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Coloring properties of corona products in signed graphs

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Abstract

This paper studies coloring and circular coloring of signed graphs. A signed graph is a graph together with a signature that assigns to each edge a sign which is either positive or negative. This paper studies coloring properties of corona products of signed graphs, including vertex corona product, edge corona product, neighbourhood corona product. Assume (G, σ) and (H, τ) are two signed graphs. The vertex corona product, edge corona product and neighborhood corona product of (G, σ) and (H, τ) are denoted by $(G \odot H, \sigma \odot \tau)$, $(G \diamond H, \sigma \diamond \tau)$, and $(G \star H, \sigma \star \tau)$, respectively. We obtain the following upper bounds for their chromatic numbers: $\chi(G \odot H, \sigma \odot \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 2\}$, $\chi(G \diamond H, \sigma \diamond \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 4\}$, and $\chi(G \star H, \sigma \star \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 2(\Delta(G) + 1)\}$. Then we study circular coloring of corona products of graphs, and obtained the following upper bounds: $\chi_c(G \odot H, \sigma \odot \tau) \leq \max\{\chi_c(G, \sigma), \chi_c(H, \tau) + 4\}$, $\chi_c(G \diamond H, \sigma \diamond \tau) \leq \max\{\chi_c(G, \sigma), \chi_c(H, \tau) + 8\}$, and $\chi_c(G \star H, \sigma \star \tau) \leq \max\{\chi_c(G, \sigma), \chi_c(H, \tau) + 4(\Delta(G) + 1)\}$. We shall show that the upper bound for the circular chromatic number of vertex corona product is attained, by constructing signed Paley graph that achieves this upper bound.

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1. Introduction

A *signed graph* (G, σ) is a graph $G = (V, E)$ endowed with a *signature function* $\sigma : E(G) \to \{-1, +1\}$. The signature on the edges of G induces a signature function (also denoted by σ) on its vertices by $\sigma(v) = \prod_{v \in e} \sigma(e)$. Coloring of signed graphs was studied first by Zaslavsky [18] in 1982. The requirement of a proper coloring ϕ of a signed graph (G, σ) is that for every edge $e = xy$ of (G, σ), $f(x) \neq \sigma(e)f(y)$. If $\sigma(e) = 1$, then this means that $f(x) \neq f(y)$. This is the same as in a proper coloring of a graph. If $\sigma = -1$, then the requirement is that $f(x) \neq -f(y)$. There are different ways of interpreting $-c$ for a color c. The following is one way of interpreting $-c$.

Definition 1. Assume (G, σ) is a signed graph and k is a positive integer. Let

$$M_k = \begin{cases} \{\pm 1, \pm 2, \ldots, \pm q\}, & \text{if } k = 2q \text{ is even,} \\ \{0, \pm 1, \pm 2, \ldots, \pm q\}, & \text{if } k = 2q + 1 \text{ is odd.} \end{cases}$$

A proper k-*coloring* of (G, σ) is a mapping $f : V(G) \to M_k$ such that for any edge $e = xy$ of G, $f(x) \neq \sigma(e)f(y)$. We say (G, σ) is k-*colorable* if there exists a k-coloring of G. The *chromatic number* $\chi(G, \sigma)$ of (G, σ) is the minimum integer k such that (G, σ) is k-colorable.

Another way of interpreting $-c$ is to use elements of an additive abelian group as the color set.

Definition 2. Assume (G, σ) is a signed graph and Γ is an additive abelian group. A proper Γ-*coloring* of (G, σ) is a mapping $f : V(G) \to \Gamma$ such that for any edge $e = xy$ of G, $f(x) \neq \sigma(e)f(y)$.

A chromatic number for signed graphs defined through additive abelian groups relies on the structure of the group. The self-inverse elements of the group are pivotal in these colorings, as the color sets formed by them are independent sets. If all elements of Γ are self-inverse, i.e., if $\Gamma = Z_2^q$ for some integer q, then for any signature σ on a graph G, a proper Γ-coloring of (G, σ) is the same as a proper coloring of G. Another group that has been used as the color set for coloring of signed graphs is Z_k. If k is an odd integer, then a proper Z_k-coloring is the same as the k-coloring of (G, σ) as in Definition 1. If k is even, then it is different. In the following, we shall use Definition 1 for coloring of signed graphs.

