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On Ramsey $(2K_2, W_n)$-minimal graphs of smallest order

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Abstract

The notation $F \to (H, G)$ means that if all edges of F are arbitrarily colored by red or blue, then either the subgraph of F induced by all red edges contains a graph H or the subgraph of F induced by all blue edges contains a graph G. Let $\mathcal{R}(H, G)$ denote the set of all graphs F satisfying $F \to (H, G)$ and for every $e \in E(F)$, $(F-e) \not\to (H, G)$. In this paper, we propose some properties of Ramsey $(2K_2, G)$-minimal graph of smallest order, where G is a graph containing a dominating vertex. We also find all members of $\mathcal{R}(2K_2, W_n)$ of smallest order for $n \in [5, 8]$.

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1. Introduction

Ramsey theory, first introduced by Ramsey [14] in 1930, investigates the unavoidable emergence of order in large or complex structures. While the classical formulation focused on complete graphs and cliques, the theory was later extended in the 1950s by Erdős and Rado [7] to a more

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general setting involving arbitrary graphs. All graphs considered in this paper are simple. The notation $F \to (H, G)$ means that each red-blue coloring of the edges of the graph F results in F containing a red subgraph that is isomorphic to H or a blue subgraph that is isomorphic to G.

The smallest integer n for which $K_n \to (H, G)$ is called the *Ramsey number* for the graphs H and G, denoted by $R(H, G)$. Zhang et al. [20] established the best known general upper bound for $R(C_4, W_n)$ and determined several exact values of $R(C_4, W_n)$ for specific values of n. Furthermore, Sudarsana [15] proved that if $n \geq f(t)$, then the Ramsey number $R(C_n, tW_4) = 2n + t - 2$ holds for all $t \geq 1$, where $f(t) = 15t^2 - 4t + 2$

Ramsey's theory evolved to become the minimal Ramsey graph theory as defined by Burr et al. [6] in 1976. A graph F is called a *Ramsey (H, G)-minimal graph* if $F \to (H, G)$ and $F - e \nrightarrow (H, G)$ for any subgraph $F - e \subset F$. $\mathcal{R}(H, G)$ is the set of all (H, G)-minimal graphs. The pair (H, G) is called Ramsey-finite if $\mathcal{R}(H, G)$ is finite and Ramsey-infinite otherwise.

Łuczak [10] showed that the set $\mathcal{R}(H, G)$ is infinite for every forest H other than a matching mK_2 and every graph G containing a cycle. Nabila et al. [13] presented several finite and infinite classes of Ramsey $(C_4, K_{1,n})$-minimal graphs for every $n \geq 3$. Furthermore, Assiyatun et al. [1] introduced a new class of Ramsey $(C_4, K_{1,n})$-minimal graphs. Burr et al. [5] proved that $\mathcal{R}(mK_2, G)$ is Ramsey finite for any graph G and a positive integer m. In this paper, we focus on the class $\mathcal{R}(mK_2, G)$, where G is a graph containing a dominating vertex (a vertex that is adjacent to any other vertices in the graph).

Mangersen and Oeckermann [11] proved that $\mathcal{R}(2K_2, K_{1,2}) = \{2K_{1,2}, C_4, C_5\}$, and presented the characterization of graphs belonging to $\mathcal{R}(2K_2, K_{1,n})$ for $n \geq 3$. Furthermore, Muhshi and Baskoro [12] proved that $\mathcal{R}(3K_2, P_3) = \{3P_3, C_4 \cup P_3, C_5 \cup P_3, C_7, C_8\}$. Baskoro and Yulianti [4] characterized all graphs in $\mathcal{R}(2K_2, P_n)$ for $n \in \{3, 4\}$, while Fajri et al. [9] determined all graphs in $\mathcal{R}(2K_2, F_n)$ with minimum order for $n \in [4, 8]$, where $F_n = K_1 + P_{n-1}$. Moreover, Yulianti et al. [19] gave the construction of some infinite class in $\mathcal{R}(K_{1,2}, P_4)$. Wijaya et al. [16] determined all graphs belonging to $\mathcal{R}(2K_2, K_4)$. Wijaya et al. [17] characterized all uncyclic graphs belonging to $\mathcal{R}(mK_2, P_3)$, and subsequently Baskoro et al. [3] characterized all uncyclic graphs belonging to $\mathcal{R}(mK_2, P_4)$.

Baskoro and Wijaya [2] derived the necessary and sufficient conditions for the graphs to be in $\mathcal{R}(2K_2, G)$ for any connected graph G. They proved the following theorem.

Theorem 1.1. *[2] Let G be any connected graph. $F \in \mathcal{R}(2K_2, G)$ if and only if the following conditions are satisfied:*

(i) for every $v \in V(F)$, $F - v \supseteq G$,
(ii) for every K_3 in F, $F - E(K_3) \supseteq G$,
(iii) for every $e \in E(F)$, there exists $v \in V(F)$ or K_3 in F such that $(F - e) - v \nsupseteq G$ or $(F - e) - E(K_3) \nsupseteq G$.

If a graph F satisfies Theorem 1.1 (i) and (ii), then $F \to (2K_2, G)$. Moreover, a graph F satisfying Theorem 1.1 (iii) means that F satisfies the minimality property of F, that is for each $e \in F$, $F - e \nrightarrow (2K_2, G)$. Furthermore, Wijaya and Baskoro [18] described the necessary and sufficient conditions for graphs in $\mathcal{R}(mK_2, G)$.

In this paper, we propose some properties of Ramsey $(2K_2, G)$-minimal graph of smallest order, where G is a graph containing a dominating vertex. We also find all members of $\mathcal{R}(2K_2, W_n)$ of smallest order for $n \in [5, 8]$.

2. Main Result

We will introduce some definitions and notations. A complete graph and a cycle on the n vertices are denoted by K_n and C_n, respectively. A wheel W_n is defined as the graph $K_1 + C_{n-1}$. A union of m disjoint copies of K_2, denoted by mK_2, is called a *matching*. A complete multipartite graph consisting of j partite sets and k vertices in each partite set is denoted by $K_{j\times k}$. Given a graph G, E_i denotes the set of i arbitrary edges of $E(G)$. The notation $G - E_i$ denotes the removal of any i edges from the graph G. The notation $[i, t]$ represents the set of all integers that lie between i and t (including i and t themselves).

The main results of this paper are presented through several theorems. The first theorem (Theorem 2.1) provides a characterization of the graphs belonging to Ramsey $(2K_2, G)$-minimal graphs of smallest order and maximum degree $n - 1$ for any graph G with n vertices, where $G = K_1 + H$ for some $H \supseteq K_{\frac{n+1}{2}\times 2} - e$. The second theorem (Theorem 2.2) establishes necessary conditions for a graph contained in F to be a member of $\mathcal{R}(2K_2, G)$ of smallest order and maximum degree n, where G has a dominating vertex. In the final theorems (Theorems 2.3-2.6), we discover and prove the complete list of Ramsey $(2K_2, W_n)$-minimal graphs of smallest order for $n \in [5, 8]$.

Let $\mathcal{G}(n)$ denote the set of finite simple graphs G that have exactly n vertices, none of which is isolated, and let $\mathcal{G}(n, c)$ denote the subset of $\mathcal{G}(n)$ with the independence number c. Faudree et al. [8] found the Ramsey number for matching and any graph G as follows.

Corollary 2.1. *[8] Let $G \in \mathcal{G}(n, c)$. Then $R(sK_2, G) = n + s - 1$ if $s \leq \lfloor \frac{2}{3}c \rfloor + 1$.*

As a result of Corollary 2.1, when we set $s = 2$, it follows that $R(2K_2, G) = n + 1$ for $c \geq 2$. Define $\mathcal{G}_D(n, i)$ as the subset of $\mathcal{G}(n, c)$ where $n \geq 2$, $c \geq 2$, and precisely i vertices have degree $n - 1$ for $i \geq 1$. Utilizing the relationship between the Ramsey number and Ramsey-minimal graphs, the smallest order of $F \in \mathcal{R}(H, G)$ is $R(H, G)$. It follows that for each $G \in \mathcal{G}_D(n, i)$, the minimum order of $F \in \mathcal{R}(2K_2, G)$ is $n+1$. Hereafter, we limit our discussion to the set of Ramsey (H, G)-minimal graphs of the smallest order, denoted by $\hat{\mathcal{R}}(H, G)$. The subsequent proposition is established.

Proposition 2.1. *Let $G \in \mathcal{G}_D(n, i)$. If $F \in \hat{\mathcal{R}}(2K_2, G)$, then $n - 1 \leq \Delta(F) \leq n$.*

Proof. Since $|V(F)| = n + 1$ and $\Delta(G) = n - 1$, then $n - 1 \leq \Delta(F) \leq n$. □

Let $\deg_G(v)$ denote the degree of the vertex v in the graph G, and let $D(G, i)$ represent the number of vertices of degree i in the graph G. Let $G \in \mathcal{G}_D(n, i)$ and denote $G = K_1 + H$ for some graph H. By Theorem 1.1 and Proposition 2.1, we obtain necessary and sufficient conditions for graphs belonging to $\hat{\mathcal{R}}(2K_2, G)$ in following observation.

Observation 2.1. *Let $G \in \mathcal{G}_D(n, i)$. Let $G = K_1 + H$, $F \in \hat{\mathcal{R}}(2K_2, G)$ if and only if the following conditions are satisfied:*

(i) For every $v \in V(F)$, there exists $v' \in V(F - v)$ where $deg_{F-v}(v') \geq n - 1$ such that $(F - v) - v' \supseteq H$.
(ii) For every K_3 in F, there exists $v' \in V(F - E(K_3))$ where $deg_{F-E(K_3)}(v') \geq n - 1$ such that $(F - E(K_3)) - v' \supseteq H$.
(iii) For every $e \in E(F)$, there exists $v \in V(F - e)$ or $K_3 \subset F - e$ such that
  (a) $\Delta((F - e) - v) < n - 1$ or $\Delta((F - e) - E(K_3)) < n - 1$, or
  (b) for every $v' \in (F - e) - v$ where $deg_{(F-e)-v}(v') \geq n - 1$, $((F - e) - v) - v' \not\supseteq H$, or
  (c) for every $v' \in (F - e) - E(K_3)$ where $deg_{(F-e)-E(K_3)}(v') \geq n - 1$, $((F - e) - E(K_3)) - v' \not\supseteq H$.

Characterizing all graphs in $R(2K_2, G)$ for a given graph $G \in \mathcal{G}_D(n, i)$ is difficult. Based on the definition of Ramsey (H, G)-minimal, we also obtain some properties for the graph belonging to $\hat{\mathcal{R}}(2K_2, G)$ in the following observation.

Observation 2.2. *Let F' be any graph. If $F \in \hat{\mathcal{R}}(H, G)$, then the following conditions are satisfied:*

(i) If $F \supset F'$ or $F' \supset F$, then $F' \notin \hat{\mathcal{R}}(H, G)$.
(ii) If $F' \in \hat{\mathcal{R}}(H, G)$, then $F \not\supseteq F'$ and $F \not\supseteq F'$.

2.1. Some Properties of Ramsey $(2K_2, G)$-Minimal Graph of Smallest Order

In this subsection, we discuss some properties of the graph F that satisfy $F \in \hat{\mathcal{R}}(2K_2, G)$ for $G \in \mathcal{G}_D(n, i)$. The following result gives the characterizations of the graph F for odd n and $i = 1$, with maximum degree $n - 1$.

Theorem 2.1. *Let $G \in \mathcal{G}_D(n, 1)$ for odd $n \geq 3$, where $G = K_1 + H$ for some $H \subseteq K_{\frac{n-1}{2} \times 2} - e$. Then $K_{\frac{n+1}{2} \times 2}$ is the unique graph in $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree $n - 1$.*

Proof. Let $v_i \in V(K_{n+1})$ for $i \in [1, n+1]$. For even $n \geq 3$, define the graph $K_{\frac{n+1}{2} \times 2} = K_{n+1} - E$, where $E = \{v_{2j-1}v_{2j} \mid j \in [1, \frac{n+1}{2}]\}$. Let $G = K_1 + H$, for some $H \subseteq K_{\frac{n-1}{2} \times 2} - e$. First, we will show that $K_{\frac{n+1}{2} \times 2} \in \hat{\mathcal{R}}(2K_2, G)$. Observe the following cases

(i) Without loss of generality, consider v_1 and choose v_2 which is in the same partite with v_1. Note that $K_{\frac{n+1}{2} \times 2} - v_1 - v_2 = K_{\frac{n-1}{2} \times 2}$.
(ii) Without loss of generality, consider a K_3 with $V(K_3) = \{v_1, v_3, v_5\}$. Choose v_2 as the dominating vertex in G. Since v_1 is not adjacent to v_2, we can disregard v_1. Observe that $K_{\frac{n+1}{2} \times 2} - E(K_3) - v_2 - v_1 = K_{\frac{n-1}{2} \times 2} - e$.

Based on both cases above, since $H \subseteq K_{\frac{n-1}{2} \times 2} - e \subset K_{\frac{n-1}{2} \times 2}$, by Observation 2.1 (i) and (ii), $K_{\frac{n+1}{2} \times 2} \rightarrow (2K_2, G)$. Next, we will show that $K_{\frac{n+1}{2} \times 2} - e \not\rightarrow (2K_2, G)$. Without loss of generality, let $e = v_1v_3$. Note that $\Delta(K_{\frac{n+1}{2} \times 2} - e - v_1) = n - 2$, thus $G \not\subseteq K_{\frac{n+1}{2} \times 2} - e - v_1$. By Observation 2.1 (iii), $K_{\frac{n+1}{2} \times 2} - e \not\rightarrow (2K_2, G)$. Therefore, we obtain $K_{\frac{n+1}{2} \times 2} \in \hat{\mathcal{R}}(2K_2, G)$.

