



Graceful labeling of zero-divisor graph $\Gamma(\mathbb{Z}_{p^2q})$ and $\Gamma(\mathbb{Z}_{p^2q})$

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Abstract

Some papers have already provided graceful labeling for some types of zero-divisor graphs. We reviewed the graceful labeling results of $\Gamma(\mathbb{Z}_{25})$, $\Gamma(\mathbb{Z}_8)$, and $\Gamma(\mathbb{Z}_{27})$, then use those results to label zero-divisors graphs $\Gamma(\mathbb{Z}_{25q})$, $\Gamma(\mathbb{Z}_{8q})$, and $\Gamma(\mathbb{Z}_{27q})$. The result is that there is graceful labeling for $\Gamma(\mathbb{Z}_{p^2q})$ for $p = 5$ and $\Gamma(\mathbb{Z}_{p^3q})$ for $p = 2, 3$, where q is prime number that is different from p . In this paper, we provide the graceful labeling of zero-divisor graph $\Gamma(\mathbb{Z}_{p^2q})$ and $\Gamma(\mathbb{Z}_{p^3q})$ with adaptation and modification of existing results.

Keywords: graceful labeling, zero-divisor, graph, congruence classes

Mathematics Subject Classification : 05C25

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1. Introduction

The concept of zero divisor graph is based on a term in algebra, namely zero-divisor.

Definition 1. [1] *An element a in a commutative ring R is called a zero-divisor if there is an element $b \in R$ and $b \neq 0$ such that $ab = 0$. The set of all zero-divisors of R is denoted by $Z(R)$.*

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The definition of zero-divisor graph was first introduced by Istvan Beck in 1988 when he was studying the coloring of several graphs. Beck proposed a conjecture, which states that the clique number of $\Gamma_0(R)$ is equal to the chromatic number of $\Gamma_0(R)$ [2]. In 1993, D. D. Anderson and M. Naseer [3] disproved the Beck's conjecture by providing a counterexample. Consequently, a new definition of zero-divisor graph was obtained. This zero-divisor graph becomes known as the Anderson-Livingston Zero-divisor Graph [4]. The following are some results for definition of zero-divisor graph.

Definition 2 (Beck's Zero-divisor Graph). [2] *Let R be a commutative ring with identity. The zero-divisor graph of R , denoted by $\Gamma_0(R)$, is a simple graph whose vertices are all the elements of R such that two distinct elements x and y are adjacent if and only if $xy = 0$.*

Definition 3 (Anderson-Livingston's Zero-divisor Graph). [4] *Let R be a commutative ring with identity and $Z(R)$ its set of zero-divisors. The zero-divisor graph of R , denoted by $\Gamma(R)$, is a simple graph with vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $xy = 0$.*

As we see above, the definition of $\Gamma(R)$ is a better illustration of the zero-divisor structure of R . In this paper, we use different notation to represent the edges of a graph. If vertices x and y are adjacent, then the edge connecting x and y is denoted by $\{x, y\}$. This is done to distinguish it from the multiplication of ring elements.

Graceful labeling is one of the many labelings mentioned by Gallian [5], defined as follows. Given $f : V(G) \rightarrow \{0, 1, \dots, m\}$ and $g : E(G) \rightarrow \{1, 2, \dots, m\}$. A graph G with size m is said to be graceful if the function f is an injection from the vertices of G to the set $\{0, 1, \dots, m\}$ such that for each edge $\{x, y\} \in E(G)$, it holds that $g(\{x, y\}) = |f(x) - f(y)|$ with g being a bijective function. The function f represents a vertex label and function g represents an edge label.

Several researchers have presented their findings on the graceful labeling of zero-divisor graphs $\Gamma(\mathbb{Z}_n)$ when $n = pq$, $n = 4q$, or $n = 9q$ [6], and some n that have maximum 14 vertices [7]. In this paper, we provide the graceful labeling of zero-divisor graph $\Gamma(\mathbb{Z}_{p^2q})$ when $p = 5$ and $\Gamma(\mathbb{Z}_{p^3q})$ when $p = 2, 3$, with q is a different prime number from p .

Another aspect to note is the adjacency illustration of the zero divisor graph. Consider the following figure.

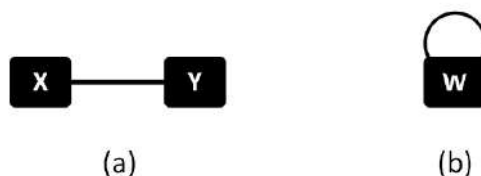


Figure 1. The adjacency illustration

From the two forms above, we have the following results. If two different partitions are connected by an edge as in Figure 1 part (a), then partitions X and Y will form a complete bipartite graph.

Therefore, the number of edges in this form is $|X||Y|$. Else, if a partition is connected by a 'loop' as in Figure 1 part (b), then partition W will form a complete graph. Therefore, the number of edges in this form is $\binom{|W|}{2}$.

2. Preliminaries

First of all, we make a notation for partition in graph. The partition containing a will be denoted by $[a]$. Some research results that will be useful for obtaining graceful labeling $\Gamma(\mathbb{Z}_{p^2q})$ when $p = 5$ and $\Gamma(\mathbb{Z}_{p^3q})$ when $p = 2, 3$ are $\Gamma(\mathbb{Z}_{25})$, $\Gamma(\mathbb{Z}_8)$, and $\Gamma(\mathbb{Z}_{27})$. First, consider the zero-divisor graph $\Gamma(\mathbb{Z}_{25})$. We know that $V(\Gamma(\mathbb{Z}_{25})) = \{5, 10, 15, 20\}$, so $|V(\Gamma(\mathbb{Z}_{25}))| = 4$. We also know that this type of zero-divisor graph is complete graph. S.W. Golomb [8] proposed a theorem stating that if $n > 4$, the complete graph K_n cannot be labeled gracefully. Therefore, we get graceful labeling for zero divisor graph $\Gamma(\mathbb{Z}_{25})$ as shown below (Figure 2).

