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On Radenković and Gutman conjecture for some classes of trees of diameter 5

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Abstract

The Radenković and Gutman conjecture establishes a relationship between the Laplacian energies of any tree T_n, the star graph S_n and the path graph P_n, i.e., $LE(P_n) \leq LE(T_n) \leq LE(S_n)$. In this paper, we focus on verifying the validity of this conjecture for some classes of trees with diameter 5. By analyzing their structural properties and the corresponding Laplacian spectra, we establish that the conjecture holds for these few subclasses.

Keywords: , Laplacian energy, Laplacian characteristics polynomial, tree, Laplacian eigenvalue
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1. Introduction

Let $G = (V, E)$ be a crisp graph having $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{e_1, e_2, \dots, e_n\}$. The numbers $n = |V(G)|$ and $m = |E(G)|$ represent the order and size of the graph G, respectively. By $d(v)$, we denote the degree of the vertex v. Vertices of degree 1 are known as pendant vertices. A tree on n vertices represented by T_n, is a connected graph without cycles. A path on n vertices indicated by P_n is a tree having 2 pendant vertices. Any tree graph having $n-1$ pendant vertices and one node with $n-1$ connected edges is called a star graph denoted by S_n. We direct readers to any standard literature, such as [3], [12], [16],and [19], for further notations and definitions of these concepts from matrix theory and graph theory. The "energy" of a graph is a concept that comes from spectral graph theory, a subfield of graph theory that focuses on the connections between the eigenvalues and eigenvectors of certain matrices associated with a graph.

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Graph energy has a wide range of applications in chemistry [24, 1], particularly in studying molecular graphs and their properties. The energy of the graph has served as a structure descriptor in the case of σ-electron systems. The graph energy gives better results than the traditional structure descriptors. Understanding the structure, stability, and reactivity of molecules in many chemical situations is made possible by useful tools provided by graph energy and LE analyses, which further advance synthetic chemistry. The LE is related to molecular stability, where molecules with lower LE are generally more stable. The eigenvalue distribution of the Laplacian matrix offers insight into molecular reactivity. Isomers with the same chemical formula can have different graph Laplacian energies, and these differences help chemists understand how structural variations affect molecular properties.

Gutman [7] first introduced the concept of the energy of a graph G denoted by $E(G)$ in 1978 and it is known as the total of the absolute eigenvalues of the adjacency matrix $\mathbf{A}(G)$ denoted by

$$E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent eigenvalues of $\mathbf{A}(G)$. The idea of energy has been extended to general matrices in addition to other kinds of matrices connected to graphs [18]. Many studies have been conducted in this area, and further research is ongoing because there are still a number of unsolved problems. We refer to [9, 17, 20] and their sources for some current research on graph energy and associated findings. In 2006, Gutman and Zhou [8] introduced the Laplacian energy (LE) of a graph G and is represented by

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|,$$

where m is total count of edges and n is total count of vertices of $L(G)$. The Laplacian matrix is defined as $L(G) = D(G) - \mathbf{A}(G)$, where $D(G)$ is the degree matrix which is a diagonal matrix that tells the degree of each vertex (count of edges connecting to it) of the graph. and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$ are the LEVs of G. Since $\sum\limits_{i=1}^{n} \mu_i = 2m$, it is easy to see that [5]

$$\mathrm{LE}(G) = 2\sum_{i=1}^{n} \left(\mu_i - \sigma\hat{d}\right) = 2\max_{1\leq k\leq n} \left(\sum_{i=1}^{k} \mu_i - k\hat{d}\right) \tag{1}$$

where σ denotes the count of Laplacian eigenvalues (LEVs) that are equal to or greater than the average degree $\hat{d}$. This number has been explored in great detail due to its uses and relationships with more significant spectral characteristics of graph G. This term is also a current area of study, and the literature has a few publications, especially on trees [23]. See [10] and its references for some new findings on LE.

Finding the maximum value of a spectral parameter and characterizing the graph or graphs that reach this maximum value are key issues in spectral graph theory. The extremal value within a class of order n graphs or within all graphs of order n. The graph's energy, or $E(G)$, for Several families of graphs and trees has been examined in this problem, and the graphs that reach the greatest and least values are fully identified. According to [6],P_n the path graph has the highest energy than trees T_n and star graph S_n.

The symmetry of a molecule is an important factor in determining its chemical properties. Graph LE can be used to quantify the symmetry of molecular graphs. Molecules with higher symmetry tend to have distinct LE signatures, which can be used in classifying and predicting chemical behavior. Similar to the energy of a graph, an easy way to get the greatest value that a parameter can have in every n-vertex graph therefore the purpose of including the LE of a graph is to identify the graph on n vertices whose LE accomplishes this maximum value.

