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NP-completeness and an ILP formulation for the P_3-convex domination problem

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Abstract

A set $S \subseteq V(G)$ of a graph G is called a P_3-convex dominating set if it is a dominating set and for every path $u - w - v$ of length two with $u, v \in S$, the vertex w belongs to S. The P_3-convex domination number $\gamma_{3c}(G)$ is the cardinality of a minimum P_3-convex dominating set (P_3-CDS). In this paper, we present several foundational results concerning this parameter. First, we show that for any integers $n \geq 6$ and $1 \leq k \leq n$, there exists a graph G of order n such that $\gamma_{3c}(G) = k$. We then show that the P_3-CONVEX DOMINATING SET decision problem is NP-complete, even when restricted to bipartite graphs. Finally, we present an integer linear programming formulation for the P_3-convex dominating set problem and prove that its optimal solutions correspond exactly to minimum P_3-convex dominating sets.

Keywords: P_3-convex domination, NP-completeness, integer linear programming
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1. Introduction

In the context of graphs, there are many instances in the literature where convex sets are defined using the concept of path, typically as follows: A subset $X \subseteq V$ is convex in G if vertices from every $(u - v)$-geodesic belong to X for each pair of vertices $u, v \in X$. Geodesic convexity was

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introduced by Harary and Nieminen [12], minimal path convexity or M-convexity is studied in [8] and triangle path convexity in [4]. Researchers have studied convexity in graphs in many context. One well received direction is to determine convexity invariants, such as the convex domination number, the convexity number and the interval number, among others. For an extensive overview on convexity invariants one may refer [19]. In this article a new convexity invariant, namely P_3-convex domination number is studied.

All the graphs considered here are simple, undirected, finite, non-trivial and connected. For standard terminologies in graphs, we refer to the book by Harary [11]. A subset $D \subseteq V(G)$ is called a dominating set of G if each vertex in $V \setminus D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the cardinality of the smallest dominating set of G. For basics on domination theory in graphs we refer the book "Topics in domination in graphs" by Haynes, Hedetniemi and Henning [13].

By a convex dominating set of a graph G we refer to a dominating set of G that is convex. The convex domination number $\gamma_{con}(G)$ is the cardinality of a minimum convex dominating set of G. The first paper on convex domination and weakly convex domination was by Magdalena Lemaska [17]. Cyman et.al.[6] studied the convex domination number close to their order. Weakly convex and convex domination numbers of torus is studied in [21]. An integer linear programming formulation for weakly convex dominating set and convex dominating set have been studied by Kratica et.al.[16]. Different parameters of convex domination has been studied such as convex doubly connected domination [1], restrained convex domination [9], outer-convex domination [7] and so on. In [20] it is shown that finding weakly convex dominating set and convex dominating set are NP-complete. Some recent decision problems related to domination shown to be NP-complete can be found in [18, 15].

A set $S \subseteq V(G)$ is P_3-convex if, for every path $u - w - v$ of length two with $u, v \in S$, the vertex w belongs to S. Equivalently, no vertex in $V(G) \setminus S$ has two neighbors in S. P_3-convexity has been studied for different convexity parameters such as Carathéodory number [2], hull number [3], radon number [14] and geodetic number [5]. We define, a dominating set D of a graph G to be a P_3-convex dominating set if D is P_3-convex. The P_3-convex domination number $\gamma_{3c}(G)$ is the cardinality of a minimum P_3-convex dominating set.

The primary motivation for studying P_3-convex domination lies in the fundamentally different nature of P_3-convexity compared to geodesic convexity. Geodesic convexity is a *global* condition requiring knowledge of the entire graph structure, whereas P_3-convexity is a *local* condition determined entirely by paths of length two and immediate neighborhoods. This distinction is not merely technical, it has deep structural consequences. A geodesically convex dominating set may fail the P_3-convexity requirement if some exterior vertex has two neighbors in the set, thereby forcing a much larger P_3-convex dominating set. Conversely, a sparse P_3-convex set may successfully dominate G without satisfying geodesic closure, yielding a smaller P_3-convex dominating set. Consequently, there is no fixed inequality between $\gamma_{con}(G)$ and $\gamma_{3c}(G)$, and the two parameters are incomparable in general.