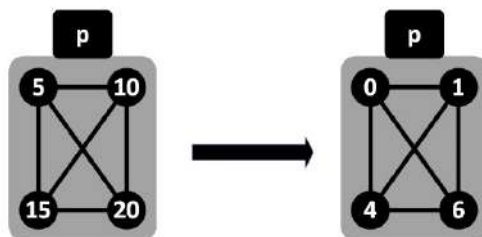


Figure 2. (Left) Zero-divisor graph $\Gamma(\mathbb{Z}_{25})$ and (Right) graceful labeling of $\Gamma(\mathbb{Z}_{25})$

Consider that from the labeling above, we can get edge label from 1 to 6 according to the definition of graceful labeling.

Next, consider the zero divisor graph $\Gamma(\mathbb{Z}_8)$. We know that $V(\Gamma(\mathbb{Z}_8)) = \{2, 4, 6\}$ and we can divide it into two partitions, those are $[2] = \{2, 6\}$ and $[4] = \{4\}$. So, we get graceful labeling for zero-divisor graph $\Gamma(\mathbb{Z}_8)$ as shown below (Figure 3) (ref: [7]).

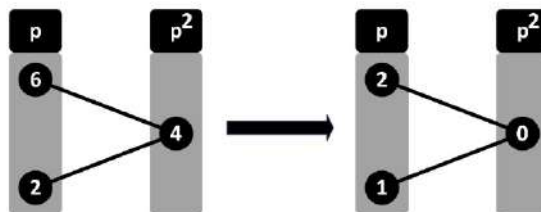


Figure 3. (Left) Zero-divisor graph $\Gamma(\mathbb{Z}_8)$ and (Right) graceful labeling of $\Gamma(\mathbb{Z}_8)$

Last, consider the zero-divisor graph $\Gamma(\mathbb{Z}_{27})$. We know that $V(\Gamma(\mathbb{Z}_{27})) = \{3, 6, 9, 12, 15, 18, 21, 24\}$ and we can divide it into two partitions, those are $[3] = \{3, 6, 12, 15, 21, 24\}$ and $[9] = \{9, 18\}$. So, we get graceful labeling for zero-divisor graph $\Gamma(\mathbb{Z}_{27})$ as shown below (Figure 4) (ref: [7], [6]).

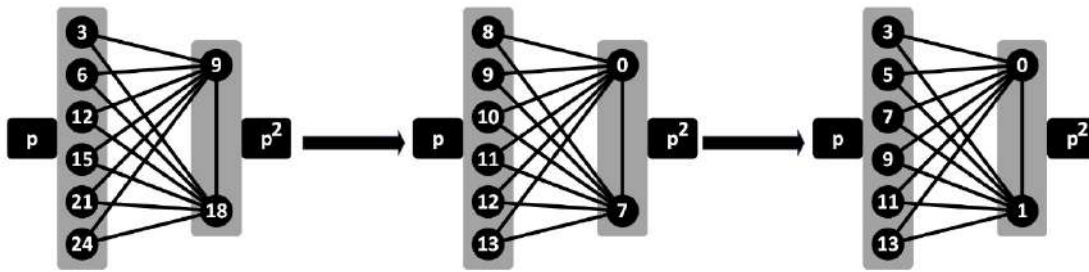


Figure 4. (Left) Zero-divisor graph $\Gamma(\mathbb{Z}_{27})$, (Center) graceful labeling of $\Gamma(\mathbb{Z}_{27})$, and (Right) modification graceful labeling of $\Gamma(\mathbb{Z}_{27})$

The reason for modification is that we want to assign smaller edge labels to the complete subgraph in partition $[p^2]$. This modification can help us to get graceful labeling for zero-divisor graph $\Gamma(\mathbb{Z}_{27q})$.

3. Main Results

There are three theorems that we obtained. We have these theorems due to some researches, modifications, and adaptations from existing results.

Theorem 3.1. *The zero divisor-graph $\Gamma(\mathbb{Z}_n)$, where $n = 25q$, with $q \neq 5$, q is prime, is graceful.*

Proof. For zero-divisor graph $\Gamma(\mathbb{Z}_n)$, where $n = 25q$, with $q \neq 5$, q is prime, there are four partitions, those are $[5]$, $[25]$, $[q]$, and $[5q]$. We make the sets that can represent those partitions, such as the set $W = \{5q, 10q, 15q, 20q\}$ is representing the partition $[5q]$, the set $B = \{25, 50, \dots, 25(q-1)\}$ is representing the partition $[25]$, the set $V = \{q, 2q, \dots, 24q\} \setminus W$ is representing the partition $[q]$, and the set $A = \{5, 10, \dots, 5(5q-1)\} \setminus (B \cup W)$ is representing the partition $[5]$. From the explanation, we know that the orders of the sets are $|A| = 4(q-1)$, $|W| = 4$, $|B| = q-1$, and $|V| = 20$. If we consider the adjacency property of each partition, we have the following result (Figure 5).



Figure 5. The adjacency of the zero-divisor graph $\Gamma(\mathbb{Z}_{25q})$

From Figure 5, it is found that the number of edges for each line in the image is shown in Table 1.