To the extent of our understanding, this appears to be a challenging issue that is still unresolved. Researchers Radenković and Gutman [22] looked into the relationship between a tree's energy and LE. Following the computation of the energy and LE of each tree with $n \geq 14$ vertices, the upcoming conjecture came into existence.

Conjecture 1. *Let T_n be the set of trees with n vertices. Then,*

$$LE(P_n) \leq LE(T_n) \leq LE(S_n).$$

In [25], it has been proved that the conjecture 1 is true for diameter three trees. They also validated that the conjecture is true for any tree of order upto 18. Fritscher et al. [5] proof establishes the validity of the Conjecture 1's right-side for all nth order trees. Chang et al. verified the first hypothesis for trees with diameters of 4 and 5 with perfect matching in [4], proving the left inequality. Rahman et al. [21] recently examined a few classes of trees with diameter 4 (which were not taken into account in [4]) and confirmed the validity of the left-side inequality of Conjecture 1 for the trees in these families. Our goal in this work is to fully demonstrate the left side of the inequality of Conjecture 1 for diameter 5 trees. Ganie et al. [11] validated the conjecture for all trees having a maximum of $\frac{9n}{25} - 2$ non-pendant vertices. In this work, we establish the conjecture 1 for some classes of diameter 5 trees. For computations in the paper, we have used the computational programs **Wolfram Mathematica**. We recommend reading the upcoming sources [23, 26] and the references contained therein for additional material in this area.

Theorem 1.1. *[5] Let T_n be a trees with n vertices provided that $T_n \neq S_n$. Then,*

$$LE(T_n) < LE(S_n).$$

This theorem suggests that the conjecture 1 upper bound holds true for each tree T_n with n vertices. Therefore, there is no need to prove it again for diameter 5 classes. However, the lower bound is still open. So we will focus to establish the lower bound of the conjecture.

The paper is organized as follows. We list a few results that are already known in Section 2. We use these results to verify the conjecture 1 for some classes of trees of diameter 5 in Section 3. Finally, a conclusion of the study is presented in Section 4.

2. Preliminaries

This section includes two algorithms and a few well-known results that are useful throughout the paper.

Lemma 2.1. *Let p and q represent G's neighbors and leaves, respectively. Then 1 is a multiplicity at least $p - q \geq 0$ Laplacian eigenvalue of G.*

We can now proceed with a discussion regarding LEVs and the Laplacian characteristic polynomial, specifically focusing on trees. There exists a notable lemma in this context, credited to Brouwer and Haemers [2].

Lemma 2.2. *Let G be a connected graph with LEVs as $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. Then,*

$$\mu_j \geq d_j - i + 2,\ i = 1, 2, \ldots, n,$$

such that $d_1 \geq d_2 \geq \ldots \geq d_n$ is the order of degree of vertices of G.

We use the technique described in references [14] and [15] to determine the LEVs of tree of the nth order T_n. This approach may be simply modified to investigate the distinctive polynomial of the Laplacian matrix because it is made to operate directly on the tree structure.
Algorithm (1): [13]
Input: tree T, scalar λ
Output: diagonal matrix D congruent to $L(T) + \lambda I$ **Diagonalization Algorithm** (T, λ)

1. Set up $\dot{a}(u) := d(u) + \lambda$ for each of the vertices in u
2. Order vertices from below
3. **for** $j = 1$ to n
   (a) **if** u_j is pendent **then** continue
   (b) **otherwise if** $\dot{a}(c) \neq 0$ for all children c of u_j **then**
      i. $\dot{a}(u_j) := \dot{a}(u_j) - \sum_c \frac{1}{a(c)}$; adding all children of u_j
   (c) **otherwise**
      i. Choose a child u_k of u_j for which $\dot{a}(u_k) = 0$
      ii. $\dot{a}(u_j) := -\frac{1}{2}$
      iii. $\dot{a}(u_k) := 2$
      iv. **if** u_j has a parent u_l, delete the edge $u_j u_l$

end loop

This approach allows us to estimate σ for a tree by calculating the quantity of the Laplacian matrix of T eigenvalues that fall inside a certain range. Notably, the values $\dot{a}(u)$ on each u of the $\mathcal{T}$ are exactly matched by the diagonal components of the output matrix. Throughout the work, the upcoming observation from Jacobs and Trevisan [13] is useful.
Algorithm (2): [14]
The algorithm functions by assigning a rational function $\dot{a}(v) = \frac{t}{c}$ to each vertex v in the tree, where both t and c are polynomials within the polynomial ring $\mathbf{Q}[\lambda]$.. Beginning with the leaves of the tree, which are originally given the value $\lambda - 1$ (assuming the tree is rooted arbitrarily), the process follows a bottom-up methodology. After all of v' children have undergone this computation, the vertex v is then provided the resulting rational function.