The figures below illustrate all three possible relationships explicitly. In Figure 1, the graph G has $\gamma_{con}(G) < \gamma_{3c}(G)$: the two marked (red) vertices form a minimum geodesically convex dominating set of size 2, but this set violates P_3-convexity since a vertex outside the set has two neighbors within it. To satisfy the local P_3-convexity condition it forces the inclusion of all six

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vertices, giving $\gamma_{3c}(G) = 6$. This example demonstrates how local convexity can demand a significantly larger dominating set than global convexity. In Figure 2, the path P_6 exhibits the reverse inequality $\gamma_{con}(G) > \gamma_{3c}(G)$: the global geodesic condition forces the inclusion of four intermediate vertices to maintain closure, while the weaker local P_3-condition is satisfied by just two vertices that have no common neighbor outside the set. Finally, Figure 3 presents a case where both parameters coincide, $\gamma_{con}(G) = \gamma_{3c}(G) = 2$, because the graph's structure ensures that local and global closure requirements are simultaneously satisfied by the same minimum set. Together, these three examples explicitly demonstrate that P_3-convex domination is structurally distinct from convex domination, neither parameter dominates the other in general and therefore warrants independent study. In Figures 1–3, highlighted (red) vertices denote the corresponding minimum convex dominating set or minimum P_3-convex dominating set shown in each subfigure.



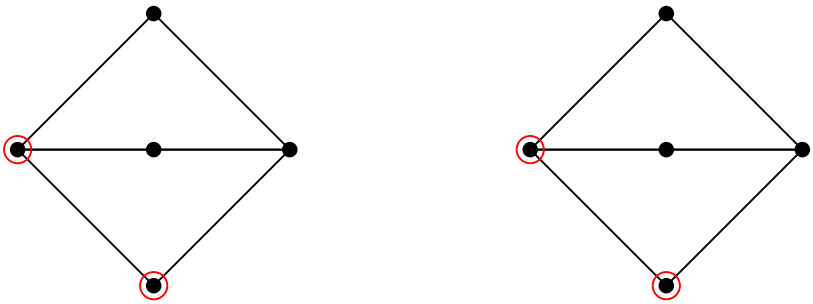
(i) Graph G with $\gamma_{con}(G) = 2$ (ii) Graph G with $\gamma_{3c}(G) = 6$

Figure 1: Graph G with $\gamma_{con}(G) < \gamma_{3c}(G)$.



(i) Graph G with $\gamma_{con}(G) = 4$ (ii) Graph G with $\gamma_{3c}(G) = 2$

Figure 2: Graph G with $\gamma_{con}(G) > \gamma_{3c}(G)$.



(i) Graph G with $\gamma_{con}(G) = 2$ (ii) Graph G with $\gamma_{3c}(G) = 2$

Figure 3: Graph G with $\gamma_{con}(G) = \gamma_{3c}(G)$.

The vertex $V(G)$ is always a dominating as well as P_3-convex dominating set for any graph G. We now show the existence of a graph G of order $n \geq 6$ for every possible P_3-convex domination number k, where $1 \leq k \leq n$.

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Definition 1.1. *[22] The semitotal-point graph $T_2(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$, where $u, v \in V(T_2(G))$ are adjacent if $u \sim v$ in G or $u \in V(G)$ is incident with $v \in E(G)$ in G. We denote the graph $T_2(C_n)$ by S_n and refer it as the sunlike graph.*

If $G(n, m)$ is a graph on n vertices and m edges then $T_2(G)$ has $m + n$ vertices and $3m$ edges.