Edges	Total of edges
$A - W$	$16(q - 1)$
$B - W$	$4(q - 1)$
$B - V$	$20(q - 1)$
$W - W$	$\binom{4}{2} = 6$
$E \Gamma(\mathbb{Z}_{25q}) $	$40q - 34$

Table 1. Total of edges of zero-divisor graph $\Gamma(\mathbb{Z}_{25q})$

Consider that $\Gamma(\mathbb{Z}_{25})$ has 4 vertices from partition [5] (the only partition from this zero-divisor graph) and the subgraph of zero-divisor graph $\Gamma(\mathbb{Z}_{25q})$, that is partition [5q], have the same number of vertices. Therefore, we can copy the graph of $\Gamma(\mathbb{Z}_{25})$ to a subgraph of $\Gamma(\mathbb{Z}_{25q})$ in partition [5q]. However, in this type of graph, we have to modify the labeling to get graceful labeling.

Now, we define the labeling function $f : V(\Gamma(\mathbb{Z}_{25q})) \rightarrow \{0, 1, \dots, m\}$, where $m = 40q - 34$ as follows:

- For partition [5] or set A , applies

$$f(a_i) = \begin{cases} 40 \left\lfloor \frac{i+3}{4} \right\rfloor - 32 \equiv 8(\text{mod } 40), & \text{when } i \equiv 1(\text{mod } 4); \\ 40 \left\lfloor \frac{i+3}{4} \right\rfloor - 24 \equiv 16(\text{mod } 40), & \text{when } i \equiv 2(\text{mod } 4); \\ 40 \left\lfloor \frac{i+3}{4} \right\rfloor - 16 \equiv 24(\text{mod } 40), & \text{when } i \equiv 3(\text{mod } 4); \\ 40 \left\lfloor \frac{i+3}{4} \right\rfloor - 8 \equiv 32(\text{mod } 40), & \text{when } i \equiv 0(\text{mod } 4); \end{cases}$$

where $i = 1, \dots, 4(q - 1)$.

- For partition [5q] or set W , applies

$$f(w_\epsilon) = \begin{cases} 39 + 40(q - 2) \equiv 39(\text{mod } 40), & \text{when } \epsilon = 1; \\ 40 + 40(q - 2) \equiv 0(\text{mod } 40), & \text{when } \epsilon = 2; \\ 43 + 40(q - 2) \equiv 3(\text{mod } 40), & \text{when } \epsilon = 3; \\ 45 + 40(q - 2) \equiv 5(\text{mod } 40), & \text{when } \epsilon = 4. \end{cases}$$

- For partition [25] or set B , applies $f(b_j) = 4(j - 1) \equiv 0(\text{mod } 40)$ where $j = 1, \dots, (q - 1)$.
- For partition [q] or set V , applies

$$f(v_\delta) = \begin{cases} 40(q - 2) + 8 \left\lfloor \frac{\delta+3}{4} \right\rfloor + 1 \equiv 1, 9, 17, 25, 33(\text{mod } 40), & \text{when } \delta \equiv 1(\text{mod } 4); \\ 40(q - 2) + 8 \left\lfloor \frac{\delta+3}{4} \right\rfloor + 2 \equiv 2, 10, 18, 26, 34(\text{mod } 40), & \text{when } \delta \equiv 2(\text{mod } 4); \\ 40(q - 2) + 8 \left\lfloor \frac{\delta+3}{4} \right\rfloor + 4 \equiv 4, 12, 20, 28, 36(\text{mod } 40), & \text{when } \delta \equiv 3(\text{mod } 4); \\ 40(q - 2) + 8 \left\lfloor \frac{\delta+3}{4} \right\rfloor + 6 \equiv 6, 14, 22, 30, 38(\text{mod } 40), & \text{when } \delta \equiv 0(\text{mod } 4); \end{cases}$$

where $\delta = 1, \dots, 20$.

By using graceful labeling rules and modulo operations, the following results are obtained (Table 2).