$$\dot{a}(v) = \lambda - d_v - \sum_{c \in C} \frac{1}{\dot{a}(c),} \tag{2}$$

where d_v is the degree of vertex v and C is the collection of its offspring. Following this procedure for each vertex, the characteristic polynomial is calculated by multiplying each function $\dot{a}(v)$ by the total count of vertices.

$$\chi(\lambda) = \prod_{v \in V} \dot{a}(v). \tag{3}$$

Lemma 2.3. *[13] Consider $\mathcal{D}$ a diagonal matrix generated by the algorithm Diagonalize$(T, -\alpha)$, and let T be a tree. Then, the following claims are true:*

1. *The count of LEVs of T larger than α is the count of positive values in $\mathcal{D}$.*
2. *The count of LEVs of T less than α is the count of negative values in $\mathcal{D}$.*
3. *A LEV of T with multiplicity j is α if there are j zero diagonal elements in D.*

Trevisan et al. [25] have established a lemma regarding the LE of path P_n.

Lemma 2.4. *[25] Let P_n be the set of trees with n vertices. Then, $LE(P_n) \leq 2 + \frac{4n}{\pi}$.*

Lemma 2.5. *[11] The LEVs of G interlace the LEVs of G_0 if $G_0 = G + e$. In other words,*

$$\mu_1(G_0) \leq \mu_1(G) \leq \mu_2(G_0) \leq \mu_2(G) \leq \cdots \leq \mu_n(G_0) \leq \mu_n(G) = 0.$$

[23] has the upcoming finding regarding the division of LEVs of trees.

Lemma 2.6. *The number of LEVs smaller than $2 - \frac{2}{n}$ will be at $\lceil \frac{n}{2} \rceil$.*

Corollary 2.1. *[11] Let T be a tree with s internal (non-pendent) vertices and an order of $n \geq 4$.* (i) For any $n \geq 9$, Conjecture 1 is valid if $s = 1$. (ii) For any $n \geq 12$, Conjecture 1 is valid, if $s = 2$. (iii) For any $n \geq 14$, Conjecture 1 is valid, if $s = 3$. (iv)For any $n \geq 17$, Conjecture 1 is valid, if $s = 4$. (v) For any $n \geq 20$, Conjecture 1 is valid, if $s = 5$. (vi) For any $n \geq 23$ Conjecture 1 is valid, if $s = 6$. (vii) For any $n \geq 25$ Conjecture 1 is valid, if $s = 7$. (viii) For all n Conjecture 1 is valid, if $s \leq \frac{9n}{25} - 2$.

Corollary 2.2. *[11] For a tree $\mathcal{T}$ with $n > 7$ and e be its non-pendant edge. The components of $\mathcal{T} - e$ are represented by T_1 and $\mathcal{T}_2$, and the quantity of LEVs of $\mathcal{T} - e$ that are more or comparable to the mean degree $d(\mathcal{T} - e)$ is represented by σ. The count of LEVs of $\mathcal{T}_j$ that are more or equal to $d(T - e)$ with $h_1 + h_2 = \sigma$ and the counts of LEVs of T_j that are higher or equal to $\hat{d}(T_j)$ are denoted by h_j and σ_i, respectively. Conjecture 1 holds for T if $\sigma_1 = h_1$ and $\sigma_2 = h_2$, given that $LE(T_j) \geq 2 + \frac{4}{n_j}$ for $j = 1, 2$.*

3. Laplacian energies of trees of diameter 5

In this section, we verify that for some classes of trees of diameter 5 tree, Conjecture 1 holds. Trees having order $n \geq 3$ and diameter d are denoted by $T_n(d)$. Specifically, considering the family denoted as $T_n(5)$ of few classes of trees of order n and diameter 5. To confirm the validity of Conjecture 1 for the family of trees $T_n(5)$, we must take into account more than 19 potential subfamilies. Here, we examine the family $T_n(5)$ as a whole and confirm that Conjecture 1 is true.

Theorem 3.1. *For the family $T_n(5)$ of trees with diameter 5, **Conjecture 1** is true.*

Proof. The theorem 1.1 suggests that the conjecture 1 upper bound holds true for each tree T_n with n vertices. Therefore, there is no need to prove it. However, the lower bound is still open. So we will focus to establish the lower bound of the conjecture.

When a tree of diameter 5 and order $n \geq 19$ is represented by T, then $T \in T_n(5)$. Let $\hat{d}(T) = 2 - \frac{2}{n}$ be the average node degree of T and the LEVs are arranged in a nondecrasing order. It is clear from Lemma 2.4 that, for every $n \geq 19$, it is sufficient to show that Conjecture 1 is true for T, the expression

$$LE(T) \leq \frac{4n}{\pi} + 2, \tag{4}$$

applies for T. T is either a multi-broom graph of diameter 5 (as shown in Figure 1, or an SNS-tree of diameter 5 (as shown in Figure 7).