Theorem 1.1. *[10] Let $G(n, m)$ be a graph with n vertices and m edges. Then $\gamma_{3c}(T_2(G)) = m+n$ if and only if G has no pendant vertex.*

For the star graph $K_{1,n-1}$, it is easy to see $\gamma_{3c}(K_{1,n-1}) = 1$, and for sunlike graph S_n, $\gamma_{3c}(S_n) = 2n$. See Figure 1(ii), where $G = T_2(C_3) = S_3$ and $\gamma_{3c}(S_3) = 6$. It is worth noting here that there exist graphs G with $\gamma_{3c}(G) = n$. Also, for any graph G with $2 \leq n \leq 5$ vertices, $\gamma_{3c}(G) < n$.

Theorem 1.2. *Let $n \geq 6$. Then for each $1 \leq k \leq n$, there exist a graph G with n vertices, such that $\gamma_{3c}(G) = k$.*

Proof. For $k = 1$, any graph G with $\Delta(G) = n - 1$ serves the purpose. For $k = 2, 3, 4$ and 5 the graphs in Figures 4, 5, 6 and 7 respectively ensure the claim. In Figures 4–7, the red-circled vertices represent a minimum P_3-convex dominating set of the graph. For $k \geq 6$, we consider the following two cases.

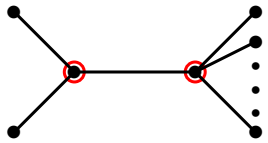


Figure 4: A graph G with $\gamma_{3c}(G) = 2$

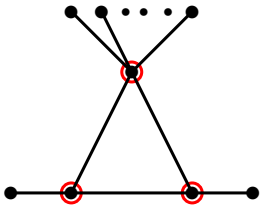


Figure 5: A graph G with $\gamma_{3c}(G) = 3$

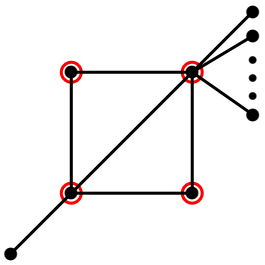


Figure 6: A graph G with $\gamma_{3c}(G) = 4$

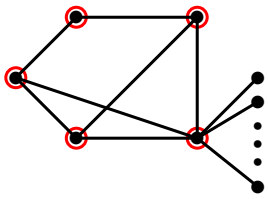


Figure 7: A graph G with $\gamma_{3c}(G) = 5$

Case 1. k is even, say $k = 2t$, where $t \geq 2$

Let G be the graph obtained by attaching $n - k$ pendant vertices with a vertex of the sunlike graph S_t. By Theorem 1.1, $\gamma_{3c}(S_t) = 2t$ with $D = V(S_t)$ as a minimum P_3-convex dominating set of S_t. The pendant vertices added to form G are each adjacent only to the single support vertex x, and to no other vertex of S_t or to each other. Hence they introduce no new common neighbor for any pair of vertices in $V(S_t)$, and no vertex of $V(S_t)$ gains a new way of being dominated through

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them beyond what it already has within S_t. Thus attaching the pendant vertices changes neither the domination nor the P_3-convexity requirement on $V(S_t)$, and $\gamma_{3c}(G) = \gamma_{3c}(S_t) = 2t = k$.

Case 2. k is odd, say $k = 2t + 1$, where $t \geq 1$.
Let G be the graph obtained from S_t by the following operations:

1. Introduce a new vertex w and edges uw, wv for some $u, v \in V(S_t)$.
2. Add $n - k$ pendant vertices at a vertex $x \in V(S_t)$.

By Theorem 1.1, $\gamma_{3c}(S_t) = 2t = |V(S_t)|$, and since this equals the full order of S_t, $V(S_t)$ is the *unique* minimum P_3-convex dominating set of S_t: no proper subset of $V(S_t)$ can dominate and satisfy P_3-convexity on S_t.

Lower bound. Let D' be any minimum P_3-convex dominating set of G. Each pendant vertex at x has degree 1, with x as its only neighbor; hence a pendant can neither dominate any vertex besides x nor serve as the common neighbor of two other vertices, so it cannot substitute for any vertex of $V(S_t)$ in D'. Consequently $D' \cap V(S_t)$ alone must dominate and satisfy P_3-convexity on S_t, which by uniqueness forces $D' \cap V(S_t) = V(S_t)$.