Afterward, we will show that $K_{\frac{n+1}{2} \times 2}$ is the unique graph in $\hat{\mathcal{R}}(2K_2, G)$ with the maximum degree $n - 1$. Let F be any graph with $n + 1$ vertices and maximum degree $n - 1$, where $F \neq$

$K_{(\frac{n+1}{2})\times 2}$. We know $F \subset K_{(\frac{n+1}{2})\times 2} \in \hat{\mathcal{R}}(2K_2, G)$. By Observation 2.2 (i), $F \notin \hat{\mathcal{R}}(2K_2, G)$. Therefore, $K_{(\frac{n+1}{2})\times 2}$ is the unique graph in $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree $n-1$ for odd $n \geq 3$. $\square$

For other conditions on n and i, the following proposition proves that no other graphs belong to $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree $n-1$.

Proposition 2.2. *Let $G \in \mathcal{G}_D(n, i)$. If n is even and $i \geq 1$ or n is odd and $i \geq 2$, there are no graphs belonging to $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree $n-1$.*

Proof. Let $v_l \in V(K_{n+1})$ for $l \in [1, n+1]$. We will show that for both following conditions, there are no graphs belonging to $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree $n-1$.

(i) For even n and $i \geq 1$.
Define a graph $K_{n+1} - E$, where $E = \{v_{2k-1}v_{2k} | k \in [1, \frac{n}{2}]\} \cup \{v_1 v_{n+1}\}$. Let F be any graph with order $n+1$ and maximum degree $n-1$, we know that $K_{n+1} - E \supseteq F$. Since $\Delta(K_{n+1} - E - v_{n+1}) = n-2$, then $K_{n+1} - E - v_{n+1} \not\supseteq G$. By Theorem 1.1 (i), $K_{n+1} - E \notin \hat{\mathcal{R}}(2K_2, G)$. Since $K_{n+1} - E \supseteq F$, by Observation 2.2 (i), $F \notin \hat{\mathcal{R}}(2K_2, G)$. Therefore, there are no graphs belonging to $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree $n-1$ for even $n \geq 2$.

(ii) For odd n and $i \geq 2$.
Define a graph $K_{(\frac{n+1}{2})\times 2} = K_{n+1} - E$, where $E = \{v_{2j-1}v_{2j} \mid j \in [1, n+1]\}$. Let F be any graph with $n+1$ vertices and maximum degree $n-1$, we know that $F \subseteq K_{(\frac{n+1}{2})\times 2}$. Consider the graph $K_{(\frac{n+1}{2})\times 2} - v_1$, since $G \in \mathcal{G}_D(n, i)$ and $\deg_{K_{(\frac{n+1}{2})\times 2} - v_1}(v_j) = n-2$ for $i \geq 2$ and $j \in [3, n+1]$, then $G \not\subseteq K_{(\frac{n+1}{2})\times 2} - v_1$. Since $F - v_1 \subseteq K_{(\frac{n+1}{2})\times 2} - v_1$, then $G \not\subseteq F - v_1$. By Theorem 1.1 (i), $F \notin \hat{\mathcal{R}}(2K_2, G)$. Therefore, there are no graphs belonging to $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree $n-1$ for odd $n \geq 3$.

Based on both conditions above, we obtain there are no graphs belonging to $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree $n-1$. $\square$

Based on Theorem 2.1 and Proposition 2.2 above, we conclude that for every $G \in \mathcal{G}_D(n, i)$, only odd n and $i = 1$ produce graphs belonging to $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree $n-1$, where $G = K_1 + H$ for some $H \subseteq K_{\frac{n-1}{2} - e \times 2}$.

The notation $D(F, n)$ denotes the number of vertices with degree n in the graph F. For the remaining of this subsection, assume that $\deg(v_i) \geq \deg(v_j)$, for $i < j$. Next, we will discuss some necessary conditions for $F \in \hat{\mathcal{R}}(2K_2, G)$ with maximum degree n for $G \in \mathcal{G}_D(n, i)$. The following result provides the properties of degrees in such graphs F.

Lemma 2.1. *Let $G \in \mathcal{G}_D(n, i)$. If $F \in \hat{\mathcal{R}}(2K_2, G)$ with $\Delta(F) = n$, then*

(i) $D(F, n) \geq i + 1$.
(ii) $D(F, n) + D(F, n-1) \geq i + 3$

Proof. Let $F \in \hat{\mathcal{R}}(2K_2, G)$ with $\Delta(F) = n$. Let $v_j \in V(F)$ for $j \in [1, n+1]$ and $\deg_F(v_1) = n$.

(i) Suppose $D(F,n) < i+1$, then $D(F-v_1, n-1) < i = D(G, n-1)$. Thus, we obtain $G \not\subseteq F - v_1$. By Theorem 1.1, $F \notin \hat{\mathcal{R}}(2K_2, G)$, a contradiction. Therefore, $D(F,n) \geq i+1$.

(ii) By Point (i), let $\deg_F(v_k) = n$, for $k \in [1, i+1]$. Observe a K_3, with $V(K_3) = \{v_1, v_2, v_{i+3}\}$. Suppose that $D(F,n) + D(F, n-1) < i+3$. By Point (i), we know that $D(F,n) = i+1$ or $i+2$. Consider the following cases.

(a) Let $D(F,n) = i+1$, then $D(F, n-1) \leq 1$. If $D(F, n-1) = 1$, let $\deg_F(v_{i+3}) = n-1$, we obtain $D(F - E(K_3), n-1) = 0$ and $D(F - E(K_3), n) = i-1$. Hence, if $D(F, n-1) = 0$, we also obtain $D(F - E(K_3), n-1) = 0$ and $D(F - E(K_3), n) = i-1$. For both possibilities, we have $D(F - E(K_3), n) + D(F - E(K_3), n-1) = i - 1 < i = D(G, n-1)$.

(b) Let $D(F,n) = i+2$, then $D(F, n-1) = 0$. We obtain $D(F - E(K_3), n) + D(F - E(K_3), n-1) = i-1 < i = D(G, n-1)$.

For both above cases, we obtain $G \not\subseteq F - E(K_3)$. By Theorem 1.1 (ii), $F \notin \hat{\mathcal{R}}(2K_2, G)$, a contradiction. Therefore, $D(F,n) + D(F, n-1) \geq i+3$.

□

Based on the lemma above, we conclude that there are at least $i+3$ vertices with degrees of at least $n-1$, and at least $i+1$ of them have degrees n. Thus, we can construct basic graphs that satisfy these conditions. Define a graph $B_{n,i,j}$, where $|V(B_{n,i,j})| = n+1$ for $n \geq 2$, $i \geq 1$, and $j \in [1,3]$, with exactly $i+1$ vertices of degree n and exactly two vertices of degree $n-1$. The graphs $B_{n,i,1}$, $B_{n,i,2}$, and $B_{n,i,3}$ have, respectively, n, $n-1$, and $n-2$ common neighbors of the two vertices of degree $n-1$. The graphs $B_{n,i,j}$, for $i \in [1,3]$, are shown in Figure 1.

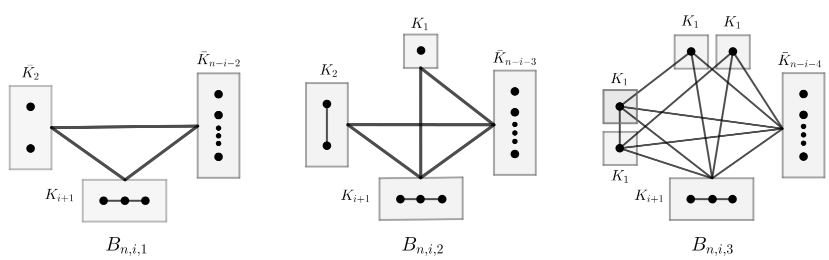

Figure 1. Graph $B_{n,i,1}$, $B_{n,i,2}$, and $B_{n,i,3}$

In the following theorem, we obtain the necessary conditions for a graph contained in F to be a member of $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree n, where $G \in \mathcal{G}_D(n,i)$.

Theorem 2.2. *Let $G \in \mathcal{G}_D(n,i)$ for $n \geq 2$. If $F \in \hat{\mathcal{R}}(2K_2, G)$ with $\Delta(F) = n$, then $F \supseteq B_{n,i,j}$ for some $j \in [1,3]$.*

Proof. We observe that every possible graph with $n+1$ vertices, where there are exactly $i+1$ vertices of degree n and two vertices of degree $n-1$, is represented by the graph $B_{n,i,j}$ for $j \in [1,3]$. By Lemma 2.1, $F \supseteq B_{n,i,j}$ for some $i \in [1,3]$. □

The graph $B_{n,i,j}$ for $j \in [1,3]$, is referred to as the basic graph that must be contained in F. By Lemma 2.1, we obtain additional necessary conditions for the graph F to be a member of $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree n, where $G \in \mathcal{G}_D(n,i)$.

Proposition 2.3. *Let $G \in \mathcal{G}_D(n,i)$ for $n \geq 2$. If $F \in \hat{\mathcal{R}}(2K_2, G)$ with $\Delta(F) = n$, then all the following conditions are necessary for graph F.*

(i) *The minimum degree of F, $\delta(F) \geq \delta(G) + 1$.*

(ii) *Any two vertices of degree $\delta(G) + 1$ are not adjacent.*

Proof. Let $F \in \hat{\mathcal{R}}(2K_2, G)$ with $\Delta(F) = n$. Let $v_k \in V(F)$ for $k \in [1, n+1]$. By Lemma 2.1, let $\deg_F(v_{k_1}) = n$ and $\deg_F(v_{k_2}) \geq n-1$ for $k_1 \in [1, i+1]$ and $k_2 \in [i+2, i+3]$.

(i) Suppose $\delta(F) < \delta(G) + 1$. Since $\delta(F - v_1) = \delta(F) - 1 < \delta(G)$, then $G \not\subseteq F - v_1$. By Theorem 1.1 (i), $F \notin \hat{\mathcal{R}}(2K_2, G)$, a contradiction. Therefore, $\delta(F) \geq \delta(G) + 1$.

(ii) Let $v_n, v_{n+1} \in V(F)$, with $\deg_F(v_n) = \deg_F(v_{n+1}) = \delta(G) + 1$. Suppose that v_n and v_{n+1} are adjacent. Observe a graph K_3, with $V(K_3) = \{v_1, v_n, v_{n+1}\}$. Consider $\delta(F - E(K_3)) = \deg_{F-E(K_3)}(v_n) = \deg_{F-E(K_3)}(v_{n+1}) = \delta(G) - 1$, hence $v_n, v_{n+1} \notin V(G)$. Since $|V(F - E(K_3))| = n+1$ and $v_n, v_{n+1} \notin V(G)$, then $F - E(K_3) \not\supseteq G$. By Theorem 1.1 (ii), $F \notin \hat{\mathcal{R}}(2K_2, G)$, a contradiction. Therefore, v_n and v_{n+1} are not adjacent.

□

If the graph G has exactly one dominating vertex, that is $G \in \mathcal{G}_D(n,1)$, then we obtain some specific necessary conditions for the graphs that are members of $\hat{\mathcal{R}}(2K_2, G)$ with maximum degree n as stated in the following proposition.

Proposition 2.4. *Let $G \in \mathcal{G}_D(n,1)$ for $n \geq 2$. If $F \in \hat{\mathcal{R}}(2K_2, G)$ with $D(F,n) = 2$, $D(F, n-1) = 2$, and $\delta(G) \geq 3$, then all the following conditions are necessary for the graph F.*

(i) *If there exists a vertex of degree $\delta(G) + t$ with $t \in [1,2]$, then the two vertices of degree $n-1$ are adjacent.*

(ii) *If there exists a vertex of degree $\delta(G) + 1$, then each vertex of degree $\delta(G) + 1$ is adjacent to at most one vertex of degree $n-1$.*

(iii) *If there exists a vertex of degree $\delta(G) + 2$ that is adjacent to both vertices of degree $n-1$, then neighbors of the vertex of degree $\delta(G) + 2$ are also neighbors of at least one vertex of degree $n-1$.*

Proof. Let $F \in \hat{\mathcal{R}}(2K_2, G)$ with $D(F,n) = 2$, $D(F, n-1) = 2$, and $\delta(G) \geq 3$. Let $v_k \in V(F)$ for $k \in [1, n+1]$. Let $\deg_F(v_{k_1}) = n$ and $\deg_F(v_{k_2}) = n-1$ for $k_1 \in [1,2]$ and $k_2 \in [3,4]$.

(i) Let $\deg_F(v_{n+1}) = \delta(G) + t$ for some $t \in [1,2]$. Suppose v_3 and v_4 are not adjacent. Since v_3 and v_4 are not adjacent, then v_3 and v_{n+1} are adjacent, and v_4 and v_{n+1} are adjacent. Observe a K_3, with $V(K_3) = \{v_1, v_2, v_{n+1}\}$. Consider $\delta(F - E(K_3)) = \deg_{F-E(K_3)}(v_{n+1}) \leq \delta(G) + t - 2$ and $\Delta(F - E(K_3)) = \deg_{F-E(K_3)}(v_3) = \deg_{F-E(K_3)}(v_4) = n-1$, hence v_3 and v_4 are the dominating vertices of G. Without loss of generality, choose vertex v_3 as the dominating vertex. Since v_3 and v_4 are not adjacent, the vertex v_4 can be ignored. Thus, we obtain

$\delta(F-E(K_3)-v_4) = \deg_{F-E(K_3)-v_4}(v_{n+1}) = \delta(G)+t-3$. Since $|V(F-E(K_3)-v_4)| = n$ and $\delta(F-E(K_3)-v_4) = \delta(G)+t-3 < \delta(G)$ for $t \in [1,2]$, then $F-E(K_3) \not\supseteq G$. By Theorem 1.1 (ii), $F \notin \hat{\mathcal{R}}(2K_2, G)$, a contradiction. Therefore, v_3 and v_4 are adjacent.