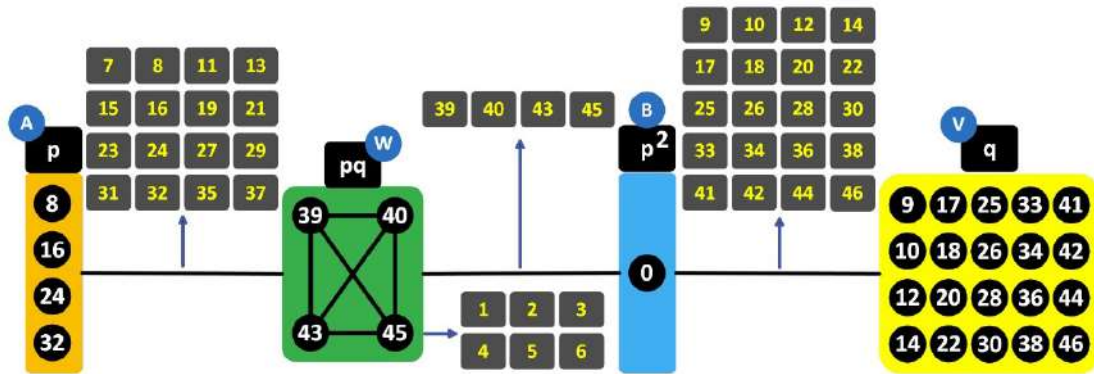
Mod	Label	Mod	Label	Mod	Label
0	$g(\{w_\epsilon, b_j\})$	14	$g(\{b_j, v_\delta\})$	27	$g(\{a_i, w_\epsilon\})$
1	$g(\{w_{\epsilon_1}, w_{\epsilon_2}\}), g(\{b_j, v_\delta\})$	15	$g(\{a_i, w_\epsilon\})$	28	$g(\{b_j, v_\delta\})$
2	$g(\{w_{\epsilon_1}, w_{\epsilon_2}\}), g(\{b_j, v_\delta\})$	16	$g(\{a_i, w_\epsilon\})$	29	$g(\{a_i, w_\epsilon\})$
3	$g(\{w_{\epsilon_1}, w_{\epsilon_2}\}), g(\{w_\epsilon, b_j\})$	17	$g(\{b_j, v_\delta\})$	30	$g(\{b_j, v_\delta\})$
4	$g(\{w_{\epsilon_1}, w_{\epsilon_2}\}), g(\{b_j, v_\delta\})$	18	$g(\{b_j, v_\delta\})$	31	$g(\{a_i, w_\epsilon\})$
5	$g(\{w_{\epsilon_1}, w_{\epsilon_2}\}), g(\{w_\epsilon, b_j\})$	19	$g(\{a_i, w_\epsilon\})$	32	$g(\{a_i, w_\epsilon\})$
6	$g(\{w_{\epsilon_1}, w_{\epsilon_2}\}), g(\{b_j, v_\delta\})$	20	$g(\{b_j, v_\delta\})$	33	$g(\{b_j, v_\delta\})$
7	$g(\{a_i, w_\epsilon\})$	21	$g(\{a_i, w_\epsilon\})$	34	$g(\{b_j, v_\delta\})$
8	$g(\{a_i, w_\epsilon\})$	22	$g(\{b_j, v_\delta\})$	35	$g(\{a_i, w_\epsilon\})$
9	$g(\{b_j, v_\delta\})$	23	$g(\{a_i, w_\epsilon\})$	36	$g(\{b_j, v_\delta\})$
10	$g(\{b_j, v_\delta\})$	24	$g(\{a_i, w_\epsilon\})$	37	$g(\{a_i, w_\epsilon\})$
11	$g(\{a_i, w_\epsilon\})$	25	$g(\{b_j, v_\delta\})$	38	$g(\{b_j, v_\delta\})$
12	$g(\{b_j, v_\delta\})$	26	$g(\{b_j, v_\delta\})$	39	$g(\{w_\epsilon, b_j\})$
13	$g(\{a_i, w_\epsilon\})$				

Table 2. Modulo 40 congruence of each edges in $\Gamma(\mathbb{Z}_{25q})$

Label	Minimum	Maximum
$g(\{a_i, w_\epsilon\})$	7	$40q - 43$
$g(\{w_{\epsilon_1}, w_{\epsilon_2}\})$	1	6
$g(\{w_\epsilon, b_j\})$	39	$40q - 35$
$g(\{b_j, v_\delta\})$	9	$40q - 34 = m$

Table 3. Minimum and maximum edges label for zero-divisor graph $\Gamma(\mathbb{Z}_{25q})$

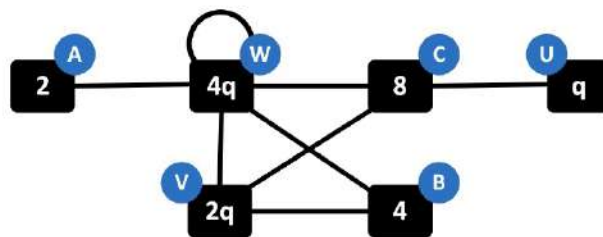
From Tables 2 and 3, it is obtained that for congruence $1, 2, 3, 4, 5, 6 \pmod{40}$, it holds that $g(\{w_{\epsilon_1}, w_{\epsilon_2}\}) \leq 6$ while the value of $g(\{b_j, v_\delta\}) \geq 9$ and $g(\{w_\epsilon, b_j\}) \geq 39$, so the two sides must have different values. It can be seen from Table 3 that $\min\{g(\{\bar{u}, \bar{v}\})\} = 1$ and $\max\{g(\{\bar{u}, \bar{v}\})\} = m$, which means that function g is bijective. Since $g(\{\bar{u}, \bar{v}\})$ is bijective for any $\{\bar{u}, \bar{v}\} \in E(\Gamma(\mathbb{Z}_{25q}))$, according to the definition of graceful labeling, the graph $\Gamma(\mathbb{Z}_{25q})$ with $q \neq 5$ for q prime can be labeled gracefully.


 Figure 6. Zero-divisor graph $\Gamma(\mathbb{Z}_{50})$

□

Theorem 3.2. The zero divisor-graph $\Gamma(\mathbb{Z}_n)$, where $n = 8q$, with $q \neq 2$, q is prime, is graceful.

Proof. For zero divisor graph $\Gamma(\mathbb{Z}_n)$, where $n = 8q$, with $q \neq 2$, q is prime, there are six partitions, those are $[2]$, $[4]$, $[8]$, $[q]$, $[2q]$, and $[4q]$. We make sets that can represent those partitions, such as the set $W = \{4q\}$ is representing the partition $[4q]$, the set $C = \{8, 16, \dots, 8(q-1)\}$ is representing the partition $[8]$, the set $V = \{2q, 6q\}$ is representing the partition $[2q]$, the set $B = \{4, 8, \dots, 4(2q-1)\} \setminus (C \cup W)$ is representing the partition $[4]$, the set $U = \{q, 3q, 5q, 7q\}$ is representing the partition $[q]$, and the set $A = \{2, 4, \dots, 2(4q-1)\} \setminus (B \cup C \cup V \cup W)$ is representing the partition $[2]$. From the explanation, we know that the order of every set is $|A| = 2(q-1)$, $|W| = 1$, $|C| = (q-1)$, $|U| = 4$, $|V| = 2$, and $|B| = (q-1)$. If we consider the adjacency property of each partition, we have the following result (Figure 7).