Let T_{n_1} be the first type of multi-broom graph of diameter 5 on n vertices as shown in Figure 1, respectively.

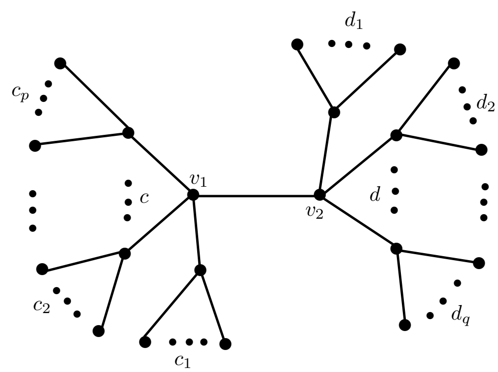

Figure 1. Multi-broom graph of diameter 5

To construct characteristic polynomial of this tree we use Algorithm (2) in the upcoming steps;

$$\begin{aligned}
X &= ((-1+x)(-1+x-c_1)-c_1)/(-1+x)\ldots\\
Y &= ((-1+x)(-1+x-c_p)-c_p)/(-1+x)\\
Z &= -3+x-(-1+x)/((-1+x)(-1+x-c_1)-c_1)-\ldots-(-1+x)/((-1+x)(-1+x-c_p)\\
&- c_p)\\
S &= ((-1+x)(-1+x-d_1)d_1)/(-1+x)\ldots\\
U &= ((-1+x)(-1+x-d_q)-d_q)/(-1+x)\\
V &= -3+x-(-1+x)/(-1+x)(-1+x-d_1)-d_1)-\ldots-(-1+x)/(-1+x)(-1+x-d_q)\\
&- d_q)-1/((-3+x)\\
&- (-1+x)/((-1+x)(-1+x-c_1)-c_1)-\ldots-(-1+x)/((-1+x)(-1+x-c_p)-c_p))\\
W &= (-1+x)^{(c+c_1+\ldots+c_p+d+d_1+\ldots+d_q)}.
\end{aligned}$$

By taking product of all of the above mentioned functions, the required polynomial can be constructed. We will discuss few subcases of this generalized class by assigning different values to c and d as shown in figures 2,3 and 4 respectively.

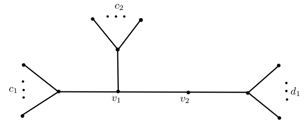

Figure 2. (c=2,d=1)

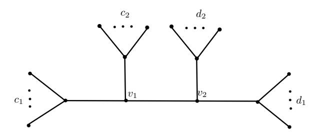

Figure 3. (c=2,d=2)

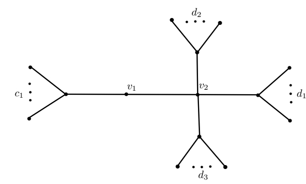

Figure 4. (c=1,d=3)

The Laplacian characteristic polynomials of these three subcasses of T_{n_1} are as follows:

(i) $\chi(T_{n_1}(c_1,c_2,d_1)) = (-1+x)^{(n-8)}x(-5+33x-89x^2+127x^3-103x^4+47x^5-11x^6+x^7-c_1+10xc_1-31x^2c_1+43x^3c_1-29x^4c_1+9x^5c_1-x^6c_1-d+11xd_1-33x^2d_1+44x^3d_1-29x^4d_1+9x^5d_1-x^6d_1+3xc_1d_1-12x^2c_1d_1+15x^3c_1d_1-7x^4c_1d_1+x^5c_1d_1-c_2+10xc_2-31x^2c_2+43x^3c_2-29x^4c_2+9x^5c_2-x^6c_2+2xc_1c-2-11x^2c_1c_2+15x^3c_1c_2-7x^4c_1c_2+x^5c_1c_2+3xd_1c_2-12x^2d_1c_2+15x^3d_1c_2-7x^4d_1c_2+x^5d_1c_2-5x^2c_1d_1c_2+5x^3c_1d_1c_2-x^4c_1d_1c_2)$;