Now consider w, whose only neighbors are $u, v \in V(S_t) \subseteq D'$. Since $u, v \in D'$ have common neighbor w, P_3-convexity forces $w \in D'$ as well. Hence $D' \supseteq V(S_t) \cup \{w\}$, so $\gamma_{3c}(G) \geq 2t + 1$.

Upper bound. Let $D = V(S_t) \cup \{w\}$, so $|D| = 2t + 1$. The only vertices of G outside D are the pendants, each adjacent solely to $x \in D$, so D dominates G. Since pendants have a single neighbor, they can never be the common neighbor of two vertices, so D is P_3-convex, giving $\gamma_{3c}(G) \leq 2t+1$.

Combining both bounds, $\gamma_{3c}(G) = 2t + 1 = k$. □

2. NP-Completeness of P_3-Convex Dominating Set

In this section first we show that P_3-CONVEX DOMINATING SET problem is in NP-class. Further, an algorithm is given that verifies an instance of the P_3-convex dominating set problem in polynomial time. Finally, it is shown that the problem of decision making for P_3-convex dominating set is NP-complete for bipartite graphs.

Algorithm 1: Verify P_3-convex dominating set

Input: Graph $G = (V, E)$ and candidate set $D \subseteq V$
Output: `True` if D is a P_3-CDS, `False` otherwise

```
for each vertex v ∈ V \ D do
    if N(v) ∩ D = ∅ then
        return False ;                          // D is not dominating
    if |N(v) ∩ D| ≥ 2 then
        return False ;                          // D is not P3-convex
return True;
```

Lemma 2.1. *For a graph G, the decision problem of P_3-CDS is in the class NP.*

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Proof. Let $D \subseteq V(G)$ be a candidate set with $|D| \leq k$. We verify that D is a P_3-CDS of G using Algorithm 1, which checks, for every vertex $v \in V \setminus D$, whether v has no neighbor in D (violating domination) or at least two neighbors in D (violating P_3-convexity). Each check requires scanning $N(v)$, so the loop runs in $O(n^2)$ time in the worst case. Hence the verification procedure is polynomial, and the decision problem of P_3-CDS belongs to NP. □

Next we show the problem P_3-CDS for bipartite graphs is NP-complete. To show that it is NP-complete, we reduce the problem EXACT-3-COVER($X3C$) to it. The EXACT-3-COVER($X3C$) problem is defined as follows: Given a set J, with $|J| = 3q$, and a collection C of 3-element subsets of J. Can we find a subset C of C where every element of J occurs in exactly one member of C?

P_3-CONVEX DOMINATING SET
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G contain a P_3-convex dominating set of cardinality at most k?

Theorem 2.1. *For bipartite graphs P_3-CDS problem is NP-complete.*

Proof. By Lemma 2.1 the problem of P_3-CDS is in NP-class. To prove our result we use polynomial reduction from EXACT-3-COVER($X3C$) problem.

Any instance of $X3C$ is reduced to P_3-CDS as follows: Let $C = \{C_1, C_2, \ldots C_p\}$ and $J = \{y_1, y_2, \ldots, y_{3s}\}$ be an instance of $X3C$, where $C_i \in C$ is 3-element subset of J. Construct a bipartite graph $G(V, E)$ with vertex set $V = C \cup J \cup \{v\}$ and edge set $E = \{vC_i : 1 \leq i \leq p\} \cup \{y_j C_i : y_j \in C_j, 1 \leq j \leq 3s, 1 \leq i \leq p\}$. It is clear from the construction that $V = C \cup (J \cup \{v\})$ forms a two set partition and obtained graph is bipartite (see Figure 8).