 Figure 7. The adjacency of the zero-divisor graph $\Gamma(\mathbb{Z}_{8q})$

From Figure 7, it is found that the number of edges for each line in the image is shown in Table 4.

Edges	Total of edges	Edges	Total of edges
$A - W$	$2(q - 1)$	$C - W$	$(q - 1)$
$C - U$	$4(q - 1)$	$B - W$	$(q - 1)$
$B - V$	$2(q - 1)$	$V - W$	2
$C - V$	$2(q - 1)$	$E \Gamma(\mathbb{Z}_{8q}) $	$12q - 10$

Table 4. Total of edges of zero-divisor graph $\Gamma(\mathbb{Z}_{8q})$

Consider that graph $\Gamma(\mathbb{Z}_8)$ can be copied to the subgraph of $\Gamma(\mathbb{Z}_{8q})$ in partition $[2q]$ and $[4q]$. Now we define the labeling function $f : V(\Gamma(\mathbb{Z}_{8q})) \rightarrow \{0, 1, 2, \dots, m\}$, where $m = 12q - 10$ as follows:

- For partition $[2]$ or set A , applies

$$f(a_i) = \begin{cases} 12 \lfloor \frac{i+1}{2} \rfloor + 1 \equiv 1 \pmod{12}, & \text{when } i \text{ is odd;} \\ 12 \lfloor \frac{i+1}{2} \rfloor + 2 \equiv 2 \pmod{12}, & \text{when } i \text{ is even;} \end{cases}$$

where $i = 1, \dots, 2(q - 1)$.

- For partition $[4q]$ or set W , applies $f(w_1) = 0$.
- For partition $[8]$ or set C , applies $f(c_k) = 12k \equiv 0 \pmod{12}$ where $k = 1, \dots, (q - 1)$.
- For partition $[q]$ or set U , applies

$$f(u_\gamma) = \begin{cases} 3 \equiv 3 \pmod{12}, & \text{when } \gamma = 1; \\ 4 \equiv 4 \pmod{12}, & \text{when } \gamma = 2; \\ 5 \equiv 5 \pmod{12}, & \text{when } \gamma = 3; \\ 9 \equiv 9 \pmod{12}, & \text{when } \gamma = 4. \end{cases}$$

- For partition $[2q]$ or set V , applies $f(v_\delta) = \delta \equiv 1, 2 \pmod{12}$ where $\delta = 1, 2$.
- For partition $[4]$ or set B , applies $f(b_j) = 12(j - 1) + 6 \equiv 6 \pmod{12}$ where $j = 1, \dots, (q - 1)$.

By using graceful labeling rules and modulo operations, the following results are obtained (Table 5).

Mod	Label	Mod	Label
0	$g(\{c_k, w_1\})$	6	$g(\{w_1, b_j\})$
1	$g(\{a_i, w_1\}), g(\{w_1, v_1\})$	7	$g(\{c_k, u_\gamma\})$
2	$g(\{a_i, w_1\}), g(\{w_1, v_2\})$	8	$g(\{c_k, u_\gamma\})$
3	$g(\{c_k, u_\gamma\})$	9	$g(\{c_k, u_\gamma\})$
4	$g(\{b_j, v_\delta\})$	10	$g(\{c_k, v_\delta\})$
5	$g(\{b_j, v_\delta\})$	11	$g(\{c_k, v_\delta\})$

Table 5. Modulo 12 congruence of each edges in $\Gamma(\mathbb{Z}_{8q})$

Label	Minimum	Maximum
$g(\{a_i, w_1\})$	13	$12q - 10 = m$
$g(\{w_1, c_k\})$	12	$12q - 12$
$g(\{c_k, u_\gamma\})$	3	$12q - 15$
$g(\{w_1, v_\delta\})$	1	2
$g(\{w_1, b_j\})$	6	$12q - 18$
$g(\{c_k, v_\delta\})$	10	$12q - 13$
$g(\{b_j, v_\delta\})$	4	$12q - 19$

Table 6. Minimum and maximum edges label for zero-divisor graph $\Gamma(\mathbb{Z}_{8q})$

From Table 5, it is obtained that for congruence $1, 2 \pmod{12}$, it holds that $g(\{a_i, w_1\}) \neq g(\{w_1, v_\delta\})$, because $f(a_i) \neq f(v_\delta)$. From Table 6, we also know that $g(\{a_i, w_1\}) \geq 13$ and $g(\{w_1, v_\delta\}) \leq 2$ for each $i = 1, \dots, 2(q-1)$ and $\delta = 1, 2$. Therefore, the two edges have different values.

It can be seen from Table 6 that $\min\{g(\{\bar{u}, \bar{v}\})\} = 1$ and $\max\{g(\{\bar{u}, \bar{v}\})\} = m$, which means that function g is bijective. Since $g(\{\bar{u}, \bar{v}\})$ is bijective for any $\{\bar{u}, \bar{v}\} \in E(\Gamma(\mathbb{Z}_{8q}))$, according to the definition of graceful labeling, the graph $\Gamma(\mathbb{Z}_{8q})$ with $q \neq 2$ for q prime can be labeled gracefully.