(ii) Let $\deg_F(v_{n+1}) = \delta(G)+1$. From conditions (1), v_3 and v_4 are adjacent. Suppose v_3 and v_{n+1} are adjacent, and v_4 and v_{n+1} are adjacent. Observe a K_3, with $V(K_3) = \{v_1, v_2, v_{n+1}\}$. Consider $\delta(F-E(K_3)) = \deg_{F-E(K_3)}(v_{n+1}) = \delta(G)-1$ and $\Delta(F-E(K_3)) = \deg_{F-E(K_3)}$ $(v_3) = \deg_{F-E(K_3)}(v_4) = n-1$,. Since v_3 and v_{n+1} are adjacent, v_4 and v_{n+1} are adjacent, and $\Delta(F-E(K_3)) = \deg_{F-E(K_3)}(v_3) = \deg_{F-E(K_3)}(v_4) = n-1$, then $v_{n+1} \in V(G)$. Since $v_{n+1} \in V(G)$ and $\delta(F-E(K_3)) = \deg_{F-E(K_3)}(v_{n+1}) = \delta(G)-1 < \delta(G)$, then $F-E(K_3) \not\supseteq G$. Theorem 1.1 (ii), $F \notin \hat{\mathcal{R}}(2K_2, G)$, a contradiction. Therefore, v_3 and v_{n+1} are not adjacent or v_4 and v_{n+1} are not adjacent.

(iii) Let $\deg_F(v_{n+1}) = \delta(G)+2$, v_3 and v_{n+1} are adjacent, and v_4 and v_{n+1} are adjacent. From conditions (1), v_3 and v_4 are neighbors. Let v_{n+1} and v_n are adjacent. Suppose v_3 and v_n are not adjacent and v_4 and v_n are not adjacent. Observe a K_3, with $V(K_3) = \{v_1, v_2, v_{n+1}\}$. Consider $\deg_{F-E(K_3)}(v_{n+1}) = \delta(G)$ and $\Delta(F-E(K_3)) = \deg_{F-E(K_3)}(v_3) = \deg_{F-E(K_3)}(v_4) = n-1$. Since v_3 and v_n are not adjacent and v_4 and v_n are not adjacent, then $v_n \notin V(G)$, so the vertex v_n can be ignored. Since $|V(F-E(K_3)-v_n)| = n$ and $\deg_{F-E(K_3)-v_n}(v_3) = \delta(G)-1 < \delta(G)$, then $G \not\subseteq F-E(K_3)$. Theorem 1.1 (ii), $F \notin \hat{\mathcal{R}}(2K_2, G)$, a contradiction. Therefore, v_3 and v_n are adjacent or v_4 and v_n are adjacent.

□

2.2. Complete List of Ramsey $(2K_2, W_n)$-Minimal Graphs of smallest Order

In this subsection, we give the complete list of $\hat{\mathcal{R}}(2K_2, W_n)$ for $n \in [5,8]$ in Theorems 2.3-2.6. Before we discuss these theorems, we introduce some notations. Let $F \in \hat{\mathcal{R}}(2K_2, W_n)$ where $v_i \in V(F)$ for $i \in [1, n+1]$ and let $d, j_1, j_2, j_3, k_1, k_2, ..., k_{n-1} \in [1, n+1]$. The notations $[\{j_1\}, (d; k_1, k_2, k_3..., k_{n-1})]$ and $[(j_1, j_2, j_3), (d; k_1, k_2, ..., k_{n-1})]$, respectively, denote the removal of the vertex v_{j_1} and the edges of the triangle K_3 with $V(K_3) = \{v_{j_1}, v_{j_2}, v_{j_3}\}$ in F such that $F - v_{j_1}, F-E(K_3) \supseteq W_n$ where $W_n = v_d + E(C_{n-1})$ with $E(C_{n-1}) = \{v_{k_1}v_{k_2}, v_{k_2}v_{k_3}, ..., v_{k_{n-2}}v_{k_{n-1}}\}$. The notations $[e = (k,l), \{j_1\}]$ and $[e = (k,l), (j_1, j_2, j_3)]$, respectively, denote the removal of the vertex v_{j_1} and edges of the triangle K_3 with $V(K_3) = \{v_{j_1}, v_{j_2}, v_{j_3}\}$ in $F - v_kv_l$ such that $F - v_kv_l - v_{j_1}, F - v_kv_l - E(K_3) \not\supseteq W_n$. All the notations we have defined above are called red-blue coloring codes of F.

In the following Theorem 2.3, we obtain the characterization of the member $\hat{\mathcal{R}}(2K_2, W_5)$.

Theorem 2.3. *The graph $K_6 - e_1$ is the unique graph in $\hat{\mathcal{R}}(2K_2, W_5)$, for any $e_1 \in E(K_6)$.*

Proof. Let $v_i \in V(K_6)$, for $i \in [1,6]$. Observe $K_6 - e_1$, without loss of generality, choose $e_1 = v_1v_2$. Consider the red-blue coloring codes on $K_6 - e_1$ and $K_6 - e_1 - e$ as shown in Table 1.

Table 1. Red-Blue Coloring Codes on Graphs $K_6 - e$ and $K_6 - E_2$

$K_6 - e_1 \rightarrow (2K_2, W_5)$		$K_6 - e_1 - e \not\rightarrow (2K_2, W_5)$
$[\{1\}, (5; 2, 3, 4, 6)]$	$[(1,5,6), (2; 3, 5, 4, 6)]$	$[e = (16), (3, 4, 5)]$
$[\{5\}, (6; 1, 4, 2, 3)]$	$[(4,5,6), (3; 1, 4, 2, 6)]$	$[e = (34), (4, 5, 6)]$

We obtain $K_6 - e_1 \in \hat{\mathcal{R}}(2K_2, W_5)$. Since $K_6 \supset K_6 - e_1$ and $K_6 - e_1 \supset K_6 - e_1 - e$, by Observation 2.2 (i), $K_6, K_6 - e_1 - E_i \notin \hat{\mathcal{R}}(2K_2, W_5)$ for $i \geq 1$. Therefore, $K_6 - e_1$ is the unique graph in $\hat{\mathcal{R}}(2K_2, W_5)$. □

As illustrative examples, the notations $[\{1\}, (5; 2, 3, 4, 6)]$ and $[(1, 5, 6), (2; 3, 5, 4, 6)]$ in Table 1 indicate, respectively, that assigning red to all edges incident to vertex v_1 yields a blue subgraph $W_5 = v_5 + C_{4,1}$, where $E(C_{4,1}) = \{v_2v_3, v_3v_4, v_5v_6, v_6v_2\}$, and that coloring the triangle formed by vertices v_1, v_5, and v_6 red produces a blue subgraph $W_5 = v_2 + C_{4,2}$, where $E(C_{4,2}) = \{v_3v_5, v_5v_4, v_4v_6, v_6v_3\}$ in the graph $K_6 - e$, as illustrated in Figure 2 (i) and (ii). In contrast, the notation $[e = (16), (3, 4, 5)]$ in Table 1 represents the removal of edge v_1v_6 and the red coloring of the triangle on vertices v_3, v_4, and v_5, which does not yield a blue subgraph isomorphic to W_5 in $K_6 - e$, as shown in Figure 2 (iii).

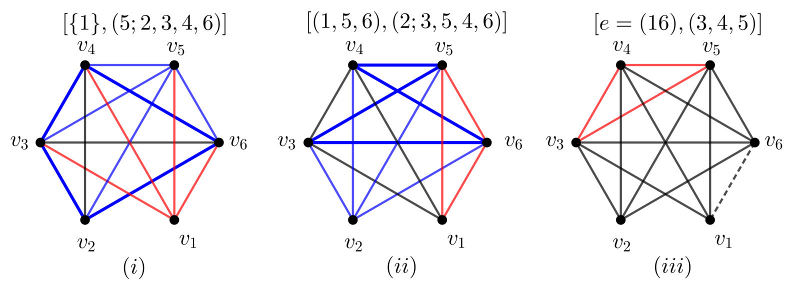

Figure 2. Examples of red edge colorings in $K_6 - e$ and $K_6 - E_2$ yielding (or failing to yield) a blue subgraph isomorphic to W_5.

In the following Theorem 2.4, we obtain the characterization of the member $\hat{\mathcal{R}}(2K_2, W_6)$ of smallest order.

Theorem 2.4. *The graph $K_7 - E(P_3)$ is the unique graph in $\hat{\mathcal{R}}(2K_2, W_6)$.*

Proof. Let $v_i \in V(K_7)$, for $i \in [1, 7]$. Observe $K_7 - E(P_3)$, where $E(P_3) = \{v_5v_6, v_6v_7\}$. The red-blue coloring codes of $K_7 - E(P_3)$ and $K_7 - E(P_3) - e$ is shown in Table 2, we obtain $K_7 - E(P_3) \in \hat{\mathcal{R}}(2K_2, W_6)$.

Table 2. Red-Blue Coloring Codes on Graphs $K_7 - E(P_3)$ and $K_7 - E(P_3) - e$

$K_7 - E(P_3) \rightarrow (2K_2, W_6)$		$K_7 - E(P_3) - e \not\rightarrow (2K_2, W_6)$
$[\{1\}, (2; 3, 6, 4, 5, 7)]$	$[(126), (3; 1, 5, 4, 2, 7)]$	$[e = (12), (3, 4, 5)]$
$[\{5\}, (1; 2, 4, 6, 3, 7)]$	$[(245), (1; 2, 6, 4, 3, 7)]$	$[e = (25), (1, 3, 4)]$
$[(123), (4; 2, 6, 3, 5, 7)]$	$[(257), (1; 2, 3, 7, 4, 6)]$	$[e = (26), \{1\}]$
$[(125), (3; 1, 6, 5, 4, 7)]$	$[(457), (1; 2, 4, 6, 3, 7)]$	$[e = (57), \{1\}]$

Next, we will show that $K_7 - E(P_3)$ is the unique graph in $\hat{\mathcal{R}}(2K_2, W_6)$.

(1) Since $K_7 - e \supset K_7 - E(P_3)$, by Observation 2.2 (i), $K_7, K_7 - e \notin \hat{\mathcal{R}}(2K_2, W_6)$. Therefore, no graph obtained by removing at most any single edge from K_7 belongs to $\hat{\mathcal{R}}(2K_2, W_6)$.

(2) The graph $K_7 - E(2K_2)$ is the only graph obtained by removing any two edges from K_7 which is not isomorphic to $K_7 - E(P_3)$. Without loss of generality, choose $E(2K_2) = \{v_3v_4, v_5v_6\}$. Observe a K_3, with $V(K_3) = \{v_1, v_2, v_3\}$. Since $K_7 - E(2K_2) - E(K_3) - v_3 \not\supseteq C_5$, by Observation 2.1 (ii), $K_7 - E(2K_2) \notin \hat{\mathcal{R}}(2K_2, W_6)$. Therefore, removing any two edges from K_7 only produces the graph $K_7 - E(P_3)$ as a member of $\hat{\mathcal{R}}(2K_2, W_6)$.

(3) Let ${\mathcal{G}_{\mathcal{R}}}^3 = \{K_7 - E_3 \mid K_7 - E_3 \in \hat{\mathcal{R}}(2K_2, W_6)\}$. By Theorem 2.2 for every $G_R^3 \in {\mathcal{G}_{\mathcal{R}}}^3$, $G_R^3 \supset B_{6,1,j}$ for some $i \in [1, 3]$. All possible graphs $K_7 - E_3 \supset B_{6,1,j}$ are represented by the graph $K_7 - E(G)$, where $G = \{K_3, K_{1,3}, K_2 \cup P_3, P_4\}$. Since $K_7 - E(P_3) \supset K_7 - E(G)$, by Observation 2.2 (i), $K_7 - E(G) \notin \hat{\mathcal{R}}(2K_2, W_6)$. Thus, ${\mathcal{G}_{\mathcal{R}}}^3 = \emptyset$. Furthermore, since $K_7 - E(P_3) \supset K_7 - E(G) - e$, by Observation 2.2 (i), no graph obtained by removing at least three edges from K_7 belongs to $\hat{\mathcal{R}}(2K_2, W_6)$.

Based on Points $(1) - (3)$, it is proved that $K_7 - E(P_3)$ is the unique graph in $\hat{\mathcal{R}}(2K_2, W_6)$. □

Before determining the characterization for wheels of larger order, we will discuss the necessary conditions for a graph to be a member of $\hat{\mathcal{R}}(2K_2, W_n)$ with maximum degree $n + 1$. Proposition 2.3, 2.4, and the following lemma are used to directly eliminate graphs that do not belong to $\hat{\mathcal{R}}(2K_2, W_n), n \geq 7$.

Lemma 2.2. *Let $F \in \hat{\mathcal{R}}(2K_2, W_n)$, with $n \geq 7$. If $\Delta(F) = n$, then all the following conditions are necessary for the graph F.*

(i) *Any two vertices of degree four have at most three common neighbors.*

(ii) *F does not contain $\bar{K}_{\lceil \frac{n+1}{2} \rceil}$.*

Proof. By Lemma 2.1, let $\deg_F(v_{k_1}) = n$ and let $\deg_F(v_{k_2}) \geq n - 1$ for $k_1 \in [1, 2]$ and $k_2 \in [3, 4]$.

(i) Let $\deg_F(v_n) = \deg_F(v_{n+1}) = \delta(F) = 4$ and v_n and v_{n+1} are not adjacent. Suppose $N(F, v_n) = N(F, v_{n+1})$. Since $\deg_F(v_3) = \deg_F(v_4) \geq n-1$, then without loss of generality $v_3 \in N(F, v_n)$ and $v_4 \in N(F, v_{n+1})$. Since $N(v_n) = N(v_{n+1})$, then $v_3 \in N(v_{n+1})$ and $v_4 \in N(v_n)$. Thus, we obtain $N(F, v_n) = N(F, v_{n+1}) = \{v_i | i \in [1, 4]\}$. Consider the graph $F - v_1$. Since $\Delta(F - v_1) = \deg_{F-v_1}(v_2) = n - 1$, then v_2 is the dominating vertex in W_n. Since $|V(F - v_1 - v_2)| = n - 1$ and $N(F - v_1 - v_2, v_n) = N(F - v_1 - v_2, v_{n+1}) = \{v_3, v_4\}$, then there is a cycle C_4 with $V(C_4) = \{v_j | j \in [3, 6]\}$ in $F - v_1 - v_2$. Since $C_{n-1} \not\supseteq C_4$, with $n \geq 7$, then $F - v_1 - v_2 \not\supseteq C_{n-1}$. By Observation 2.1 (i), $F \notin \hat{\mathcal{R}}(2K_2, W_n)$, a contradiction. Therefore, $N(v_n) \neq N(v_{n+1})$.

