.

We finish this section by the observation that Theorem 1.3 follows immediately from Theorem 1.2. Indeed, the only exceptional multi-graphs of the conclusion of Theorem 1.2 satisfying

1. Two vertices of are on at most two common edges,
2. If $v, w$ are two vertices which are on two common edges, then one of the vertices is on no other edges,

are the twelve graphs occurring in the conclusion of Theorem 1.3.

## 3. Find the root graph of a line graph

Let $\Gamma = (V, E)$ be a finite connected graph. In [1] it has been shown that $\Gamma$ is the $\geq 1$-line graph of a multi-graph $\Delta$ if and only if $\Gamma$ does not contain one of $7$ graphs on at most $7$ vertices as induced subgraph. A similar result for 1-line graphs has been obtained in [5], where a complete finite list of $32$ forbidden graphs, all on $6$ vertices, has been found. This implies that there is an algorithm with polynomial time in the number of vertices of $\Gamma$ to determine whether $\Gamma$ is indeed a line graph. Just check the $\binom{|V|}{7}$ or $\binom{|V|}{6}$ subgraphs on $7$ or $6$ vertices, respectively, against the lists of forbidden subgraphs.

The results of the previous section imply that, in case $\Gamma$ is indeed a line graph, there is a unique connected root graph $\Delta$, not on four vertices, if $\Gamma$ is a 1-line graph, or which is $\Delta_0$-free, if $\Gamma$ is a $\geq 1$-line graph.

The following algorithm has, on input of a finite connected graph $\Gamma$, as output this unique root graph $\Delta$ if $\Gamma$ is a 1- (or $\geq 1$-)line graph, or returns the message that the graph $\Gamma$ is not a line graph.

1. Find the false (or true) twin decomposition of $\Gamma$. A decomposition into the $\equiv$-equivalence (or $\bowtie$-) classes can be found by a standard partition refinement. See for example Algorithm 2 in [7].
2. Determine $\overline{\Gamma}$ (or $\widehat{\Gamma}$). Take in each equivalence class a unique vertex and form the induced subgraph on these vertices.



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3. Find a root graph $[\Delta]$ not on $4$ vertices for the graph $\overline{\Gamma}$ (or $\langle\Delta\rangle$ which is $\Delta_0$-free for $\widehat{\Gamma}$), or return the message that the graph $\Gamma$ is not a line graph. Several efficient algorithms exist. See for example [10, 11, 12, 13].
4. Determine and output $\Delta$. The vertex set of $\Delta$ equals that of $[\Delta]$ (or $\langle\Delta\rangle$). Each vertex $v$ of $\Gamma$ determines a unique vertex $[v]$ (or $\langle v\rangle$) of $\overline{\Gamma}$ (or $\widehat{\Gamma}$) and hence a unique edge $[v]$ (or $\langle v\rangle$) of $[\Delta]$ (or $\langle\Delta\rangle$). So, we can assign to $w \in [v]$ (or $w \in \langle v\rangle$) a unique edge of $\Delta$ which is on the same two vertices as the corresponding edge $[v]$ (or $\langle v\rangle$) of $[\Delta]$ (or $\langle\Delta\rangle$).

Correctness of the algorithm follows easily from the results of the previous section.

The above algorithm can of course also be used to determine a root graph of a generalized line graph, or decide that a graph is not a generalized line graph. Indeed, after step (i) one can check whether all $\equiv$-classes have size at most 2, and, in step (iv) whether for two vertices of $\Delta$ which are on two common edges, one is on no other edges. In [14] a different algorithm is discussed to find a root graph of a generalized line graph.

Finally we notice that each of the steps in the above algorithm can be done with at most $O(|V|+|E|)$ elementary computations. Indeed, [7] proves that the first step can be done by $O(|V|+|E|)$ elementary computations. The second step can clearly be done in $O(|V|+|E|)$ elementary computations, just as the third step as follows from for example [10, 11, 12, 13]. It remains the last step. In this step we already have all the vertices of $\Delta$, we only have to add all vertices of $[v]$ (or $\langle v\rangle$) to $\Delta$ for each vertex $v$ of $\Gamma$ and assign them to the same edges as $[v]$ (or $\langle v\rangle$). This last step can be done in $O(|V|)$ computations. So, the complexity of the complete algorithm is $O(|V|+|E|)$.

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