(ii) $\chi(T_{n_1}(c_1,d_1,c_2,d_2)) = (-1+x)^{(n-10)}x(-6+53x-200x^2+424x^3-556x^4+466x^5-248x^6+80x^7-14x^8+x^9-c_1+14xc_1-64x^2c_1+144x^3c_1-182x^4c_1+134x^5c_1-56x^6c_1+12x^7c_1-x^8c_1-d_1+14xd_1-64x^2d_1+144x^3d_1-182x^4d_1+134x^5d_1-56x^6d_1+12x^7d_1-x^8d_1+3xc_1d_1-20x^2c_1d_1+50x^3c_1d_1-60x^4c_1d_1+36x^5c_1d_1-10x^6c_1d_1+x^7c_1d_1-c_2+14xc_2-64x^2c_2+144x^3c_2-182x^4c_2+134x^5c_2-56x^6c_2+12x^7c_2-x^8c_2+2xc_1c_2-18x^2c_1c_2+49x^3c_1c_2-60x^4c_1c_2+36x^5c_1c_2-10x^6c_1c_2+x^7c_1c_2+3xd_1c_2-20x^2d_1c_2+50x^3d_1c_2-60x^4d_1c_2+36x^5d_1c_2-10x^6d_1c_2+x^7d_1c_2-5x^2c_1d_1c_2+18x^3c_1d_1c_2-20x^4c_1d_1c_2+8x^5c_1d_1c_2-x^6c_1d_1c_2-d_2+14xd_2-64x^2d_2+144x^3d_2-182x^4d_2+134x^5d_2-56x^6d_2+12x^7d_2-x^8d_2+3xc_1d_2-20x^2c_1d_2+50x^3c_1d_2-60x^4c_1d_2+36x^5c_1d_2-10x^6c_1d_2+x^7c_1d_2+2xd_1d_2-18x^2d_1d_2+49x^3d_1d_2-60x^4d_1d_2+36x^5d_1d_2-10x^6d_1d_2+x^7d_1d_2-5x^2c_1d_1d_2+18x^3c_1d_1d_2-20x^4c_1d_1d_2+8x^5c_1d_1d_2-x^6c_1d_1d_2+3xc_2d_2-20x^2c_2d_2+50x^3c_2d_2-60x^4c_2d_2+36x^5c_2d_2-10x^6c_2d_2+x^7c_2d_2-5x^2c_1c_2d_2+18x^3c_1c_2d_2-20x^4c_1c_2d_2+8x^5c_1c_2d_2-x^6c_1c_2d_2-5x^2d_1c_2d_2+18x^3d_1c_2d_2-20x^4d_1c_2d_2+8x^5d_1c_2d_2-x^6d_1c_2d_2+8x^3c_1d_1c_2d_2-6x^4c_1d_1c_2d_2+x^5c_1d_1c_2d_2)$;

(iii) $\chi(T_{n_1}(c_1,d_1,d_2,d_3)) = (-1+x)^{(n-10)}x(-6+52x-194x^2+409x^3-536x^4+451x^5-242x^6+79x^7-14x^8+x^9-c_1+15xc_1-67x^2c_1+146x^3c_1-180x^4c_1+131x^5c_1-55x^6c_1+12x^7c_1-x^8c_1-d_1+13xd_1-59x^2d_1+134x^3d_1-172x^4d_1+129x^5d_1-55x^6d_1+12x^7d_1-x^8d_1+3xc_1d_1-20x^2c_1d_1+49x^3c_1d_1-58x^4c_1d_1+35x^5c_1d_1-10x^6c_1d_1+x^7c_1d_1-d_2+13xd_2-59x^2d_2+134x^3d_2-172x^4d_2+129x^5d_2-55x^6d_2+12x^7d_2-x^8d_2+3xc_1d_2-20x^2c_1d_2+49x^3c_1d_2-58x^4c_1d_2+35x^5c_1d_2-10x^6c_1d_2+x^7c_1d_2+2xd_1d_2-16x^2d_1d_2+44x^3d_1d_2-56x^4d_1d_2+35x^5d_1d_2-10x^6d_1d_2+x^7d_1d_2-5x^2c_1d_1d_2+17x^3c_1d_1d_2-19x^4c_1d_1d_2+8x^5c_1d_1d_2-x^6c_1d_1d_2-d_3+13xd_3-59x^2d_3+134x^3d_3-172x^4d_3+129x^5d_3-55x^6d_3+12x^7d_3-x^8d_3+3xc_1d_3-20x^2c_1d_3+49x^3c_1d_3-58x^4c_1d_3+35x^5c_1d_3-10x^6c_1d_3+x^7c_1d_3+2xd_1d_3-16x^2d_1d_3+44x^3d_1d_3-56x^4d_1d_3+35x^5d_1d_3-10x^6d_1d_3+x^7d_1d_3-5x^2c_1d_1d_3+17x^3c_1d_1d_3-19x^4c_1d_1d_3+8x^5c_1d_1d_3-x^6c_1d_1d_3+2xd_2d_3-16x^2d_2d_3+44x^3d_2d_3-$