The circular chromatic number of a graph was introduced by Vince in 1988 [15], where it is called the *star-chromatic number*.

Definition 3. Let k and d be positive integers such that $k \geq 2d$. A (k, d)-coloring of a signed graph (G, σ) is a mapping $g : V(G) \to \mathbb{Z}_k$ such that for each edge $e = ab$,

$$d_{(\text{mod } k)}(g(a), g(b)) \geq d,$$

where for $i, j \in \mathbb{Z}_k$, $d_{(\text{mod } k)}(i, j) = \min\{|i - j|, k - |i - j|\}$. The *circular chromatic number* $\chi_c(G)$ of a graph G is defined as

$$\chi_c(G) = \inf\{k/d : \text{ there is a } (k, d)\text{-coloring of } G\}.$$

Below is an alternate definition of the circular coloring of a graph.

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Definition 4. Given a real number $r \geq 2$, let $\mathcal{C}^r$ represent the circle with a circumference of r, created by identifying the two ends of the interval $[0, r]$. The *circular distance* of two points $x, y \in \mathcal{C}^r$ is defined as $d_{\mathcal{C}^r}(x, y) = \min\{|x - y|, r - |x - y|\}$. A circular r-coloring of G is a mapping $f : V(G) \to \mathcal{C}^r$ such that for any edge xy of G, $d_{\mathcal{C}^r}(f(x), f(y)) \geq 1$.

It is known [20, 22] and easy to see that if $r = k/d$, then a graph G is circular r-colorable if and only if G is (k, d)-colorings. Hence, the circular chromatic number of a graph G can be equivalently defined as

$$\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ is } C^r\text{-colorable}\}.$$

It is also known that the infimum in the definition above is attained and hence can be replaced by minimum.

The concept of circular coloring of graphs was generalized to signed graphs by Naserasr, Wang, and Zhu in [12].

For a point $x \in \mathcal{C}^r$, the *antipodal point* $\overline{x}$ of x is the unique point at a distance of $r/2$ away from x, i.e., $\overline{x} = x + r/2 \mod r$.

Definition 5. Assume (G, σ) is a signed graph and $r \geq 2$ is a real number. A *circular r-coloring* of (G, σ) is a map $f : V(G) \to C^r$ such that for each positive edge uv, $d_{\mathcal{C}^r}(f(u), f(v)) \geq 1$, and for each negative edge uv, $d_{\mathcal{C}^r}(f(u), \overline{f(v)}) \geq 1$. We say that (G, σ) is *r-colorable* if there is a circular r-coloring of (G, σ).

Definition 6. Given a graph (G, σ), its *circular chromatic number* is

$$\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ is circular } r\text{-colorable}\}.$$

It is known [12] that the infimum in the definition above is attained and hence can be replaced by the minimum.

Proposition 1.1. *If (H, τ) and (H, τ') are switch equivalent, then $\chi_c(H, \tau) = \chi_c(H, \tau')$.*

Proof. Assume (H, τ') is obtained from (H, τ) by switching a set S of vertices. Suppose ϕ is a circular r-coloring of (H, τ). Let $\phi' : V(H) \to \mathcal{C}^r$:

$$\phi'(b) = \begin{cases} \phi(b), & \text{if } b \in V(H) \setminus S, \\ \overline{\phi(b)}, & \text{if } b \in S. \end{cases}$$