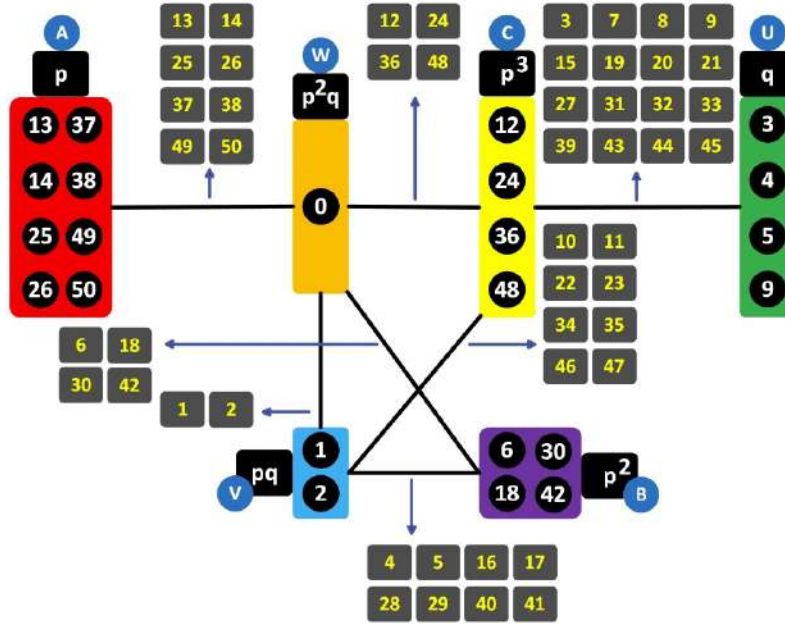


Figure 8. Zero-divisor graph $\Gamma(\mathbb{Z}_{40})$

□

Theorem 3.3. The zero divisor-graph $\Gamma(\mathbb{Z}_n)$, where $n = 27q$, with $q \neq 3$, q is prime, is graceful.

Proof. For zero-divisor graph $\Gamma(\mathbb{Z}_n)$, where $n = 27q$, with $q \neq 3$, q is prime, there are six partitions, those are $[3]$, $[9]$, $[27]$, $[q]$, $[3q]$, and $[9q]$. We make sets that can represent those partitions, such as the set $W = \{9q, 18q\}$ is representing the partition $[9q]$, the set $C = \{27, 54, \dots, 27(q-1)\}$ is representing the partition $[27]$, the set $V = \{3q, 6q, \dots, 24q\} \setminus W$ is representing the partition $[3q]$, the set $B = \{9, 18, \dots, 9(3q-1)\} \setminus (C \cup W)$ is representing the partition $[9]$, the set $U = \{q, 2q, \dots, 26q\} \setminus (V \cup W)$ is representing the partition $[q]$, and the set $A = \{3, 6, \dots, 3(9q-1)\} \setminus (B \cup C \cup V \cup W)$ is representing the partition $[3]$. From the explanation, we know that the order of every set is $|A| = 6(q-1)$, $|W| = 2$, $|C| = (q-1)$, $|U| = 18$, $|V| = 6$, and $|B| = 2(q-1)$. If we consider the adjacency property of each partition, we have the following result (Figure 9).

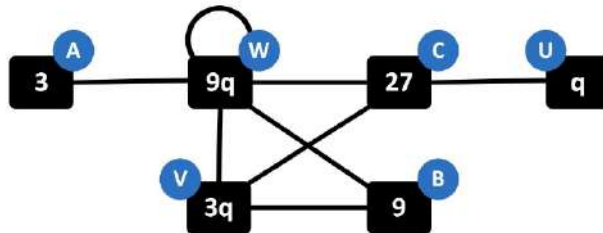


Figure 9. The adjacency of the zero-divisor graph $\Gamma(\mathbb{Z}_{27q})$

From Figure 9, it is found that the number of edges for each line in the image is shown in Table 7.

Edges	Total of edges	Edges	Total of edges
$A - W$	$12(q - 1)$	$C - W$	$2(q - 1)$
$C - U$	$18(q - 1)$	$B - W$	$4(q - 1)$
$B - V$	$12(q - 1)$	$V - W$	12
$C - V$	$6(q - 1)$	$W - W$	1
$E \Gamma(\mathbb{Z}_{27q}) $	$54q - 41$		

Table 7. Total of edges of zero-divisor graph $\Gamma(\mathbb{Z}_{27q})$

Consider that graph $\Gamma(\mathbb{Z}_{27})$ can be copied to the subgraph of $\Gamma(\mathbb{Z}_{27q})$ in partition $[3q]$ and $[9q]$. Now, we define the labeling function $f : V(\Gamma(\mathbb{Z}_{27q})) \rightarrow \{0, 1, 2, \dots, m\}$, where $m = 12q - 10$ as follows:

- For partition $[3]$ or set A , applies

$$f(a_i) = \begin{cases} 54 \left\lfloor \frac{i+1}{6} \right\rfloor + 29 \equiv 29 \pmod{54}, & \text{when } i \equiv 1 \pmod{6}; \\ 54 \left\lfloor \frac{i+1}{6} \right\rfloor + 31 \equiv 31 \pmod{54}, & \text{when } i \equiv 2 \pmod{6}; \\ 54 \left\lfloor \frac{i+1}{6} \right\rfloor + 47 \equiv 47 \pmod{54}, & \text{when } i \equiv 3 \pmod{6}; \\ 54 \left\lfloor \frac{i+1}{6} \right\rfloor + 49 \equiv 49 \pmod{54}, & \text{when } i \equiv 4 \pmod{6}; \\ 54 \left\lfloor \frac{i+1}{6} \right\rfloor + 11 \equiv 11 \pmod{54}, & \text{when } i \equiv 5 \pmod{6}; \\ 54 \left\lfloor \frac{i+1}{6} \right\rfloor + 13 \equiv 13 \pmod{54}, & \text{when } i \equiv 0 \pmod{6}; \end{cases}$$

where $i = 1, \dots, 6(q - 1)$.