(ii) Suppose $\bar{K}_{\lceil \frac{n+1}{2} \rceil} \supseteq F$. Consider $\Delta(F - v_1) = \deg F - v_1, v_2) = n - 1$, thus v_2 is the dominating vertex of the wheel W_n. Since $\bar{K}_{\lceil \frac{n+1}{2} \rceil} \supseteq F$, there exists a partition with $\lceil \frac{n+1}{2} \rceil$ vertices in the graph F. Since $|V(F - v_1 - v_2)| = n - 1$ and there is a partition with $\lceil \frac{n+1}{2} \rceil$ vertices in $F - v_1 - v_2$, then $\lceil \frac{n+1}{2} \rceil > n - \lceil \frac{n+1}{2} \rceil - 2$. Consequently, $F - v_1 - v_2 \not\supseteq C_{n-1}$. By Observation 2.1 (i), $F \notin \hat{\mathcal{R}}(2K_2, W_n)$, a contradiction. Therefore, $K_{\lceil \frac{n+1}{2} \rceil} \nsubseteq \bar{F}$.

□

Lemma 2.3. *Let $F \in \hat{\mathcal{R}}(2K_2, W_n)$ for $n \geq 7$. Then $|E(F)| \geq 4n - 7$.*

Proof. By Theorem 2.2, there exists $B_{n,1,i}$ for $i \in [1,3]$ such that $F \supseteq B_{n,1,j}$. Consider $|E(B_{n,1,1})| = 4n-8 > |E(B_{n,1,j})| = 4n-7$ for $j \in [2,3]$. By Proposition 2.3 (i), $B_{n,1,2}, B_{n,1,3} \notin \hat{\mathcal{R}}(2K_2, W_n)$. Since $F \supseteq B_{n,1,i}$ for some $i \in [1,3]$ and $B_{n,1,2}, B_{n,1,3} \notin \hat{\mathcal{R}}(2K_2, W_n)$, then $E(F) \geq |E(B_{n,1,2})| + 1 = |E(B_{n,1,3})| + 1 = 4n-7$. □

The graphs H_i for $i \in [1,4]$ are shown in Figure 3.

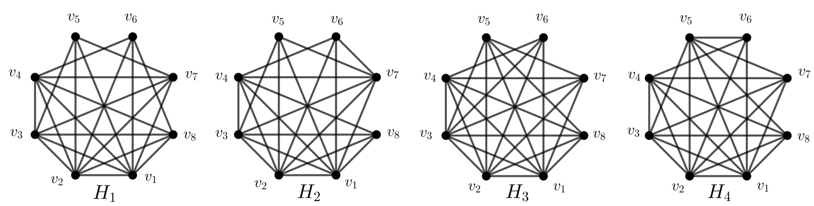

Figure 3. H_i for $i \in [1,4]$

In the following Theorem 2.5, we obtain the characterization of the member $\hat{\mathcal{R}}(2K_2, W_7)$.

Theorem 2.5. *The graphs in $\hat{\mathcal{R}}(2K_2, W_7)$ are precisely $K_{5\times 2}$ and H_i for $i \in [1,4]$.*

Proof. By Theorem 2.1, $K_{4\times 2} \in \hat{\mathcal{R}}(2K_2, W_7)$. The red-blue colorings codes of H_i and $H_i - e$ for $i \in [1,4]$, respectively, as shown in Table 3 and 4. Thus, $H_i \in \hat{\mathcal{R}}(2K_2, W_7)$.

Table 3. Red-Blue Coloring Codes of Graphs H_i for $i \in [1,4]$

i	$H_i \to (2K_2, W_7)$			
1	$[\{2\},(1;3,5,7,4,6,8)]$	$[\{8\},(1;2,6,4,3,5,7)]$	$[(1,2,8),(4;1,3,8,6,2,7)]$	$[(2,5,7),(1;2,6,4,7,3,8)]$
	$[\{4\},(1;2,3,7,5,8,6)]$	$[(1,2,3),(4;1,6,2,8,3,7)]$	$[(2,3,5),(1;2,6,4,7,3,8)]$	$[(3,4,8),(1;2,3,7,4,6,8)]$
	$[\{6\},(3;1,7,5,2,4,8)]$	$[(1,2,6),(3;1,4,8,2,5,7)]$	$[(2,3,8),(1;2,6,4,7,3,5)]$	
2	$[\{1\},(2;3,5,7,6,4,8)]$	$[(1,2,3),(4;1,6,2,7,3,8)]$	$[(2,3,5),(4;1,3,7,6,2,8)]$	
	$[\{3\},(1;2,5,7,6,4,8)]$	$[(1,2,5),(1;2,6,4,7,3,5)]$	$[(3,4,7),(1;4,8,3,6,7,5)]$	
	$[\{5\},(1;2,3,7,6,4,8)]$	$[(2,3,4),(1;2,6,4,7,3,8)]$	$[(3,4,8),(1;2,3,5,7,4,6)]$	
3	$[\{1\},(2;3,6,4,7,5,8)]$	$[\{7\},(1;2,6,5,3,4,8)]$	$[(2,3,6),(1;2,4,3,7,5,8)]$	$[(3,5,7),(1;2,5,8,3,4,7)]$
	$[\{4\},(1;2,6,3,8,5,7)]$	$[(1,2,3),(4;1,7,2,6,3,8)]$	$[(2,3,7),(1;2,5,7,4,2,6)]$	
	$[\{5\},(1;2,6,3,8,4,7)]$	$[(2,3,4),(1;3,6,4,8,5,7)]$	$[(3,4,6),(1;2,3,8,5,7,4)]$	
	$[\{6\},(1;2,5,3,7,4,8)]$	$[(2,3,5),(1;3,6,4,8,5,7)]$	$[(3,4,7),(1;2,4,8,3,5,7)]$	
4	$[\{1\},(2;3,7,5,6,4,8)]$	$[(1,2,3),(4;1,6,2,8,3,7)]$	$[(2,3,5),(1;2,6,4,7,3,8)]$	$[(2,5,7),(1;2,6,5,3,4,8)]$
	$[\{3\},(1;2,6,5,7,4,8)]$	$[(1,2,4),(3;1,5,2,7,4,8)]$	$[(2,3,7),(1;2,4,6,5,4,8)]$	$[(3,4,7),(1;2,4,6,5,3,8)]$
	$[\{5\},(1;2,6,4,8,3,7)]$	$[(1,2,6),(3;1,5,2,8,4,7)]$	$[(2,4,6),(1;2,7,4,3,5,8)]$	
	$[\{6\},(1;2,5,3,7,4,8)]$	$[(1,2,7),(3;1,4,7,5,2,8)]$	$[(2,4,7),(1;2,6,5,7,3,8)]$	
	$[\{7\},(1;2,6,5,3,4,8)]$	$[(2,3,4),(1;2,7,5,6,4,8)]$	$[(2,5,6),(1;2,7,5,3,4,8)]$	

Table 4. Red-Blue Coloring Codes of Graphs $H_i - e$ for $i \in [1,4]$

i	$H_i - e \not\to (2K_2, W_7)$				
1	$[e=(12),\{7\}]$	$[e=(25),\{1\}]$	$[e=(34),(1,2,3)]$	$[e=(37),(1,2,4)]$	
	$[e=(23),\{1\}]$	$[e=(27),\{1\}]$	$[e=(35),(1,2,4)]$	$[e=(57),(1,2,7)]$	
2	$[e=(12),\{3\}]$	$[e=(23),\{1\})]$	$[e=(25),\{1\}]$	$[e=(34),(1,2,8)]$	$[e=(3,5),\{1\}]$
3	$[e=(12),\{8\}]$	$[e=(25),\{1\}]$	$[e=(27),\{1\}]$	$[e=(47),(1,2,3)]$	
	$[e=(24),\{1\}]$	$[e=(26),\{1\}]$	$[e=(46),(1,2,3)]$	$[e=(57),\{1\}]$	
4	$[e=(12),\{8\}]$	$[e=(24),\{1\}]$	$[e=(27),\{1\}]$	$[e=(35),(1,2,4)]$	$[e=(46),\{1\}]$
	$[e=(23),\{1\}]$	$[e=(26),\{1\}]$	$[e=(34),(1,2,5)]$	$[e=(37),(1,2,7)]$	$[e=(47),(1,2,7)]$

By Proposition 2.2, $K_{4\times 2}$ is only the graph in $\hat{\mathcal{R}}(2K_2, W_7)$ with maximum degree 6. Thus, we only need to prove that H_i for $i \in [1,4]$ are the only graphs with eight vertices in $\hat{\mathcal{R}}(2K_2, W_7)$ with maximum degree 7. Let $v_i \in V(K_8)$ for $i \in [1,8]$. Define the graphs $H_{i,j} = K_8 - E(H_{8,i,j})$, where the data from the set of edges $H_{8,i,j}$ is shown in Table 5 for $i \in [3,7]$ and some j.

Table 5. The Set of Edges $E_{8,i,j}$ for $i \in [3,7]$ and some j

$E(H_{8,7,1}) = \{45, 56, 57, 58, 67, 68, 78\}$	$E(H_{8,6,9}) = \{56, 57, 58, 67, 68, 78\}$	$E(H_{8,4,1}) = \{57, 58, 67, 68\}$
$E(H_{8,7,2}) = \{35, 45, 57, 58, 67, 68, 78\}$	$E(H_{8,6,10}) = \{36, 45, 57, 67, 68, 89\}$	$E(H_{8,4,2}) = \{46, 56, 67, 78\}$
$E(H_{8,7,3}) = \{35, 45, 56, 57, 58, 67, 68\}$	$E(H_{8,6,11}) = \{35, 45, 58, 67, 68, 89\}$	$E(H_{8,4,3}) = \{35, 45, 67, 68\}$
$E(H_{8,7,4}) = \{36, 45, 56, 57, 58, 67, 78\}$	$E(H_{8,5,1}) = \{45, 57, 58, 67, 68\}$	$E(H_{8,4,4}) = \{45, 58, 67, 68\}$
$E(H_{8,7,5}) = \{36, 45, 56, 57, 58, 67, 68\}$	$E(H_{8,5,2}) = \{36, 45, 56, 67, 78\}$	$E(H_{8,4,5}) = \{56, 67, 68, 78\}$
$E(H_{8,7,6}) = \{36, 45, 57, 58, 67, 68, 78\}$	$E(H_{8,5,3}) = \{35, 45, 57, 67, 78\}$	$E(H_{8,4,6}) = \{34, 57, 58, 67\}$
$E(H_{8,7,7}) = \{34, 56, 57, 58, 67, 68, 78\}$	$E(H_{8,5,4}) = \{36, 45, 57, 68, 78\}$	$E(H_{8,4,7}) = \{35, 45, 59, 67\}$
$E(H_{8,6,1}) = \{45, 56, 57, 58, 67, 68\}$	$E(H_{8,5,5}) = \{45, 57, 58, 67, 78\}$	$E(H_{8,4,8}) = \{45, 67, 68, 78\}$
$E(H_{8,6,2}) = \{35, 45, 57, 58, 67, 68\}$	$E(H_{8,5,6}) = \{46, 56, 67, 68, 78\}$	$E(H_{8,4,9}) = \{35, 45, 56, 57\}$
$E(H_{8,6,3}) = \{36, 45, 56, 67, 68, 78\}$	$E(H_{8,5,7}) = \{35, 45, 57, 58, 67\}$	$E(H_{8,3,1}) = \{36, 46, 56\}$
$E(H_{8,6,4}) = \{35, 45, 56, 57, 58, 67\}$	$E(H_{8,5,8}) = \{35, 45, 56, 57, 58\}$	$E(H_{8,3,2}) = \{34, 45, 56\}$
$E(H_{8,6,5}) = \{35, 45, 57, 58, 67, 78\}$	$E(H_{8,5,9}) = \{57, 58, 67, 68, 78\}$	$E(H_{8,3,3}) = \{45, 56, 78\}$
$E(H_{8,6,6}) = \{36, 45, 57, 58, 67, 68\}$	$E(H_{8,5,10}) = \{34, 57, 58, 67, 68\}$	$E(H_{8,3,4}) = \{34, 56, 78\}$
$E(H_{8,6,7}) = \{45, 57, 58, 67, 68, 78\}$	$E(H_{8,5,11}) = \{35, 45, 58, 67, 68\}$	$E(H_{8,3,5}) = \{45, 46, 56\}$
$E(H_{8,6,8}) = \{34, 57, 58, 67, 68, 78\}$	$E(H_{8,5,12}) = \{35, 45, 67, 68, 78\}$	

Let $\mathcal{H_R}^y = \{K_8 - E_y \mid K_8 - E_y \in \hat{\mathcal{R}}(2K_2, W_7)\}$. By Lemma 2.3, since $|E(K_8 - E_y)| \geq 21$, then $y \leq 7$. In the following cases, we will prove that H_i for $i \in [1,4]$ are the only graphs with eight vertices in $\hat{\mathcal{R}}(2K_2, W_7)$ with maximum degree 7.

Case 1. $K_8 - E_7$.

By Theorem 2.2 for every $H_R^7 \in \mathcal{H_R}^7$, $H_R^7 \supset B_{7,1,i}$ for some $i \in [1,3]$. All possible graphs $K_8 - E_7 \supset B_{7,1,i}$ are represented by the graphs $H_{7,j}$ for $j \in [1,7]$. According to Proposition 2.3 (i), (ii), and Lemma 2.2 (ii), respectively, $H_{7,j}$, $H_{7,6}$, and $H_{7,7}$ do not belong to $R(2K_2, W_7)$, for $j \in [1,5]$. Thus, $\mathcal{H_R}^7 = \emptyset$. Therefore, there are no graphs obtained by removing any seven edges from K_8 that belonging to $\hat{\mathcal{R}}(2K_2, W_7)$.