$56x^4d_2d_3 + 35x^5d_2d_3 - 10x^6d_2d_3 + x^7d_2d_3 - 5x^2c_1d_2d_3 + 17x^3c_1d_2d_3 - 19x^4c_1d_2d_3 + 8x^5c_1d_2d_3 - x^6c_1d_2d_3 - 3x^2d_1d_2d_3 + 15x^3d_1d_2d_3 - 19x^4d_1d_2d_3 + 8x^5d_1d_2d_3 - x^6d_1d_2d_3 + 7x^3c_1d_1d_2d_3 - 6x^4c_1d_1d_2d_3 + x^5c_1d_1d_2d_3)$;

Initially, it is important to acknowledge that Trevisan et al. [25], asserted the validity of the inequality $LE(P_n) \leq LE(T_n)$ for n vertices in the context of any tree T_n, specifically for cases where $n \leq 18$. In the course of our proof, we can safely suppose that $n \geq 19$. Consider the case $T_{n_1}(c_1, c_2, d_1)$ since, $(T_{n_1}(c_1, c_2, d_1))$ has exactly 4 LEVs greater than $\hat{d} = 2 - \frac{2}{n}$, so from Eq. 2, we see that

$$\begin{aligned} LE(T_{n_1}(c_1, c_2, d_1)) &= |x_1 - \hat{d}| + |x_2 - \hat{d}| + |x_3 - \hat{d}| + |x_4 - \hat{d}| + (n-7)|1 - \hat{d}| + |x_{n-2} - \hat{d}| \\ &\quad + |x_{n-1} - \hat{d}| + |x_n - \hat{d}| \\ &= \hat{d} + (n-7)(\hat{d} - 1) + (x_1 - \hat{d}) + (x_2 - \hat{d}) + (x_3 - \hat{d}) + (\hat{d} - x_4) \\ &\quad + (\hat{d} - x_{n-2}) + (\hat{d} - x_{n-1}) \\ &= \hat{d} + (n-7)(\hat{d} - 1) + (x_1 + x_2 + x_3) - (x_4 + x_{n-2} + x_{n-1}). \end{aligned} \tag{5}$$

Through the characteristic polynomial of $(T_{n_1}(c_1, c_2, d_1))$ and Eq. 3, we observe that $x_1 + x_2 + x_3 + x_4 + x_{n-2} + x_{n-1} = n + 5$ and substituting $\hat{d} = 2 - \frac{2}{n}$, Eq. 5 implies that:

$$LE(T_{n_1}(c_1, c_2, d_1)) = 2n + \frac{10}{n} - 4 - 2(x_4 + x_{n-2} + x_{n-1}). \tag{6}$$

Using Lemma 2.4 and Eq. 6, the above equation becomes:

$$LE(P_n) - LE(T_{n_1}(c_1, c_2, d_1)) \leq n(\frac{4}{\pi} - 2) - \frac{10}{n} + 4 + 2(x_4 + x_{n-2} + x_{n-1}).$$

Since $x_4 < 2, \quad x_{n-2}, x_{n-1} < 1$ and $\frac{10}{n} > 0$, so it follows;

$$LE(P_n) - LE(T_{n_1}(c_1, c_2, d_1)) \leq n(\frac{4}{\pi} - 2) - \frac{10}{n} + 10,$$

provided that $\frac{4n}{\pi} + 2 \leq 2n + \frac{10}{n} - 8$. which is negative for $n > 18$. In similar steps, the same results can be proven for remaining cases.

Let T_{n_2} be the second class of multi-broom graph of diameter 5 on n vertices having 2 sub-classes as shown in Figures 5 and 6 respectively. The Laplacian characteristic polynomials of all 2

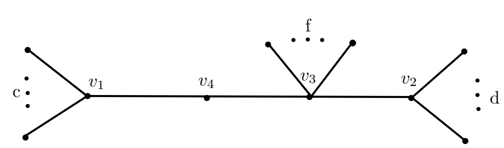

Figure 5. (i)

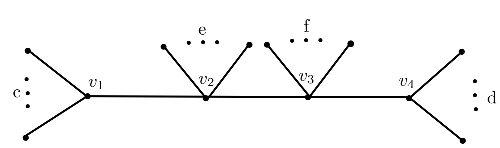

Figure 6. (ii)

sub-classes of T_{n_2} are as follows:

(i) $\chi(T_{n_2}(c,d,f)) = (-1+x)^{(n-7)}x(4-22x+48x^2-53x^3+31x^4-9x^5+x^6+c-8xc+18x^2c-17x^3c+7x^4c-x^5c+d-8xd+18x^2d-17x^3d+7x^4d-x^5d-3xcd+7x^2cd-5x^3cd+x^4cd+f-6xf+13x^2f-13x^3f+6x^4f-x^5f-2xcf+5x^2cf-4x^3cf+x^4cf-xdf+4x^2df-4x^3df+x^4df+2x^2cdf-x^3cdf)$;