- For partition $[9q]$ or set W , applies $f(w_\epsilon) = \epsilon - 1 \equiv 0, 1 \pmod{54}$ where $\epsilon = 1, 2$.
- For partition $[27]$ or set C , applies $f(c_k) = 54k + 9 \equiv 9 \pmod{54}$ where $k = 1, \dots, (q - 1)$.
- For partition $[q]$ or set U , applies

$$f(u_\gamma) = \begin{cases} 18 \left\lfloor \frac{\gamma-1}{6} \right\rfloor + 2 \equiv 2, 20, 38 \pmod{54}, & \text{when } \gamma \equiv 1 \pmod{6}; \\ 18 \left\lfloor \frac{\gamma-1}{6} \right\rfloor + 4 \equiv 4, 22, 40 \pmod{54}, & \text{when } \gamma \equiv 2 \pmod{6}; \\ 18 \left\lfloor \frac{\gamma-1}{6} \right\rfloor + 6 \equiv 6, 24, 42 \pmod{54}, & \text{when } \gamma \equiv 3 \pmod{6}; \\ 18 \left\lfloor \frac{\gamma-1}{6} \right\rfloor + 8 \equiv 8, 26, 44 \pmod{54}, & \text{when } \gamma \equiv 4 \pmod{6}; \\ 18 \left\lfloor \frac{\gamma-1}{6} \right\rfloor + 10 \equiv 10, 28, 46 \pmod{54}, & \text{when } \gamma \equiv 5 \pmod{6}; \\ 18 \left\lfloor \frac{\gamma-1}{6} \right\rfloor + 12 \equiv 12, 30, 48 \pmod{54}, & \text{when } \gamma \equiv 0 \pmod{6}; \end{cases}$$

where $\gamma = 1, \dots, 18$.

- For partition $[3q]$ or set V , applies

$$f(v_\delta) = 2\delta + 1 \equiv 3, 5, 7, 9, 11, 13 \pmod{54}$$

where $\delta = 1, \dots, 6$.

- For partition $[9]$ or set B , applies

$$f(b_j) = \begin{cases} 54 \lfloor \frac{j+1}{2} \rfloor - 27 \equiv 27 \pmod{54}, & \text{when } j \text{ is odd;} \\ 54 \lfloor \frac{j+1}{2} \rfloor - 9 \equiv 45 \pmod{54}, & \text{when } j \text{ is even;} \end{cases}$$

where $j = 1, \dots, 2(q-1)$.

By using graceful labeling rules and modulo operations, the following results are obtained (Table 8).

Mod	Label	Mod	Label	Mod	Label
0	$g(\{c_k, v_\delta\})$	18	$g(\{b_j, v_\delta\})$	36	$g(\{b_j, v_\delta\})$
1	$g(\{w_1, w_2\}), g(\{c_k, u_\gamma\})$	19	$g(\{c_k, u_\gamma\})$	37	$g(\{c_k, u_\gamma\})$
2	$g(\{w_\epsilon, v_\delta\}), g(\{c_k, v_\delta\})$	20	$g(\{b_j, v_\delta\})$	38	$g(\{b_j, v_\delta\})$
3	$g(\{w_\epsilon, v_\delta\}), g(\{c_k, u_\gamma\})$	21	$g(\{c_k, u_\gamma\})$	39	$g(\{c_k, u_\gamma\})$
4	$g(\{w_\epsilon, v_\delta\}), g(\{c_k, v_\delta\})$	22	$g(\{b_j, v_\delta\})$	40	$g(\{b_j, v_\delta\})$
5	$g(\{w_\epsilon, v_\delta\}), g(\{c_k, u_\gamma\})$	23	$g(\{c_k, u_\gamma\})$	41	$g(\{c_k, u_\gamma\})$
6	$g(\{w_\epsilon, v_\delta\}), g(\{c_k, v_\delta\})$	24	$g(\{b_j, v_\delta\})$	42	$g(\{b_j, v_\delta\})$
7	$g(\{w_\epsilon, v_\delta\}), g(\{c_k, u_\gamma\})$	25	$g(\{c_k, u_\gamma\})$	43	$g(\{c_k, u_\gamma\})$
8	$g(\{w_\epsilon, v_\delta\}), g(\{w_\epsilon, c_k\})$	26	$g(\{w_\epsilon, b_j\})$	44	$g(\{w_\epsilon, b_j\})$
9	$g(\{w_\epsilon, v_\delta\}), g(\{w_\epsilon, c_k\})$	27	$g(\{w_\epsilon, b_j\})$	45	$g(\{w_\epsilon, b_j\})$
10	$g(\{w_\epsilon, v_\delta\}), g(\{a_i, w_\epsilon\})$	28	$g(\{a_i, w_\epsilon\})$	46	$g(\{a_i, w_\epsilon\})$
11	$g(\{w_\epsilon, v_\delta\}), g(\{a_i, w_\epsilon\})$	29	$g(\{a_i, w_\epsilon\})$	47	$g(\{a_i, w_\epsilon\})$
12	$g(\{w_\epsilon, v_\delta\}), g(\{a_i, w_\epsilon\})$	30	$g(\{a_i, w_\epsilon\})$	48	$g(\{a_i, w_\epsilon\})$
13	$g(\{w_\epsilon, v_\delta\}), g(\{a_i, w_\epsilon\})$	31	$g(\{a_i, w_\epsilon\})$	49	$g(\{a_i, w_\epsilon\})$
14	$g(\{b_j, v_\delta\})$	32	$g(\{b_j, v_\delta\})$	50	$g(\{c_k, v_\delta\})$
15	$g(\{c_k, u_\gamma\})$	33	$g(\{c_k, u_\gamma\})$	51	$g(\{c_k, u_\gamma\})$
16	$g(\{b_j, v_\delta\})$	34	$g(\{b_j, v_\delta\})$	52	$g(\{c_k, v_\delta\})$
17	$g(\{c_k, u_\gamma\})$	35	$g(\{c_k, u_\gamma\})$	53	$g(\{c_k, u_\gamma\})$