Case 2. $K_8 - E_6$.

All possible graphs $H_{7,t}+e$ for $t \in [1,7]$ that are not isomorphic to H_1 and H_2 are represented by the graphs $H_{6,j}$ for $j \in [1,11]$. According to Proposition 2.3 (i), (ii), Lemma 2.2 (i), and Proposition 2.4 (ii), respectively $H_{6,j_1}, H_{6,6}, H_{6,j_2}$, and H_{6,j_3} do not belong to $R(2K_2, W_7)$, for $j_1 \in [1,5]$, $j_2 \in [7,9]$, and $j_3 \in [10,11]$. Thus, $\mathcal{H_R}^6 = \{H_1, H_2\}$. Therefore, removing any six edges from the graph K_8 only produces the graphs H_1 and H_2 as a member of $\hat{\mathcal{R}}(2K_2, W_7)$.

Case 3. $K_8 - E_5$.

All possible graphs $H_{6,t} + e$ for $t \in [1,11]$ that are not isomorphic to H_3 and H_4 are represented by the graphs $H_{5,j}$ for $j \in [1,12]$. Consider $H_{5,j} \supset H_1$ and $H_{5,5} \supset H_2$ for $j \in [1,4]$. By Observation 2.2 (i), $H_{5,t_1} \notin \hat{\mathcal{R}}(2K_2, W_7)$ for $t_1 \in [1,5]$. According to Proposition 2.3 (i), Lemma 2.2 (i), Proposition 2.4 (i), and (iii), respectively $H_{5,k_1}, H_{5,9}, H_{5,10}$, and H_{5,k_2} do not belong to $R(2K_2, W_7)$, for $k_1 \in [6,8]$ and $k_2 \in [11,12]$. Thus, $\mathcal{H_R}^5 = \{H_3, H_4\}$. Therefore, removing any five edges from the graph K_8 only produces the graphs H_3 and H_4 as a member of $\hat{\mathcal{R}}(2K_2, W_7)$.

Case 4. $K_8 - E_4$.

All possible graphs $H_{5,t} + e$ for $t \in [6,12]$ are represented by the graphs $H_{4,j}$ for $j \in [1,9]$.

Consider $H_{4,j_1} \supset H_1$, $H_{4,4} \supset H_2$, $H_{4,5} \supset H_3$, and $H_{4,j_2} \supset H_4$ for $j_1 \in [1,3]$ and $j_2 \in [6,8]$. By Observation 2.2 (i), $H_{4,t_1} \notin \hat{\mathcal{R}}(2K_2, W_7)$ for $t_1 \in [1,8]$. By Proposition 2.3 (i), $H_{4,9} \notin \hat{\mathcal{R}}(2K_2, W_7)$. Thus, ${\mathcal{H}_{\mathcal{R}}}^4 = \emptyset$. Therefore, there are no graphs obtained by removing any four edges from K_8 that belonging to $\hat{\mathcal{R}}(2K_2, W_7)$.

Case 5. $K_8 - E_3$.
All possible graphs $H_{4,9} + e$ are represented by the graphs $H_{3,i}$ for $i \in [1,5]$. Since $H_{3,i} \supset H_3$, then $H_{3,i} \notin \hat{\mathcal{R}}(2K_2, W_7)$. Thus, ${\mathcal{H}_{\mathcal{R}}}^3 = \emptyset$. Therefore, there are no graphs obtained by removing any three edges from K_8 that belonging to $\hat{\mathcal{R}}(2K_2, W_7)$.

Case 6. $K_8 - E_t$ for $t \leq 2$.
Since $H_{3,i} + e \supset H_3$, by Observation 2.2 (i), ${\mathcal{H}_{\mathcal{R}}}^t = \emptyset$ for $t \leq 2$. Therefore, there are no graphs obtained from removing at most two edges from K_8 as a member of $\hat{\mathcal{R}}(2K_2, W_7)$.

Based on cases $1-6$ above, the graphs with maximum degree 7 in $\hat{\mathcal{R}}(2K_2, W_7)$ are only H_i for $i \in [1,4]$. Thus, we have proven that the graphs in $\hat{\mathcal{R}}(2K_2, W_7)$ are precisely $K_{5\times 2}$ and H_i for $i \in [1,4]$. $\square$

The graphs F_i for $i \in [1,10]$ are shown in Figure 4.

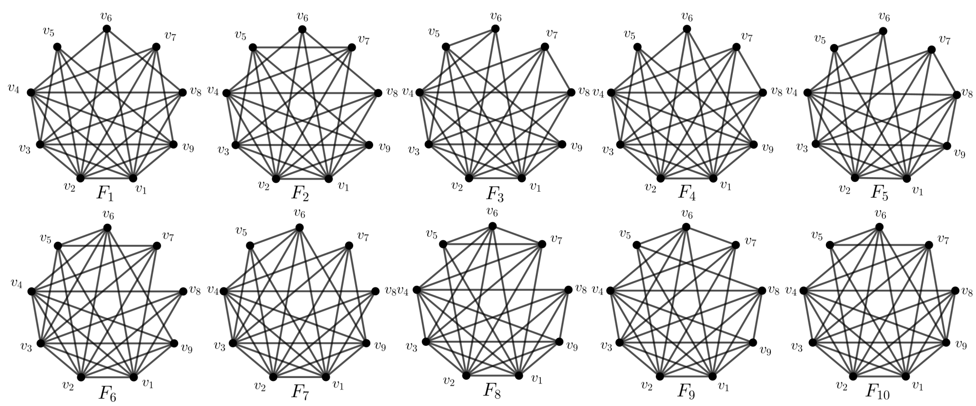

Figure 4. F_i for $i \in [1,10]$

In the following Theorem 2.6, we obtain the characterization of the member $\hat{\mathcal{R}}(2K_2, W_8)$.

Theorem 2.6. *The graphs with nine vertices in $\hat{\mathcal{R}}(2K_2, W_8)$ are precisely F_i for $i \in [1,10]$.*

Proof. The red-blue coloring codes of F_i and $F_i - e$ for $i \in [1,10]$, respectively, as shown in Table 6 and 7. Thus, $F_i \in \hat{\mathcal{R}}(2K_2, W_8)$.

Table 6. Red-Blue Coloring Codes of Graphs F_i for $i \in [1, 10]$

i	$F_i \to (2K_2, W_8)$			
	$[\{1\}, (2; 3, 6, 5, 4, 9, 7, 8)]$	$[(1, 2, 7), (3; 1, 6, 2, 5, 9, 7, 8)]$	$[(1, 4, 9), (3; 1, 6, 5, 9, 2, 8, 7)]$	$[(3, 6, 9), (1; 2, 4, 9, 5, 3, 7, 8)]$
	$[\{3\}, (2; 1, 6, 5, 4, 9, 7, 8)]$	$[(1, 2, 6), (3; 1, 7, 9, 6, 5, 2, 8)]$	$[(1, 5, 9), (2; 1, 3, 5, 4, 9, 7, 8)]$	$[(3, 7, 9), (1; 1, 5, 9, 6, 3, 8, 7)]$
1	$[\{4\}, (1; 2, 3, 5, 6, 9, 7, 8)]$	$[(1, 2, 8), (9; 1, 6, 2, 4, 5, 3, 7)]$	$[(1, 5, 6), (2; 1, 4, 5, 3, 6, 9, 7)]$	$[(4, 5, 9), (2; 1, 5, 3, 8, 7, 9, 6)]$
	$[\{5\}, (2; 1, 4, 9, 7, 8, 3, 6)]$	$[(1, 2, 9), (3; 1, 5, 2, 6, 9, 7, 8)]$	$[(1, 6, 9), (2; 3, 6, 5, 4, 9, 7, 8)]$	$[(5, 6, 9), (2; 1, 3, 5, 4, 9, 7, 8)]$
	$[\{6\}, (2; 1, 7, 8, 3, 5, 4, 9)]$	$[(1, 3, 9), (2; 1, 5, 3, 6, 9, 7, 8)]$	$[(3, 4, 9), (1; 2, 5, 4, 9, 6, 3, 7)]$	
	$[\{1\}, (2; 3, 8, 6, 4, 7, 5, 9)]$	$[(1, 2, 3), (4; 1, 6, 8, 3, 7, 2, 9)]$	$[(2, 3, 4), (1; 2, 5, 7, 4, 8, 3, 9)]$	$[(2.6.8), (1; 2, 5, 7, 4, 8, 3, 9)]$
	$[\{3\}, (2; 1, 6, 8, 4, 7, 5, 9)]$	$[(1, 2, 4), (3; 1, 7, 5, 2, 8, 4, 9)]$	$[(2, 3, 7), (1; 2, 8, 4, 7, 5, 3, 9)]$	$[(3, 4, 7), (1; 2, 6, 4, 8, 3, 5, 9)]$
	$[\{4\}, (2; 1, 6, 8, 3, 7, 5, 9)]$	$[(1, 2, 5), (4; 1, 7, 3, 9, 2, 6, 8)]$	$[(2, 4, 6), (1; 2, 8, 4, 7, 5, 3, 9)]$	$[(3, 4, 8), (1; 2, 6, 4, 7, 5, 3, 9)]$
2	$[\{5\}, (4; 1, 7, 2, 6, 8, 3, 9)]$	$[(1, 2, 6), (3; 1, 5, 7, 2, 9, 4, 8)]$	$[(2, 4, 7), (1; 2, 3, 8, 6, 4, 9, 5)]$	$[(3, 5, 7), (1; 2, 5, 9, 3, 8, 4, 6)]$
	$[\{6\}, (3; 1, 8, 2, 4, 7, 5, 9)]$	$[(1, 2, 7), (3; 1, 8, 4, 7, 5, 2, 9)]$	$[(2, 4, 8), (1; 2, 7, 4, 6, 8, 3, 9)]$	$[(4, 6, 8), (1; 2, 8, 3, 4, 7, 5, 9)]$
	$[\{7\}, (2; 1, 4, 6, 8, 3, 5, 9)]$	$[(1, 2, 8), (4; 1, 6, 8, 3, 7, 2, 9)]$	$[(2, 5, 7), (1; 2, 6, 4, 8, 3, 5, 9)]$	
	$[\{8\}, (2; 1, 6, 4, 9, 3, 5, 7)]$	$[(2, 3, 5), (4; 1, 7, 3, 8, 6, 2, 9)]$	$[(2, 5, 9), (1; 2, 7, 5, 3, 4, 6, 8)]$	