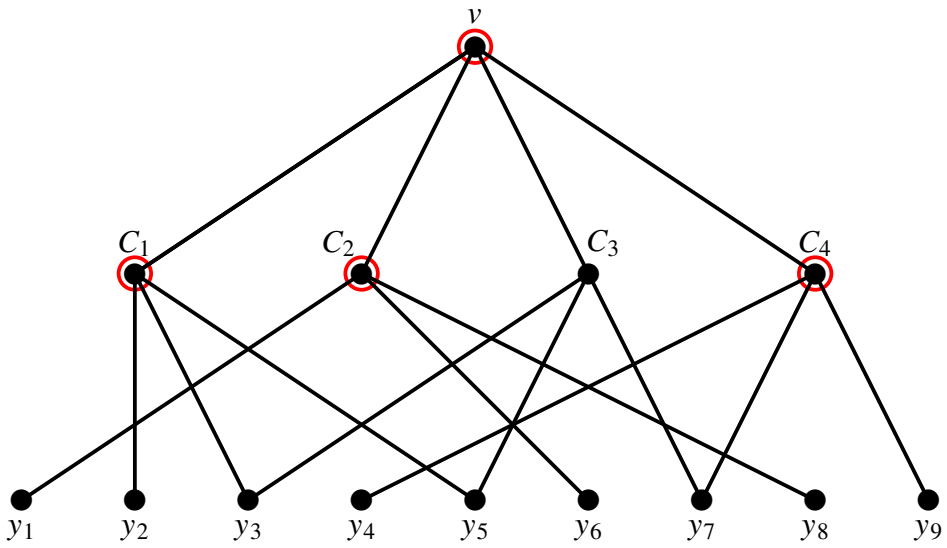


Figure 8: Bipartite Graph with minimum P_3-convex dominatin set(marked in red) $D = \{v, C_1, C_2, C_4\}$

Next we analyze the time complexity of Algorithm 2. Initially, the algorithm constructs the vertex set V, which includes the sets C, J, and a new vertex v, where the cardinality of C and J are

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Algorithm 2: Polynomial Reduction from X3C to P_3-CDS

Input: A set $J = \{y_1, y_2, \ldots, y_{3s}\}$ and a collection $C = \{C_1, C_2, \ldots, C_p\}$ of 3-element subsets of J

Output: A bipartite graph $G = (V, E)$

```
Initialize the vertex set V as follows:
    V = C ∪ J ∪ {v}, where v is a new vertex;
Initialize the edge set E as follows:
    E = ∅;
    for each C_i ∈ C do
        Add the edge vC_i to E;
    for each y_j ∈ J do
        for each C_i ∈ C such that y_j ∈ C_i do
            Add the edge y_jC_i to E;
    return G = (V, E)
```

p and $3s$ respectively. This step involves combining these sets, resulting in a time complexity of $O(p + 3s)$.

Next, adding edges between the vertex v and each subset in C involves iterating over each subset C_i, resulting in a time complexity of $O(p)$.

Next part of the algorithm is adding edges between elements of J and C. For each element y_j in J, the algorithm iterates over each subset C_i to check if y_j belongs to C_i. This involves checking all subsets for each element in J, leading to a time complexity of $O(3s \cdot p)$, which simplifies to $O(sp)$.

Combining all these steps, the overall time complexity of the algorithm is dominated by the $O(sp)$ term. Thus, the polynomial reduction from X3C to P_3-CDS operates in $O(sp)$ time, indicating that the reduction process is polynomial.

Claim. *C has $X3C$ of size s if and only if G has a P_3-CDS of size at most $s + 1$.*

Proof of the Claim: Let $D' = \{C_{q_1}, C_{q_2}, \ldots, C_{q_s}\}$ be an $X3C$ of J. Since $v \sim C_i$, $1 \leq i \leq p$ and D' dominates all the vertices of J, thus $D = D' \cup \{v\}$ forms a dominating set of G. Since $N(C_{q_i}) \cap N(C_{q_j}) = v, \forall\, i, j \in \{1, 2, \ldots, s\}, i \neq j$ and $v \in D$, thus D forms a P_3-CDS of G.