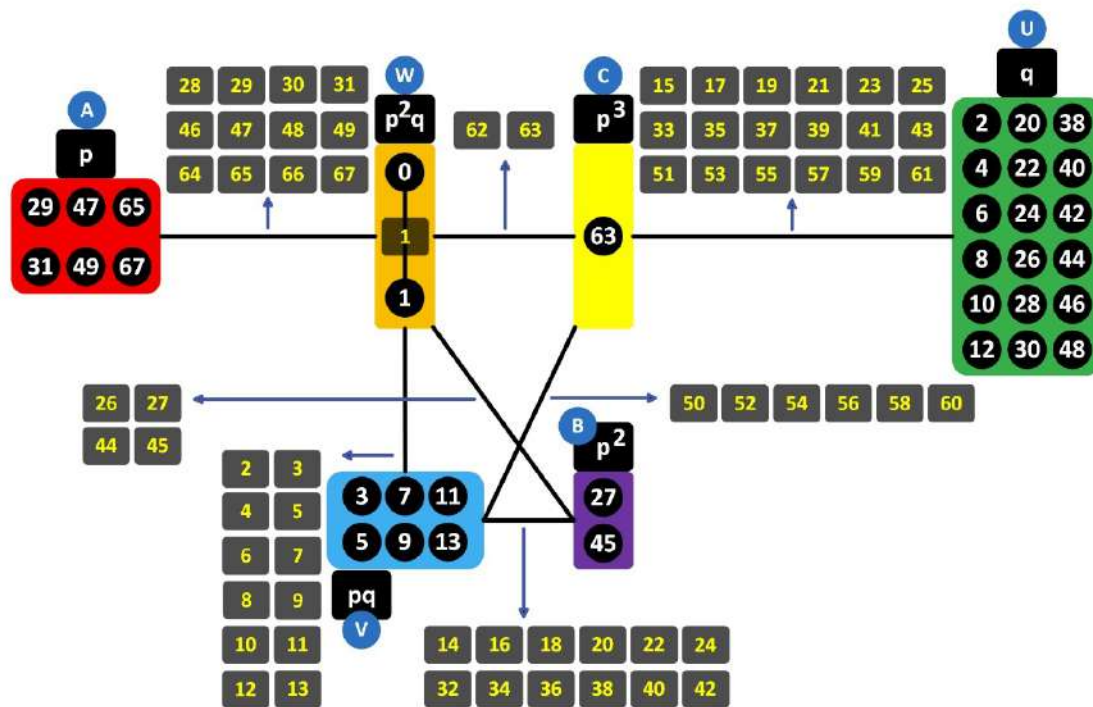
Table 8. Modulo 54 congruence of each edges in $\Gamma(\mathbb{Z}_{27q})$

From Table 8, it is obtained that the value of $g(\{w_1, w_2\})$ and $g(\{w_\epsilon, v_\delta\})$ are less or equal to 13 while the value of $g(\{c_k, u_\gamma\})$, $g(\{c_k, v_\delta\})$, $g(\{w_\epsilon, c_k\})$, and $g(\{a_i, w_\epsilon\})$ are greater or equal to 15. Therefore, the edges with congruence 1 to 13 have different values.

Label	Minimum	Maximum
$g(\{a_i, w_\epsilon\})$	28	$54q - 41 = m$
$g(\{w_1, w_2\})$	1	
$g(\{w_\epsilon, c_k\})$	62	$54q - 45$
$g(\{c_k, u_\gamma\})$	15	$54q - 47$
$g(\{w_\epsilon, v_\delta\})$	2	13
$g(\{w_\epsilon, b_j\})$	26	$54q - 63$
$g(\{c_k, v_\delta\})$	50	$54q - 48$
$g(\{b_j, v_\delta\})$	14	$54q - 66$

Table 9. Minimum and maximum edges label for zero-divisor graph $\Gamma(\mathbb{Z}_{27q})$

It can be seen from Table 9 that $\min\{g(\{u, v\})\} = 1$ and $\max\{g(\{u, v\})\} = m$, which means that function g is bijective. Since $g(\{u, v\})$ is bijective for any $\{u, v\} \in E(\Gamma(\mathbb{Z}_{27q}))$, according to the definition of graceful labeling, the graph $\Gamma(\mathbb{Z}_{27q})$ with $q \neq 3$ for q prime can be labeled gracefully.


Figure 10. Zero-divisor graph $\Gamma(\mathbb{Z}_{54})$

□

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