i	$F_i \to (2K_2, W_8)$			
3	$[\{1\},(2;3,8,7,4,6,5,9)]$	$[(1,2,3),(4;1,6,2,7,6,3,8)]$	$[(2,3,5),(4;1,6,2,7,8,3,9)]$	$[(2,5,6),(1;2,7,3,5,9,4,8)]$
	$[\{3\},(2;1,8,7,4,6,5,9)]$	$[(1,2,4),(3;1,5,2,9,4,7,8)]$	$[(2,3,7),(1;2,6,4,7,8,3,9)]$	$[(2,5,9),(1;2,6,5,3,7,4,8)]$
	$[\{4\},(1;2,6,5,9,3,8,7)]$	$[(1,2,5),(4;1,6,2,8,7,3,9)]$	$[(2,3,8),(1;2,5,3,7,8,4,9)]$	$[(2,7,8),(1;2,3,7,4,9,5,6)]$
	$[\{5\},(1;2,5,4,9,2,8,7)]$	$[(1,2,6),(3;1,5,2,9,4,7,8)]$	$[(2,3,9),(1;2,4,7,8,3,5,6)]$	$[(3,4,7),(1;2,5,6,4,9,3,8)]$
	$[\{6\},(3;1,8,7,4,2,5,9)]$	$[(1,2,7),(4;1,6,2,8,7,3,9)]$	$[(2,4,6),(1;2,8,4,7,3,5,9)]$	$[(3,4,8),(1;2,5,6,4,7,3,9)]$
	$[\{7\},(1;2,5,9,3,8,4,6)]$	$[(1,2,8),(3;1,5,2,7,8,4,9)]$	$[(2,4,7),(1;2,6,4,8,3,5,9)]$	$[(3,4,9),(1;2,8,7,4,7,6,9)]$
	$[\{8\},(1;2,6,5,3,9,4,7)]$	$[(1,2,9),(3;1,5,9,4,2,7,8)]$	$[(2,4,8),(1;2,6,5,9,3,8,7)]$	$[(3,7,8),(1;2,7,4,6,5,3,9)]$
	$[\{9\},(1;2,7,4,6,5,3,8)]$	$[(2,3,4),(1;2,7,3,5,6,4,9)]$	$[(2,4,9),(1;2,6,5,9,3,8,7)]$	$[(4,7,8),(1;2,8,3,5,6,4,9)]$
4	$[\{2\},(1;3,5,9,6,4,7,8)]$	$[(1,2,3),(4;1,6,2,8,7,3,9)]$	$[(2,3,7),(1;2,8,7,4,3,5,9)]$	$[(3,4,7),(1;2,6,9,5,3,8,7)]$
	$[\{4\},(1;2,6,9,5,3,7,8)]$	$[(1,2,5),(4;1,7,8,3,2,6,9)]$	$[(2,3,9),(1;2,6,9,5,3,7,8)]$	$[(3,4,9),(1;2,6,9,5,3,7,8)]$
	$[\{6\},(1;2,8,7,4,3,5,9)]$	$[(1,2,7),(3;1,5,9,2,4,7,8)]$	$[(2,4,9),(1;2,8,7,3,4,6,9)]$	$[(3,7,8),(1;2,6,9,5,3,4,7)]$
	$[\{8\},(1;2,6,4,7,3,5,9)]$	$[(1,2,9),(3;1,5,9,6,4,7,8)]$	$[(2,5,9),(4;1,8,7,3,2,6,9)]$	
	$[\{9\},(1;2,5,3,8,7,5,6)]$	$[(2,3,5),(1;2,7,8,3,4,6,9)]$	$[(2,7,8),(1;2,5,3,7,4,9,6)]$	
5	$[\{1\},(2;3,5,6,4,7,8,9)]$	$[\{8\},(1;2,7,4,6,5,3,9)]$	$[(1,2,8),(3;1,5,2,4,7,8,9)]$	$[(2,3,8),(4;1,6,2,9,3,7,8)]$
	$[\{3\},(2;1,5,6,4,7,8,9)]$	$[(1,2,3),(4;1,6,2,7,3,8,9)]$	$[(2,3,4),(1;2,6,4,7,8,3,9)]$	$[(3,4,7),(1;2,4,6,5,3,8,7)]$
	$[\{6\},(3;1,5,2,4,7,8,9)]$	$[(1,2,5),(4;1,6,2,3,7,8,9)]$	$[(2,3,5),(4;1,6,2,7,3,8,9)]$	$[(3,4,8),(1;1,6,5,3,7,8,9)]$
	$[\{7\},(1;2,6,5,3,4,8,9)]$	$[(1,2,7),(3;1,5,2,4,7,8,9)]$	$[(2,3,7),(4;1,6,2,9,3,8,7)]$	$[(3,7,8),(1;2,3,5,6,4,8,9)]$
6	$[\{1\},(2;3,8,4,7,5,6,9)]$	$[\{9\},(1;2,4,6,5,7,3,8)]$	$[(1,2,8),(3;1,5,6,4,7,2,9)]$	$[(3,4,9),(1;2,3,8,4,7,5,6)]$
	$[\{4\},(1;2,8,3,7,5,6,9)]$	$[(1,2,3),(4;1,8,2,7,3,6,9)]$	$[(1,2,9),(3;1,4,6,5,7,2,8)]$	$[(3,5,6),(1;2,3,9,6,4,7,5)]$
	$[\{5\},(4;1,8,2,7,3,6,9)]$	$[(1,2,4),(3;1,5,6,4,7,2,8)]$	$[(1,5,6),(2;1,3,5,7,4,6,9)]$	$[(3,5,7),(4;1,7,2,8,3,6,9)]$
	$[\{6\},(1;2,5,7,4,8,3,9)]$	$[(1,2,5),(3;1,7,5,6,2,4,8)]$	$[(3,4,6),(1;2,3,8,4,7,5,6)]$	$[(3,6,9),(1;2,6,5,3,7,4,9)]$
	$[\{7\},(1;2,8,4,6,5,3,9)]$	$[(1,2,6),(3;1,5,6,9,2,4,8)]$	$[(3,4,7),(1;2,3,9,4,6,5,7)]$	$[(4,6,9),(1;2,3,8,4,7,5,6)]$
	$[\{8\},(1;2,4,6,5,7,3,9)]$	$[(1,2,7),(3;1,4,7,5,6,2,8)]$	$[(3,4,8),(1;2,3,9,6,5,7,4)]$	
7	$[\{3\},(1;2,5,6,9,7,4,8)]$	$[\{9\},(3;1,5,6,4,7,2,8)]$	$[(2,3,8),(1;2,5,6,4,7,3,9)]$	$[(3,5,6),(1;2,6,9,4,7,3,8)]$
	$[\{4\},(3;1,5,6,9,7,2,8)]$	$[(1,2,3),(4;1,8,3,6,2,7,9)]$	$[(2,3,9),(1;2,5,6,3,4,9,7)]$	$[(3,6,9),(1;2,6,5,3,4,9,7)]$
	$[\{5\},(4;1,7,9,6,3,2,8)]$	$[(2,3,4),(1;2,7,4,6,5,3,8)]$	$[(3,4,6),(1;2,5,3,7,4,9,6)]$	$[(3,7,9),(1;2,7,4,6,5,3,8)]$
	$[\{6\},(3;1,5,2,8,4,9,7)]$	$[(2,3,5),(1;2,8,4,7,3,6,9)]$	$[(3,4,7),(1;2,6,4,8,3,9,7)]$	$[(4,6,9),(1;2,4,7,9,3,5,6)]$
	$[\{7\},(3;1,5,2,8,4,9,6)]$	$[(2,3,6),(1;2,7,4,6,5,3,8)]$	$[(3,4,8),(1;2,5,6,4,7,3,9)]$	$[(4,7,9),(1;2,5,6,9,3,4,8)]$
	$[\{8\},(3;1,7,4,6,5,2,9)]$	$[(2,3,7),(1;2,5,3,6,4,7,9)]$	$[(3,4,9),(1;2,7,4,6,5,3,8)]$	
8	$[\{2\},(1;3,6,5,7,4,8,9)]$	$[(1,2,4),(3;1,8,2,7,6,4,9)]$	$[(2,4,8),(1;2,7,6,4,3,8,9)]$	$[(3,4,6),(1;2,6,7,4,8,3,9)]$
	$[\{4\},(1;2,5,6,7,3,9,8)]$	$[(1,2,5),(3;1,8,4,7,6,2,9)]$	$[(2,5,6),(1;2,7,6,4,8,3,9)]$	$[(3,4,8),(1;2,3,6,7,4,9,8)]$
	$[\{5\},(1;2,3,7,6,4,8,7)]$	$[(1,2,7),(3;1,6,7,4,8,2,9)]$	$[(2,5,7),(1;2,4,6,7,3,9,8)]$	$[(4,6,7),(1;2,5,6,3,4,8,9)]$
	$[\{7\},(1;2,3,9,8,4,7,6)]$	$[(1,2,9),(3;1,7,6,2,4,9,8)]$	$[(2,6,7),(1;2,5,6,4,7,3,8)]$	$[(4,8,9),(1;2,5,6,7,4,3,8)]$
	$[\{9\},(1;2,3,9,8,4,6,5)]$	$[(2,4,6),(1;2,5,6,3,7,4,8)]$	$[(2,8,9),(1;2,3,8,4,7,6,5)]$	$[(5,6,7),(1;2,7,3,6,4,9,8)]$
9	$[\{2\},(1;3,8,5,6,7,4,9)]$	$[(1,2,3),(4;1,6,7,4,8,2,9)]$	$[(2,3,6),(1;2,5,6,7,3,8,9)]$	$[(2,6,7),(1;2,5,6,3,4,8,9)]$
	$[\{4\},(1;2,5,6,7,3,8,9)]$	$[(1,2,5),(4;1,6,7,4,8,2,9)]$	$[(2,3,7),(1;3,6,7,4,8,3,9)]$	$[(3,4,6),(1;2,6,7,4,8,3,9)]$
	$[\{5\},(1;2,1,6,7,3,8,9)]$	$[(1,2,6),(4;1,3,6,7,2,8,9)]$	$[(2,4,6),(1;2,3,8,4,7,6,5)]$	$[(3,4,7),(1;2,3,9,8,5,6,7)]$
	$[\{6\},(1;2,5,8,9,3,4,7)]$	$[(1,2,7),(4;1,3,7,6,2,8,9)]$	$[(2,4,7),(1;2,8,3,7,6,4,9)]$	$[(4,6,7),(1;2,7,3,6,5,8,9)]$
	$[\{7\},(1;2,5,6,3,4,8,9)]$	$[(2,3,4),(1;2,5,6,7,3,8,9)]$	$[(2,5,6),(1;2,8,3,7,6,4,9)]$	
10	$[\{1\},(2;3,4,7,5,6,9,8)]$	$[\{8\},(1;2,3,4,7,9,6,5)]$	$[(1,2,8),(9;1,3,8,4,6,2,7)]$	$[(2,5,6),(1;2,4,8,9,6,3,7)]$
	$[\{3\},(1;2,4,6,5,7,9,8)]$	$[(1,2,3),(9;1,6,2,7,3,4,8)]$	$[(2,3,4),(1;2,7,5,6,3,9,8)]$	$[(2,4,8),(1;2,6,5,7,3,8,9)]$
	$[\{5\},(3;1,7,2,6,4,8,9)]$	$[(1,2,5),(9;1,7,4,3,6,2,8)]$	$[(2,3,6),(1;2,8,4,7,5,6,9)]$	$[(3,4,6),(1;3,8,4,7,5,6,9)]$
	$[\{6\},(1;2,3,9,8,4,7,5)]$	$[(1,2,6),(9;1,3,6,4,7,2,8)]$	$[(2,3,8),(1;3,6,5,7,4,8,9)]$	$[(3,4,8),(1;2,6,5,7,4,9,8)]$

Table 7. Red-Blue Coloring Codes of Graphs $F_i - e$ for $i \in [1, 10]$

i	$F_i - e \not\to (2K_2, W_8)$					
1	$[e=(12),\{6\}]$	$[e=(15),\{2\}]$	$[e=(39),(1,2,6)]$	$[e=(56),\{3\}]$	$[e=(69),(1,2,3)]$	$[e=(78),(1,2,7)]$
	$[e=(13),\{9\}]$	$[e=(16),\{2\}]$	$[e=(45),\{2\}]$	$[e=(59),(1,2,3)]$		
	$[e=(14),\{2\}]$	$[e=(19),(1,2,3)]$	$[e=(49),(1,2,3)]$			
2	$[e=(12),\{4\}]$	$[e=(25),\{1\}]$	$[e=(28),\{1\}]$	$[e=(38),(1,2,4)]$	$[e=(48),(1,2,4)]$	$[e=(68),\{1\}]$
	$[e=(23),\{1\}]$	$[e=(26),\{1\}]$	$[e=(35),(1,2,4)]$	$[e=(46),(1,2,4)]$	$[e=(57),(1,2,7)]$	
	$[e=(24),\{1\}]$	$[e=(27),\{1\}]$	$[e=(37),(1,2,4)]$	$[e=(47),(1,2,4)]$	$[e=(59),(1,2,4)]$	
3	$[e=(12),\{3\}]$	$[e=(16),\{2\}]$	$[e=(34),(1,2,4)]$	$[e=(46),(1,2,3)]$	$[e=(56),\{2\}]$	
	$[e=(13),\{2\}]$	$[e=(17),\{2\}]$	$[e=(37),(1,2,4)]$	$[e=(47),(1,2,7)]$	$[e=(59),(1,2,9)]$	
	$[e=(14),\{2\}]$	$[e=(18),\{2\}]$	$[e=(38),(1,2,4)]$	$[e=(48),(1,2,8)]$	$[e=(78),(1,2,7)]$	
	$[e=(15),\{2\}]$	$[e=(19),\{2\}]$	$[e=(39),(1,2,4)]$	$[e=(49),(1,2,9)]$		
4	$[e=(12),\{3\}]$	$[e=(25),\{1\}]$	$[e=(29),\{1\}]$	$[e=(35),(1,2,4)]$	$[e=(39),(1,2,4)]$	$[e=(78),\{1\}]$
	$[e=(23),\{1\}]$	$[e=(27),\{1\}]$	$[e=(34),(1,2,6)]$	$[e=(37),(1,2,4)]$	$[e=(59),\{1\}]$	
5	$[e=(12),\{4\}]$	$[e=(26),\{1\}]$	$[e=(28),\{1\}]$	$[e=(35),(1,2,4)]$	$[e=(38),(1,2,4)]$	$[e=(78),(1,2,7)]$
	$[e=(23),\{1\}]$	$[e=(27),\{1\}]$	$[e=(34),(1,2,6)]$	$[e=(37),(1,2,4)]$	$[e=(56),\{1\}]$	

i	$F_i - e \not\to (2K_2, W_8)$					
6	$[e=(23),\{1\}]$	$[e=(36),(1,2,8)]$	$[e=(39),(1,2,8)]$	$[e=(48),(1,2,3)]$	$[e=(57),\{1\}]$	
	$[e=(34),(1,2,8)]$	$[e=(37),(1,2,8)]$	$[e=(46),(1,2,3)]$	$[e=(49),(1,2,3)]$	$[e=(59),\{1\}]$	
	$[e=(35),(1,2,8)]$	$[e=(38),\{1\}]$	$[e=(47),(1,2,3)]$	$[e=(56),\{1\}]$		
7	$[e=(23),\{1\}]$	$[e=(36),(1,2,8)]$	$[e=(39),(1,2,8)]$	$[e=(48),(1,2,3)]$	$[e=(69),\{1\}]$	
	$[e=(34),(1,2,3)]$	$[e=(37),(1,2,8)]$	$[e=(46),(1,2,3)]$	$[e=(49),(1,2,3)]$	$[e=(79),\{1\}]$	
	$[e=(35),(1,2,8)]$	$[e=(38),(1,2,5)]$	$[e=(47),(1,2,3)]$	$[e=(56),\{1\}]$		
8	$[e=(12),\{3\}]$	$[e=(25),\{1\}]$	$[e=(28),\{1\}]$	$[e=(36),(1,2,4)]$	$[e=(56),\{7\}]$	$[e=(89),\{1\}]$
	$[e=(23),\{1\}]$	$[e=(26),\{1\}]$	$[e=(34),(1,2,3)]$	$[e=(38),(1,2,4)]$	$[e=(67),(1,2,6)]$	
9	$[e=(12),\{3\}]$	$[e=(25),\{1\}]$	$[e=(27),\{1\}]$	$[e=(46),(1,2,3)]$	$[e=(56),\{8\}]$	
	$[e=(23),\{1\}]$	$[e=(26),\{1\}]$	$[e=(34),(1,2,3)]$	$[e=(47),(1,2,3)]$	$[e=(67),(1,2,7)]$	
10	$[e=(12),\{6\}]$	$[e=(25),\{1\}]$	$[e=(28),\{1\}]$	$[e=(36),(1,2,6)]$	$[e=(56),\{7\}]$	
	$[e=(23),\{1\}]$	$[e=(26),\{1\}]$	$[e=(34),(1,2,9)]$	$[e=(38),(1,2,8)]$		

By Proposition 2.2, there is no graph belonging to $\hat{\mathcal{R}}(2K_2, W_8)$ with maximum degree 7. Thus, we only need to prove that F_i for $i \in [1, 10]$ are the only graphs with eight vertices in $\hat{\mathcal{R}}(2K_2, W_8)$ with maximum degree 8. Let $v_i \in V(K_9)$ for $i \in [1, 9]$. Define the graphs $F_{i,j} = K_9 - E(F_{9,i,j})$, where the data from the set of edges $F_{9,i,j}$ is shown in Table 8 for $i \in [5, 11]$ and some j.