Conversely, suppose G has a P_3-CDS D with $|D| \leq s + 1$.

Step 1. $v \in D$.

If D contains at least two vertices $C_i, C_j \in C$, then $v \in N(C_i) \cap N(C_j)$. By P_3-convexity of D, $v \in D$. Thus it suffices to prove that D must contain at least two vertices of C. If $v \in D$, there is nothing to prove, so assume $v \notin D$. Since $v \notin J$ and the only neighbors of v in G are the vertices of C, at least one vertex of C must belong to D in order to dominate v. If D contains exactly one $C_i \in D \cap C$, then $|D| - 1 \leq s$ of the remaining vertices of D must lie in J. Hence the total number of elements of J dominated or contained in D is at most $3 + s$. Since all $3s$ elements of J must be

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accounted for, we require

$$3s \leq s + 3 \quad \Longrightarrow \quad 2s \leq 3 \quad \Longrightarrow \quad s \leq \tfrac{3}{2}.$$

As s is a positive integer, this forces $s = 1$, which is trivial. We therefore assume $s > 1$, in which case the above inequality cannot hold, and so D must contain at least two vertices $C_i, C_j \in C$.

Step 2. D contains no element-vertices.
Let $x = |D \cap C|$ and $y = |D \cap J|$, so $|D| = 1 + x + y \leq s + 1$, giving

$$x + y \leq s. \tag{1}$$

Every vertex $y_j \in J$ must be dominated by D. Since y_j is adjacent only to vertices of C (not to v or to other elements of J), each $y_j \notin D$ must have a neighbor among the x chosen set-vertices. The x set-vertices in $D \cap C$ collectively cover at most $3x$ elements of J (each C_i has exactly 3 neighbors in J). The y vertices of J already in D account for y number of the $3s$ elements. The remaining $3s - y$ elements of J must be dominated by the x vertices of $D \cap C$, each of which is adjacent to exactly 3 elements of J, so together they dominate at most $3x$ elements of J. Hence

$$3s - y \leq 3x, \tag{2}$$
$$3s \leq 3x + y. \tag{3}$$

From (1), $y \leq s - x$, substituting into (3):

$$3s \leq 3x + (s - x) = 2x + s \quad \Longrightarrow \quad x \geq s.$$

On the other hand, (1) gives $x \leq s - y \leq s$. Combining both bounds forces $x = s$, and consequently $y = 0$ from (1). Thus D contains exactly s vertices of C and no vertices of J.

Step 3. the s set-vertices form an exact cover.
Since $y = 0$, all $3s$ vertices of J must be dominated entirely by the $x = s$ set-vertices in $D \cap C$. Each such set-vertex is adjacent to exactly 3 elements of J, so the total coverage capacity of these s sets, counted with multiplicity, is exactly $3s$. As all $3s$ distinct elements of J must be covered, this capacity is used entirely on distinct elements, leaving no room for any element to be covered more than once. Hence the s chosen subsets $C_{q_1}, \ldots, C_{q_s}$ are pairwise disjoint and their union is J, i.e., they form an exact 3-cover of J.

□

3. Integer Linear Programming Formulation

Let $G(V, E)$ be a graph with $V = \{1, 2, \ldots, n\}$ and $|E| = m$. Let x_i be a decision variable indicating vertex i belongs to a P_3-convex dominating set D.

$$x_i = \begin{cases} 1, & i \in D, \\ 0, & i \notin D. \end{cases} \tag{4}$$

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An integer linear programming (ILP) model for the P_3-CDS can be formulated as follows:

$$\min \sum_{i=1}^{n} x_i \tag{5}$$

subject to

$$x_i + \sum_{j \in N(i)} x_j \geq 1 \qquad 1 \leq i \leq n, \tag{6}$$

$$x_k - x_i - x_j \geq -1 \qquad 1 \leq i < j \leq n, \quad k \in N(i) \cap N(j). \tag{7}$$

$$x_i \in \{0, 1\} \qquad 1 \leq i \leq n. \tag{8}$$

The objective function is given by (5), while domination is ensured by constraints (6). Constraints (7) enforce P_3-convexity, and constraints (8) ensure that the decision variables are binary.