(ii) $\chi(T_{n_2}(c,d,e,f)) = (-1+x)^{(n-8)}x(-4+26x-70x^2+101x^3-84x^4+40x^5-10x^6+x^7-c+9xc-26x^2c+35x^3c-24x^4c+8x^5c-x^6c-d+9xd-26x^2d+35x^3d-24x^4d+8x^5d-x^6d+3xcd-10x^2cd+12x^3cd-6x^4cd+x^5cd-e+7xe-19x^2e+26x^3e-19x^4e+7x^5e-x^6e+xce-5x^2ce+8x^3ce-5x^4ce+x^5ce+2xde-7x^2de+9x^3de-5x^4de+x^5de-2x^2cde+3x^3cde-x^4cde-f+7xf-19x^2f+26x^3f-19x^4f+7x^5f-x^6f+2xcf-7x^2cf+9x^3cf-5x^4cf+x^5cf+xdf-5x^2df+8x^3df-5x^4df+x^5df-2x^2cdf+3x^3cdf-x^4cdf+xef-4x^2ef+6x^3ef-4x^4ef+x^5ef-x^2cef+2x^3cef-x^4cef-x^2def+2x^3def-x^4def+x^3cdef)$;

The Algorithm (2) depicted in the previously mentioned section 2 is applied here. It is clear from Figures 5 and 6 that $(T_{n_2}(c,d,f))$ and $(T_{n_2}(c,d,e,f))$ have internal vertices $s=4$, respectively. Therefore, through the support of of part (iv) of Corollary 2.1, the upcoming expression

$$LE(T) \geq \frac{4n}{\pi} + 2$$

consistently valid, provided that it is also valid for this particular Scenario.

Now, assume the second case, where $T \in T_n(5)$ is a tree with root vertex v_0 of order $n > 18$ having $p \geq 0$ pendent vertices of stage 0 and $k \geq 2$ vertices of stage 1 provided that each v_k has s_k pendent vertices and at least two s_j of them are non-zero. while v_{k+1} a vertex of stage 2 has t pendent vertices as shown in Figure 7. Clearly, the order n of T in this case is

$$n = p + k + t + 1 + \sum_{j=1}^{k} s_j.$$

We verify Conjecture on T by applying induction on k. If $k=2$ then T has obviously $s=3$ internal vertices, Corollary 2.1 (iv) applied here, thus the solution is valid. Let us assume that the outcome is valid for all trees having $k=h+1$ vertices. 1 vertex of stage 2 and h vertices of stage 1. It is demonstrated that the result is true for trees with ($h+2$ vertices, 1 vertex of stage 2 and $h+1$ of stage 1). Given a tree T having $k=h+1$ vertices $v_1, v_2, \ldots, v_h$ of stage 1 and v_{h+1} of stage 2. Assume that $e = v_h v_{h+1}$ is the edge between the vertex v_{h+1} and v_h. Remove the edge e, and designate the elements of $T-e$ as T_1 and T_2. Let $j=1,2$, with $n_1 \geq n_2$, be the order of T_j and $\hat{d}(T_j) = 2 - \frac{2}{n_j}$ be the average vertex degree of T_j. Clearly, T_1 is tree with order n_1 and $k=h$ vertices, $v_1, v_2, \ldots, v_h$ of stage 1 as shown in Figure 8, since $k \geq 3$. On the other hand, T_2 is a star $K_{v_{h+1},t}$ as shown in Figure 9. The inequality 4 holds for both T_1 and T_2, according to the induction hypothesis. If $\hat{d}(T_j) = 2 - \frac{2}{n_j}$, then σ_j represents the count of Laplace eigenvalues of tree T_j that are higher than or equal $\hat{d}$. $\sigma_2 = 1$ follows from T_2 is the star graph having at least two vertices, and for any star graph, With the exception of the spectral radius, all non-zero LEVs are equal to 1. Now to compute σ_1, we apply Algorithm (1) with $\alpha = -2 + \frac{2}{n_1}$ to the tree T_1. Let the pendant vertices in T_1 be represented by w. We have