Table 8. The set of edges $E_{9,i,j}$ for $i \in [5, 11]$ and some j

$E(F_{9,1,11}) = \{35, 45, 57, 58, 59, 67, 68, 69, 78, 78, 89\}$	$E(F_{9,4,8}) = \{36, 45, 56, 57, 67, 68, 78, 89\}$	$E(F_{9,18,7}) = \{35, 45, 59, 67, 78, 79, 89\}$
$E(F_{9,2,11}) = \{34, 45, 56, 57, 58, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,5,8}) = \{36, 45, 57, 58, 68, 69, 79, 89\}$	$E(F_{9,19,7}) = \{35, 45, 58, 67, 68, 78, 89\}$
$E(F_{9,3,11}) = \{36, 45, 56, 57, 58, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,6,8}) = \{36, 45, 56, 57, 58, 67, 69, 89\}$	$E(F_{9,20,7}) = \{36, 45, 56, 57, 67, 78, 79\}$
$E(F_{9,4,11}) = \{36, 46, 56, 57, 58, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,7,8}) = \{36, 45, 57, 58, 67, 68, 78, 89\}$	$E(F_{9,21,7}) = \{35, 45, 58, 59, 68, 78, 89\}$
$E(F_{9,5,11}) = \{36, 45, 57, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,8,8}) = \{36, 45, 56, 57, 67, 69, 78, 89\}$	$E(F_{9,22,7}) = \{36, 45, 68, 69, 78, 79, 89\}$
$E(F_{9,6,11}) = \{34, 56, 57, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,9,8}) = \{36, 45, 56, 57, 58, 68, 69, 78\}$	$E(F_{9,23,7}) = \{45, 59, 67, 68, 78, 79, 89\}$
$E(F_{9,7,11}) = \{45, 56, 57, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,10,8}) = \{45, 58, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,24,7}) = \{35, 45, 59, 67, 68, 78, 89\}$
$E(F_{9,1,10}) = \{45, 56, 57, 58, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,13,8}) = \{36, 45, 56, 58, 67, 78, 79, 89\}$	$E(F_{9,25,7}) = \{36, 45, 56, 67, 78, 79, 89\}$
$E(F_{9,2,10}) = \{36, 45, 56, 58, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,11,8}) = \{45, 58, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,26,7}) = \{36, 45, 58, 67, 69, 78, 89\}$
$E(F_{9,3,10}) = \{36, 45, 56, 58, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,12,8}) = \{36, 45, 57, 68, 69, 78, 79, 89\}$	$E(F_{9,27,7}) = \{36, 45, 58, 69, 78, 79, 89\}$
$E(F_{9,4,10}) = \{36, 45, 56, 57, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,14,8}) = \{36, 45, 57, 58, 67, 68, 79, 89\}$	$E(F_{9,28,7}) = \{36, 45, 57, 68, 69, 78, 89\}$
$E(F_{9,5,10}) = \{36, 45, 56, 57, 58, 59, 67, 68, 69, 78\}$	$E(F_{9,15,8}) = \{36, 45, 57, 58, 67, 68, 69, 89\}$	$E(F_{9,29,7}) = \{36, 45, 67, 68, 69, 79, 89\}$
$E(F_{9,6,10}) = \{35, 45, 57, 58, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,16,8}) = \{36, 45, 58, 67, 68, 69, 79, 89\}$	$E(F_{9,30,7}) = \{45, 57, 58, 59, 67, 68, 69\}$
$E(F_{9,7,10}) = \{35, 45, 57, 58, 59, 67, 68, 78, 79, 89\}$	$E(F_{9,17,8}) = \{36, 45, 57, 58, 59, 67, 68, 69\}$	$E(F_{9,31,7}) = \{36, 45, 58, 67, 68, 69, 79\}$
$E(F_{9,8,10}) = \{35, 45, 56, 57, 58, 59, 68, 69, 78, 79\}$	$E(F_{9,18,8}) = \{36, 45, 57, 58, 67, 68, 69, 78\}$	$E(F_{9,32,7}) = \{58, 59, 67, 68, 69, 78, 79\}$
$E(F_{9,9,10}) = \{35, 45, 56, 57, 58, 59, 67, 68, 78, 79\}$	$E(F_{9,19,8}) = \{36, 45, 56, 58, 67, 68, 69, 78\}$	$E(F_{9,33,7}) = \{45, 56, 67, 68, 69, 79, 89\}$
$E(F_{9,10,10}) = \{36, 45, 57, 58, 59, 67, 68, 69, 78, 89\}$	$E(F_{9,20,8}) = \{36, 45, 56, 57, 58, 67, 68, 69\}$	$E(F_{9,34,7}) = \{45, 56, 57, 67, 78, 79, 89\}$
$E(F_{9,11,10}) = \{56, 57, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,21,8}) = \{45, 56, 57, 58, 59, 67, 68, 69\}$	$E(F_{9,35,7}) = \{45, 56, 58, 67, 68, 69, 79\}$
$E(F_{9,12,10}) = \{34, 57, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,22,8}) = \{45, 56, 57, 58, 59, 67, 68, 79\}$	$E(F_{9,36,7}) = \{45, 56, 58, 67, 68, 69, 79\}$
$E(F_{9,13,10}) = \{45, 57, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,23,8}) = \{36, 45, 56, 67, 68, 69, 79, 89\}$	$E(F_{9,37,7}) = \{35, 45, 59, 68, 69, 78, 79\}$
$E(F_{9,14,10}) = \{36, 45, 56, 57, 58, 67, 68, 78, 79, 89\}$	$E(F_{9,24,8}) = \{36, 45, 56, 57, 67, 68, 69, 78\}$	$E(F_{9,38,7}) = \{35, 45, 59, 68, 69, 78, 79\}$
$E(F_{9,15,10}) = \{36, 45, 57, 58, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,25,8}) = \{35, 45, 57, 58, 59, 68, 69, 79\}$	$E(F_{9,39,7}) = \{35, 45, 58, 59, 67, 78, 89\}$
$E(F_{9,16,10}) = \{35, 45, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,26,8}) = \{35, 45, 57, 58, 59, 78, 79, 89\}$	$E(F_{9,40,7}) = \{45, 57, 58, 67, 69, 78, 89\}$
$E(F_{9,17,10}) = \{36, 45, 56, 57, 58, 67, 69, 78, 79, 89\}$	$E(F_{9,27,8}) = \{35, 45, 56, 57, 58, 59, 68, 79\}$	$E(F_{9,41,7}) = \{35, 45, 57, 58, 59, 67, 68\}$
$E(F_{9,18,10}) = \{36, 45, 56, 57, 58, 68, 69, 78, 79, 89\}$	$E(F_{9,28,8}) = \{35, 45, 57, 58, 59, 67, 68, 78\}$	$E(F_{9,42,7}) = \{35, 45, 56, 58, 59, 67, 89\}$
$E(F_{9,1,9}) = \{36, 45, 56, 57, 58, 67, 68, 69, 78\}$	$E(F_{9,29,8}) = \{35, 45, 56, 57, 58, 59, 67, 68\}$	$E(F_{9,43,7}) = \{35, 45, 56, 57, 59, 67, 68\}$
$E(F_{9,2,9}) = \{36, 45, 56, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,30,8}) = \{36, 57, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,44,7}) = \{35, 45, 56, 57, 58, 59, 67\}$
$E(F_{9,3,9}) = \{36, 45, 56, 57, 58, 67, 68, 69, 89\}$	$E(F_{9,31,8}) = \{58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,45,7}) = \{45, 56, 57, 58, 59, 68, 79\}$
$E(F_{9,4,9}) = \{35, 45, 56, 57, 58, 59, 67, 68, 78\}$	$E(F_{9,32,8}) = \{45, 58, 59, 68, 69, 78, 79, 89\}$	$E(F_{9,46,7}) = \{36, 56, 67, 68, 69, 79, 89\}$
$E(F_{9,5,9}) = \{36, 56, 57, 58, 67, 68, 69, 79, 89\}$	$E(F_{9,33,8}) = \{34, 58, 59, 68, 69, 78, 79, 89\}$	$E(F_{9,47,7}) = \{58, 59, 68, 69, 78, 79, 89\}$
$E(F_{9,6,9}) = \{36, 45, 56, 58, 67, 68, 69, 79, 89\}$	$E(F_{9,34,8}) = \{34, 58, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,48,7}) = \{34, 58, 59, 67, 69, 79, 89\}$
$E(F_{9,7,9}) = \{45, 56, 57, 58, 59, 67, 68, 69, 79\}$	$E(F_{9,35,8}) = \{34, 58, 59, 67, 68, 69, 79, 89\}$	$E(F_{9,49,7}) = \{35, 58, 59, 67, 68, 69, 78\}$
$E(F_{9,8,9}) = \{45, 56, 57, 58, 59, 67, 69, 78, 79\}$	$E(F_{9,36,8}) = \{36, 45, 58, 67, 68, 69, 78, 89\}$	$E(F_{9,50,7}) = \{36, 45, 57, 67, 78, 79, 89\}$
$E(F_{9,9,9}) = \{35, 45, 56, 57, 58, 59, 68, 69, 79\}$	$E(F_{9,37,8}) = \{36, 45, 58, 68, 69, 78, 79, 89\}$	$E(F_{9,51,7}) = \{36, 45, 57, 67, 78, 79, 89\}$
$E(F_{9,10,9}) = \{35, 45, 57, 58, 59, 67, 68, 78, 79\}$	$E(F_{9,38,8}) = \{36, 45, 57, 58, 68, 69, 78, 89\}$	$E(F_{9,52,7}) = \{35, 45, 67, 68, 78, 79, 89\}$
$E(F_{9,11,9}) = \{35, 45, 57, 58, 59, 68, 69, 78, 79\}$	$E(F_{9,39,8}) = \{36, 45, 56, 57, 67, 78, 79, 89\}$	$E(F_{9,53,7}) = \{45, 67, 68, 69, 78, 79, 89\}$
$E(F_{9,12,9}) = \{35, 45, 57, 58, 59, 67, 78, 79, 89\}$	$E(F_{9,40,8}) = \{36, 45, 56, 57, 58, 69, 78, 89\}$	$E(F_{9,54,7}) = \{56, 67, 68, 69, 78, 79, 89\}$
$E(F_{9,13,9}) = \{45, 56, 57, 58, 59, 68, 69, 78, 79\}$	$E(F_{9,41,8}) = \{35, 45, 58, 67, 68, 78, 79, 89\}$	$E(F_{9,1,6}) = \{36, 56, 68, 69, 79, 89\}$
$E(F_{9,14,9}) = \{36, 56, 58, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,42,8}) = \{34, 45, 59, 68, 69, 78, 79, 89\}$	$E(F_{9,2,6}) = \{45, 56, 58, 59, 67, 89\}$
$E(F_{9,15,9}) = \{36, 45, 56, 57, 58, 59, 67, 68, 69\}$	$E(F_{9,43,8}) = \{35, 45, 58, 59, 67, 68, 78, 89\}$	$E(F_{9,3,6}) = \{36, 45, 57, 67, 79, 89\}$
$E(F_{9,16,9}) = \{36, 56, 57, 58, 67, 68, 69, 78, 89\}$	$E(F_{9,44,8}) = \{35, 36, 69, 67, 68, 78, 79, 89\}$	$E(F_{9,4,6}) = \{56, 59, 67, 68, 78, 89\}$
$E(F_{9,17,9}) = \{36, 45, 56, 58, 67, 68, 69, 78, 89\}$	$E(F_{9,45,8}) = \{35, 45, 58, 59, 67, 68, 78, 79\}$	$E(F_{9,5,6}) = \{36, 56, 67, 68, 79, 89\}$
$E(F_{9,18,9}) = \{57, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,46,8}) = \{45, 56, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,6,6}) = \{34, 59, 67, 68, 78, 89\}$
$E(F_{9,19,9}) = \{45, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,47,8}) = \{36, 45, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,7,6}) = \{35, 45, 68, 78, 79, 89\}$
$E(F_{9,20,9}) = \{45, 57, 58, 59, 67, 68, 78, 79, 89\}$	$E(F_{9,48,8}) = \{36, 58, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,8,6}) = \{36, 45, 57, 67, 78, 79\}$
$E(F_{9,21,9}) = \{46, 56, 57, 58, 67, 68, 78, 79, 89\}$	$E(F_{9,49,8}) = \{35, 45, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,9,6}) = \{34, 56, 59, 68, 78, 89\}$
$E(F_{9,22,9}) = \{45, 56, 58, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,50,8}) = \{45, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,10,6}) = \{34, 56, 67, 68, 78, 89\}$