Let $A = (a_{ij})$ be the adjacency matrix of a graph G and $A^2 = \left(a_{ij}^{(2)}\right)$ be the square of A. It is well known that $a_{ij}^{(2)}$ counts the number of walks of length 2 from vertex v_i to vertex v_j. For $i \neq j$, each such walk corresponds to a unique common neighbor $v_k \in N(v_i) \cap N(v_j)$, and hence $a_{ij}^{(2)} = |N(v_i) \cap N(v_j)|$. The P_3-convexity constraints in the ILP formulation arise from each unordered pair v_i, v_j and each of their common neighbors $v_k \in N(v_i) \cap N(v_j)$: for each such triple, one constraint is required to ensure that if both v_i and v_j belong to the dominating set, then v_k must also belong to it. Therefore, the total number of P_3-convexity constraints is

$$\sum_{1 \leq i < j \leq n} |N(v_i) \cap N(v_j)| = \sum_{i<j} a_{ij}^{(2)},$$

which is precisely the total number of paths of length two in G. Consequently, this ILP model has n binary variables and the total number of constraints is of order

$$3n + \sum_{1 \leq i < j \leq n} |N(v_i) \cap N(v_j)|.$$

Next, we present the polynomial time Algorithm 3, that is designed to construct all the necessary constraints for an ILPP addressing the P_3-CDS problem. Given a graph $G = (V, E)$, Algorithm 3 formulates ILP constraints to identify a P_3-CDS. The constraints are constructed to enforce the P_3-convex domination property for every vertex in the graph.

The Algorithm 3 has time complexity of $O(n^3)$. This has been shown in the following lines. Basically, the two loops has dominant input in overall time complexity. The loop in the lines 2-3 varies for each i from 1 to n. Further for each i, this loops sums over all neighbors $j \in N(i)$. Thus in worst case overall time required for this loop is $O(n^2)$. Subsequently, for the nested loop in between the lines 4-7, i runs from 1 to n and also for $j > i$ it runs from 1 to n. The complexity of these loops is $O(n^2)$. Further inner loop checks $N(i) \cap N(j)$ and iterates over this intersection. This iteration is bounded by $min(deg(i), deg(j))$, which can be as high as $n-1$, i.e. approximately $O(n)$ in worst case. Thus, overall complexity of this nested loop is $O(n^3)$. First and the last loop of the algorithm runs in $O(n)$ time. Thus overall time complexity of the Algorithm 3 is $O(n^3)$.

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Algorithm 3: CONSTRUCTION OF ILPP CONSTRAINTS FOR P_3-CDS

Input: A graph G of order n

Output: Constraints of ILPP for P_3CDS

$N(i) = \{j : ij \in E(G)\}, \forall\ i \in V(G)$

for *i from 1 to n* **do**

$$x_i + \sum_{j \in N(i)} x_j \geq 1$$

for *i from 1 to n* **do**

  for $j > i$ *from 1 to n* **do**

    if $a_{ij}^{(2)} > 0$ *and* $N(i) \cap N(j) \neq \phi$ **then**

      $x_k - x_i - x_j \geq -1, \forall\ k \in N(i) \cap N(j)$

for *i from 1 to n* **do**

  $x_i \leq 1$ and $x_i \geq 0$

Theorem 3.1. *A set $D \subseteq V(G)$ is a P_3-convex dominating set of G if and only if its incidence vector satisfies the domination and P_3-convexity constraints. Consequently, an optimal solution of the ILP gives a minimum P_3-convex dominating set.*