$$a(w) = d(w) + \alpha = 1 - 2 + \frac{2}{n_1} = -\frac{n_1 - 2}{n_1} < 0,$$

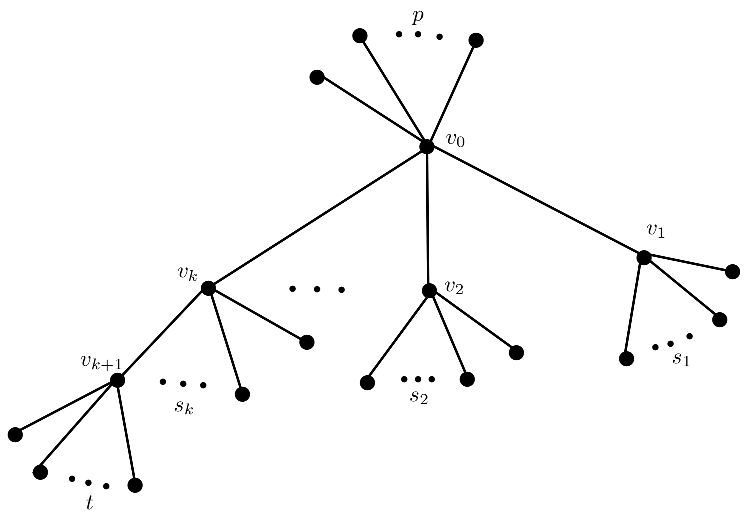

Figure 7. An SNS tree T of diameter 5

as $n_1 \geq 4$.This indicates that each pendant vertex's diagonal entries in the resulting diagonal matrix are negative. Regarding the vertices of stage 1, v_i, $1 \leq j \leq h$, we have

$$a(v_j) = s_j + 1 - 2 + \frac{2}{n_1} - \frac{s_j}{a(w)} = 2s_j - 1 + \frac{2s_j}{n_1 - 2} - \frac{2}{n_1} > 0,$$

indicating that each vertex v_j of stage 1 has a diagonal entry in the resulting diagonal matrix that is positive. $\sigma_1 = h$ or $h + 1$, depending on whether $a(v_0) < 0$ or $a(v_0) > 0$, is evident from Lemma 2.3. We have for the root vertex v_0,

$$a(v_0) = h + p - 2 + \frac{2}{n_1} + \frac{p(n_1 - 2)}{n_1} - \sum_{j=1}^{h} \frac{1}{a(v_j)}.$$

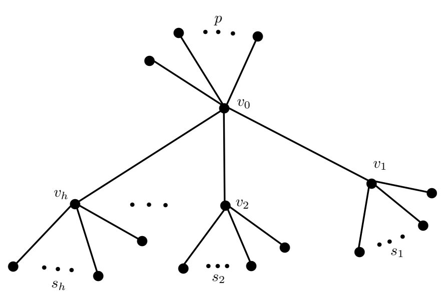

Figure 8. T_1

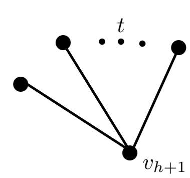

Figure 9. T_2

Lemma 2.3 indicates that $\sigma_1 = h + 1$ if $p \geq 1$, which is easily observed. Hence, let us assume that $p = 0$ in T_1. Lemma 2.3 states that $\sigma_1 = h + 1$ also holds true in the scenario where at least three of s_i are basically higher than or equal to 2. This is proven by the observation that $a(v_0) > 0$.

The induction process is thus completed for T with $r = h + 1$ (h vertices of stage 1 and 1 vertex of stage 2) vertices. Consequently, we have $\sigma_1 = h + 1$ non-pendent vertices for the tree T_1. The LEVs of T_1 higher or equal to $d(T_1 \cup T_2) = 2 - \frac{4}{n}$ can be denoted by h_1. Then, $h_1 \geq \sigma_1$, since $d(T_1) \geq d(T_1 \cup T_2)$. As we assert, $h_1 = \sigma_1$. It follows that $h_1 \leq h + 1$ since $\sigma_1 = h + 1$. Using algorithm (II) on the tree T_1, we obtain $\alpha = -2 + \frac{4}{n}$.

$$a(w) = d(w) + \alpha = 1 - 2 + \frac{4}{n} = -\frac{n-4}{n} < 0,$$

for all pendant vertices w of T_1. Accordingly, $h_1 \geq h + 1$ therefore, $h_1 = h + 1 = \sigma_1$ must exist, supporting the assertion in this instance. Moreover, let h_2 represent the count of LEVs for the tree T_2 that not exceed $d(T_1 \cup T_2) = 2 - \frac{4}{n}$. $h_2 = \sigma_2 = 1$ follows since $\sigma_2 = 1$ and T_2 have at least one edge. We have thereby shown that the inequality 4 holds for the components T_1 and T_2 of $T - e$ and that they meet the characteristic that $h_1 = \sigma_1$ and $h_2 = \sigma_2$. Using Corollary 2.2, we may deduce that inequality 4 also valid for T. Thus, we conclude that the result holds for every $k \geq 2$ with the help of induction. The proof of the Conjecture 1 is now completed for all classes of diameter 5 trees. □

4. Conclusion

We partitioned a family of diameter 5 tree classes into subclasses to test Conjecture 1's validity. Through established results and, in some cases, induction hypotheses, we confirmed Conjecture 1: the LE of such family of diameter 5 trees $T_n(5)$ falls between that of the star graph S_n and the path graph P_n.

5. Authors Contribution Statement

All the authors contributed equally to this article.

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