$E(F_{9,23,9}) = \{45, 56, 57, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,51,8}) = \{36, 45, 56, 57, 67, 68, 78, 89\}$	$E(F_{9,11,6}) = \{56, 68, 69, 78, 79, 89\}$
$E(F_{9,24,9}) = \{34, 58, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,52,8}) = \{57, 58, 59, 67, 68, 69, 78, 89\}$	$E(F_{9,12,6}) = \{56, 67, 68, 69, 78, 79\}$
$E(F_{9,25,9}) = \{35, 45, 58, 59, 68, 69, 78, 79, 89\}$	$E(F_{9,1,7}) = \{58, 59, 67, 68, 69, 78, 89\}$	$E(F_{9,13,6}) = \{58, 59, 67, 69, 79, 89\}$
$E(F_{9,26,9}) = \{34, 57, 58, 59, 67, 68, 69, 79, 89\}$	$E(F_{9,2,7}) = \{45, 58, 59, 68, 69, 78, 79\}$	$E(F_{9,14,6}) = \{45, 68, 69, 78, 79, 89\}$
$E(F_{9,27,9}) = \{36, 45, 57, 58, 67, 68, 69, 78, 89\}$	$E(F_{9,3,7}) = \{45, 58, 59, 67, 68, 78, 79\}$	$E(F_{9,15,6}) = \{58, 59, 68.69.78.89\}$
$E(F_{9,28,9}) = \{36, 45, 58, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,4,7}) = \{45, 57, 58, 59, 67, 68, 79\}$	$E(F_{9,16,6}) = \{35, 45, 67, 68, 79, 89\}$
$E(F_{9,29,9}) = \{36, 45, 57, 58, 68, 69, 78, 79, 89\}$	$E(F_{9,5,7}) = \{45, 56, 58, 67, 68, 78, 79\}$	$E(F_{9,17,6}) = \{35, 45, 58, 59, 67, 89\}$
$E(F_{9,30,9}) = \{36, 45, 57, 58, 67, 68, 78, 79, 89\}$	$E(F_{9,6,7}) = \{45, 56, 58, 59, 67, 68, 69\}$	$E(F_{9,18,6}) = \{35, 45, 58, 59, 67, 68\}$
$E(F_{9,31,9}) = \{36, 45, 56, 57, 58, 68, 69, 78, 89\}$	$E(F_{9,7,7}) = \{36, 48, 68, 69, 78, 79, 89\}$	$E(F_{9,19,6}) = \{35, 45, 56, 59, 67, 68\}$
$E(F_{9,32,9}) = \{36, 45, 56, 57, 67, 69, 78, 79, 89\}$	$E(F_{9,8,7}) = \{36, 56, 68, 69, 78, 79, 89\}$	$E(F_{9,20,6}) = \{35, 45, 56, 58, 59, 89\}$
$E(F_{9,33,9}) = \{36, 45, 56, 58, 67, 68, 78, 79, 89\}$	$E(F_{9,9,7}) = \{36, 45, 57, 58, 67, 68, 89\}$	$E(F_{9,21,6}) = \{35, 45, 57, 58, 59, 67\}$
$E(F_{9,34,9}) = \{35, 45, 59, 67, 68, 69, 78, 79, 89\}$	$E(F_{9,10,7}) = \{36, 45, 56, 57, 69, 78, 89\}$	$E(F_{9,22,6}) = \{35, 45, 56, 57, 58, 59\}$
$E(F_{9,35,9}) = \{35, 45, 58, 59, 67, 68, 78, 79, 89\}$	$E(F_{9,11,7}) = \{36, 45, 56, 67, 68, 79, 89\}$	$E(F_{9,23,6}) = \{67, 68, 69, 78, 79, 89\}$
$E(F_{9,36,9}) = \{35, 45, 58, 59, 67, 68, 69, 78, 79\}$	$E(F_{9,12,7}) = \{36, 45, 57, 67, 69, 78, 89\}$	$E(F_{9,1,5}) = \{68, 69, 78, 79, 89\}$
$E(F_{9,37,9}) = \{45, 57, 58, 59, 67, 68, 69, 78, 89\}$	$E(F_{9,13,7}) = \{36, 45, 56, 58, 67, 78, 89\}$	$E(F_{9,2,5}) = \{35, 45, 58, 59, 89\}$
$E(F_{9,38,9}) = \{36, 45, 56, 57, 58, 67, 68, 78, 89\}$	$E(F_{9,14,7}) = \{36, 45, 56, 58, 67, 69, 79\}$	$E(F_{9,3,5}) = \{35, 45, 56, 58, 89\}$
$E(F_{9,1,8}) = \{45, 57, 58, 59, 67, 69, 78, 89\}$	$E(F_{9,15,7}) = \{37, 45, 56, 58, 67, 78, 79\}$	$E(F_{9,4,5}) = \{35, 45, 58, 59, 67\}$
$E(F_{9,2,8}) = \{45, 57, 58, 59, 67, 68, 69, 79\}$	$E(F_{9,16,7}) = \{36, 45, 56, 57, 67, 79, 89\}$	$E(F_{9,5,5}) = \{34, 35, 56, 58, 59\}$
$E(F_{9,3,8}) = \{45, 56, 58, 67, 68, 69, 79, 89\}$	$E(F_{9,17,7}) = \{36, 45, 56, 67, 68, 78, 89\}$	

Let ${\mathcal{F}_{\mathcal{R}}}^y = \{K_9 - E_y | E_y = \{e_1, e_2, \ldots, e_y\} \in E(K_9)$ and $K_9 - E_y \in \hat{\mathcal{R}}(2K_2, W_8)\}$. By Lemma 2.3, since $|E(K_9 - E_y)| \geq 25$, then $y \leq 11$. In the following cases, we will prove that F_i for $i \in [1, 10]$ are the only graphs with eight vertices in $\hat{\mathcal{R}}(2K_2, W_8)$ with maximum degree 8.

Case 1. $K_9 - E_{11}$.
By Theorem 2.2 for every $F_R^{11} \in {\mathcal{F}_{\mathcal{R}}}^{11}$, $F_R^{11} \supseteq B_{8,1,i}$ for some $i \in [1, 3]$. All possible graphs $K_9 - E_{11} \supseteq B_{8,1,i}$ are represented by the graphs $F_{11,j}$ for $j \in [1, 7]$. According to Proposition 2.3, and Lemma 2.2 (ii), respectively, F_{11,j_1}, and F_{11,j_2} do not belong to $R(2K_2, W_8)$, for $j_1 \in [1, 5]$ and $j_2 \in [6, 7]$. Thus, ${\mathcal{F}_{\mathcal{R}}}^{11} = \emptyset$. Therefore, there are no graphs obtained by removing any eleven edges from K_9 that belonging to $\hat{\mathcal{R}}(2K_2, W_8)$.

Case 2. $K_9 - E_{10}$.
All possible graphs $F_{11,t} + e$ for $t \in [1, 7]$ are represented by the graphs $F_{10,j}$ for $j \in [1, 18]$. According to Proposition 2.3, Lemma 2.2 (i), and Proposition 2.4 (ii), respectively, F_{10,j_1}, F_{10,j_2}, and F_{10,j_3} do not belong to $\hat{\mathcal{R}}(2K_2, W_8)$, for $j_1 \in [1, 10]$, $j_2 \in [11, 16]$, and $j_3 \in [17, 18]$. Thus, ${\mathcal{F}_{\mathcal{R}}}^{10} = \emptyset$. Therefore, there are no graphs obtained by removing any ten edges from K_9 that belonging to $\hat{\mathcal{R}}(2K_2, W_8)$.

Case 3. $K_9 - E_9$.
All possible graphs $F_{10,t} + e$ for $t \in [1, 18]$ are represented by the graphs $F_{9,j}$ for $j \in [1, 38]$. According to Proposition 2.3 (i), Lemma 2.2 (i), and Proposition 2.4, respectively, F_{9,j_1}, F_{9,j_2}, and $F_{9,36}$ do not belong to $R(2K_2, W_8)$, for $j_1 \in [1, 26]$, and $j_2 \in [27, 35]$. Next, since $F_{9,37} - v_1 - v_2, F_{9,38} - v_1 - v_2 \not\supseteq W_n$, by Observation 2.1 (i) $F_{9,37}, F_{9,38} \notin \hat{\mathcal{R}}(2K_2, W_8)$. Thus, ${\mathcal{H}_{\mathcal{R}}}^6 = \{F_i | i \in [1, 5]\}$. Therefore, removing any nine edges from the graph K_9 only produces the graphs F_i for $i \in [1, 5]$ as members of $\hat{\mathcal{R}}(2K_2, W_8)$.

Case 4. $K_9 - E_8$.
All possible graphs $F_{9,t} + e$ for $t \in [1, 38]$ are represented by the graphs $F_{8,j}$ for $j \in [1, 52]$. Consider $F_{8,j_1} \supset F_1$, $F_{8,j_2} \supset F_2$, $F_{8,j_3} \supset F_3$, and $F_{8,j_4} \supset F_5$ for $j_1 \in [1, 9], j_2 \in [10, 13], j_3 \in [14, 16]$, and $j_4 \in [17, 18]$. By Observation 2.2 (i), $F_{8,t_1} \notin \hat{\mathcal{R}}(2K_2, W_8)$ for $t_1 \in [1, 18]$. According to Proposition 2.3 (i), Lemma 2.2, and Proposition 2.4, respectively, F_{8,k_1}, F_{8,k_2}, and F_{8,k_3} do not belong to $R(2K_2, W_8)$, for $k_1 \in [19, 30]$, $k_2 \in [31, 33] \cup [46, 50]$,

and $k_3 \in [34, 45]$. Afterward, since $F_{8,51} - v_1 - v_2, F_{8,52} - v_1 - v_2 \not\supseteq W_n$, by Observation 2.1 (i) $F_{8,51}, F_{8,52} \notin \hat{\mathcal{R}}(2K_2, W_8)$. Thus, ${\mathcal{H}_{\mathcal{R}}}^8 = \{F_i | i \in [6, 9]\}$. Therefore, removing any eight edges from the graph K_9 only produces the graphs F_i for $i \in [6, 9]$ as members of $\hat{\mathcal{R}}(2K_2, W_8)$.

Case 5. $K_9 - E_7$.
All possible graphs $F_{8,t} + e$ for $t \in [19, 52]$ are represented by the graphs $F_{7,j}$ for $j \in [1, 54]$. Consider $F_{7,j_1} \supset F_1$, $F_{7,j_2} \supset F_2$, $F_{7,j_3} \supset F_3$, $F_{7,j_4} \supset F_6$, $F_{7,j_5} \supset F_7$, $F_{7,37} \supset F_8$, and $F_{7,j_6} \supset F_9$ for $j_1 \in [1, 21], j_2 \in [22, 28], j_3 \in [29, 32]$, $j_4 \in [33, 35]$, and $j_5 \in [38, 40]$. By Observation 2.2 (i), $F_{7,t_1} \notin \hat{\mathcal{R}}(2K_2, W_8)$ for $t_1 \in [1, 40]$. According to Proposition 2.3 (i), Lemma 2.2, and Proposition 2.4, respectively, F_{7,k_1}, F_{7,k_2}, and F_{7,k_3} do not belong to $R(2K_2, W_8)$, for $k_1 \in [41, 46], k_2 \in \{47, 53, 54\}$, and $k_3 \in [48, 52]$. Thus, ${\mathcal{H}_{\mathcal{R}}}^8 = \{F_{10}\}$. Therefore, removing any seven edges from the graph K_9 only produces the graph F_{10} as a member of $\hat{\mathcal{R}}(2K_2, W_8)$.

Case 6. $K_9 - E_6$.
All possible graphs $F_{7,t} + e$ for $t \in [41, 54]$ are represented by the graphs $F_{6,j}$ for $j \in [1, 23]$. Consider $F_{6,j_1} \supset F_1$, $F_{6,j_2} \supset F_2$, $F_{6,9} \supset F_3$, $F_{6,10} \supset F_4$, $F_{6,j_3} \supset F_6$, $F_{6,j_4} \supset F_7$, $F_{6,j_5} \supset F_8$, $F_{6,17} \supset F_9$, and $F_{6,j_6} \supset F_{10}$ for $j_1 \in [1, 3], j_2 \in [4, 8], j_3 \in [11, 12]$, $j_4 \in [13, 14]$, $j_5 \in [15, 16]$, and $j_6 \in [18, 20]$. By Observation 2.2 (i), $F_{6,t_1} \notin \hat{\mathcal{R}}(2K_2, W_8)$ for $t_1 \in [1, 20]$. Next, According to Proposition 2.3 (i), and Lemma 2.2 (ii), respectively, $F_{6,j}$ and $F_{6,23}$ do not belong to $R(2K_2, W_8)$, for $j \in [20, 22]$.
By Theorem 2.2, $F_{6,t_2} \notin \hat{\mathcal{R}}(2K_2, W_8)$ for $t_2 \in [20, 23]$. Thus, ${\mathcal{H}_{\mathcal{R}}}^4 = \emptyset$. Therefore, there are no graphs obtained by removing any six edges from K_9 that belonging to $\hat{\mathcal{R}}(2K_2, W_8)$.

Case 7. $K_9 - E_5$.
All possible graphs $F_{7,t} + e$ for $t \in [20, 23]$ are represented by the graphs $F_{5,j}$ for $j \in [1, 5]$. Consider $F_{5,1} \supset F_6$, $F_{5,2} \supset F_9$, and $F_{5,j} \supset F_3$ for $j \in [3, 4]$. By Observation 2.2 (i), $H_{5,t_3} \notin \hat{\mathcal{R}}(2K_2, W_8)$ for $t_3 \in [1, 4]$. Next, by Proposition 2.4 (ii), $F_{5,5} \notin \hat{\mathcal{R}}(2K_2, W_8)$. Thus, ${\mathcal{F}_{\mathcal{R}}}^5 = \emptyset$. Therefore, there are no graphs obtained by removing any five edges from K_9 that belonging to $\hat{\mathcal{R}}(2K_2, W_8)$.

Case 8. $K_9 - E_t$ for $t \leq 4$.
Since $F_{5,5} + e \supset F_9$, by Observation 2.2 (i), ${\mathcal{F}_{\mathcal{R}}}^t = \emptyset$ for $t \leq 4$. Therefore, there are no graphs obtained from removing at most four edges from K_9 as a member of $\hat{\mathcal{R}}(2K_2, W_8)$.

Based on cases $1 - 8$ above, the graphs with maximum degree 8 in $\hat{\mathcal{R}}(2K_2, W_8)$ are only F_i for $i \in [1, 10]$. Thus, we have proven that the graphs with nine vertices in $\hat{\mathcal{R}}(2K_2, W_8)$ are precisely F_i for $i \in [1, 10]$. □

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