Proof. Let D be a minimum P_3-convex dominating set of G and decision variable x are defined by

$$x_i = \begin{cases} 1, & i \in D, \\ 0, & i \notin D. \end{cases} \tag{9}$$

Since D is a dominating set, thus

$$\begin{aligned} & \forall\ i \in V(G),\ \exists\ j \in N(i) && \text{such that } i \in D \text{ or } j \in D. \\ \text{Further,} \quad & \forall\ i \in V(G) \text{ and } j \in N(i), && x_i = 1 \text{ or } x_j = 1. \\ \text{Implies,} \quad & x_i + \textstyle\sum_{j \in N(i)} x_j \geq 1, && \forall\ i \in V(G). \end{aligned}$$

Constraints (6) are satisfied in this way. Without loss of generality, let $1 \leq i < j \leq n$, for $i, j \in V(G)$. We have two cases. **Case 1.** $i \notin D$ or $j \notin D$.
In this case, either $x_i = 0$ or $x_j = 0$ and $x_i, x_j \in \{0, 1\}$. Thus, $x_i + x_j \leq 1$, implies $-x_i - x_j \geq -1$. Since, $x_k \geq 0,\ \forall\ k \in V(G)$, therefore, $x_k - x_i - x_j \geq -1$.

Case 2. $i \in D$ and $j \in D$.
Let $k \in N(i) \cap N(j)$. Since D is P_3-convex dominating set, so $k \in D$. Therefore, $x_i = x_j = x_k = 1$, implies $x_k - x_i - x_j \geq -1$.

Constraints (7) are satisfied in both the cases. From (9), it holds

$$|D| = \sum_{i=1}^{n} x_i.$$

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As decision variable x represents a feasible solution of ILP (5)-(8), thus its optimal solution must be $\leq |D|$.

Conversely, let constraints (5)-(8) are satisfied. Let $X = \{i | x_i = 1\}$. From (6) we can say

$$\begin{aligned} & x_i = 1 \text{ or } && \textstyle\sum_{j \in N(i)} x_j \geq 1 && \forall\ i. \\ \implies & x_i = 1 \text{ or } && \exists\ j \in N(i) \text{ such that } x_j = 1 && \forall\ i. \\ \implies & i \in D \text{ or } && \exists\ j \in N(i) \text{ such that } j \in D && \forall\ i. \end{aligned}$$

This implies D is a dominating set of G. Let $i, j \in D$ and without the loss of generality, $i < j$. Let $k \in N(i) \cap N(j)$. By (7),

$$\begin{aligned} & x_k - x_i - x_j && \geq && -1 \\ \implies & x_k && \geq && x_i + x_j - 1. \\ \because\ & x_i = x_j = 1 && \text{thus} && x_k \geq 1. \end{aligned}$$

By (8) $x_k = 1$ holds, i.e. $k \in D$. Therefore for every $i, j \in D$ any vertex $k \in N(i) \cap N(j)$ will be in D. Thus D is P_3-convex dominating set. Since D is proven to be P_3-convex dominating set, minimum P_3-convex domination number has to be

$$\leq \sum_{i=1}^{n} x_i.$$

□

4. Conclusion

This article introduced the concept of a P_3-convex dominating set and studied several of its fundamental properties. It is shown that for every pair of integers $n \geq 6$ and $1 \leq k \leq n$, there exists a graph G of order n with $\gamma_{3c}(G) = k$. Further it is proved that the P_3-CONVEX DOMINATING SET problem belongs to NP and is NP-complete even for bipartite graphs. Further, an integer linear programming formulation for the problem is presented that correctly determine minimum P_3-convex dominating sets.

These results establish both the structural richness and the computational complexity of P_3-convex domination. Moreover, the parameter is shown to be fundamentally different from geodesic convex domination, indicating that it merits independent study. Future work may focus on characterizing graph classes satisfying $\gamma_{3c}(G) < \gamma_{con}(G)$, $\gamma_{3c}(G) = \gamma_{con}(G)$, and $\gamma_{3c}(G) > \gamma_{con}(G)$, as well as developing efficient algorithms for computing $\gamma_{3c}(G)$.

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