Now we verify that ϕ' is a circular r-coloring of (H, τ'). Assume $e = xy$ is an edge of G. If $x, y \in S$ or $x, y \in V(G) - S$, then $\tau'(e) = \tau(e)$ and $d_{\mathcal{C}^r}(\phi'(x), \phi'(y)) = d_{\mathcal{C}^r}(\phi(x), \phi(y))$. If $x \in S$ and $y \in V(G) - S$, then $\tau'(e) = -\tau(e)$ and $d_{\mathcal{C}^r}(\phi'(x), \phi'(y)) = d_{\mathcal{C}^r}(\phi(x), \phi(y)) + r/2 \pmod r$. Thus if $\tau'(e) = 1$, then $d_{\mathcal{C}^r}(\phi'(x), \phi'(y)) \geq 1$ and if $\tau'(e) = -1$, then $d_{\mathcal{C}^r}(\phi'(x), \overline{\phi'(y)}) \geq 1$. □

As $\mathcal{C}^r$ is obtained from the interval $[0, r]$ by identifying 0 and r, a circular r-coloring of a signed graph (G, σ) is actually a mapping $\phi : V(G) \to [0, r)$. For a real number x, let $x \mod (r)$ be the unique real number $x' \in [0, r)$ such that $x - x'$ is a multiple of r. Thus a circular r-coloring of a signed graph can be defined as follows:

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Definition 7. Assume (G, σ) is a signed graph and $r \geq 2$ is a real number. A *circular r-coloring* of (G, σ) is a map $\phi : V(G) \to [0, r)]$ such that for each positive edge uv, $1 \leq (\phi(u) - \phi(v)) \pmod r \leq r - 1$, and for each negative edge uv, $1 \leq (\phi(u) + r/2 - \phi(v)) \pmod r \leq r - 1$, or equivalently, either $(\phi(u) - \phi(v)) \pmod r \geq r/2 + 1$, or $(\phi(u) - \phi(v)) \pmod r \leq r/2 - 1$.

A useful tool in the calculation of the circular chromatic number of a signed graph is the concept of tight cycle.

Definition 8. Assume (G, σ) is a signed graph and ϕ is a circular r-coloring of (G, σ). We define a partial orientation D of G, denoted by $D = D_\phi(G, \sigma)$, with respect to ϕ as follows: (x, y) is an arc of D if and only if one of the following holds:

- xy is a positive edge and $(\phi(y) - \phi(x))(\pmod r) = 1$.
- xy is a negative edge and $(\phi(y) - \phi(x))(\pmod r) = r/2 + 1$.

Definition 9. Assume (G, σ) is a signed graph and ϕ is a circular r-coloring of (G, σ). A directed cycle in $D_\phi(G, \sigma)$ is called a *tight cycle* with respect to ϕ.

The following lemma is proved in [12].

Lemma 1.1. *For a signed graph (G, σ), $\chi_c(G, \sigma) = r$ if and only if (G, σ) is circular r-colorable, and for any circular r-coloring ϕ of (G, σ), there is a tight cycle with respect to ϕ.*

The circular chromatic number of graphs has been studied extensively in the literature (see, e.g., [21, 20, 22, 12]). It follows from the definition that for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. So the circular chromatic number offers a more precise measure of graph coloring compared to the chromatic number.

In 2009, Brewster, Richard C., et al. in [3] state that the circular chromatic number of graphs can be defined in various ways, and some definitions can be extended naturally to signed graphs. Researchers have been exploring extensions of circular coloring to signed graphs. Kang and Steffen [9] defined an extension of circular coloring to signed graphs, which is different from the one we shall study in this thesis. The circular coloring of signed graphs studied in this thesis is given by Naserasr, Wang, and Zhu [12]. The relation between circular coloring and the coloring of signed graphs is more delicate. Note that in coloring a signed graph with an odd number of colors, the color 0 is used, and it is a color whose antipodal color is itself. In the circular coloring of signed graphs, all the colors are symmetric. There is no particular color. The coloring of signed graphs using an even number of colors, i.e., the color set is $\{\pm 1, \pm 2, \ldots, \pm q\}$, does correspond to circular coloring of signed graphs. Such a coloring of signed graphs is called 0-free coloring of signed graphs by Zaslavsky [18].

Proposition 1.2. *Given a positive integer k and a signed graph (G, σ), (G, σ) is 0-free $2k$-colorable if and only if (G, σ) is circular $2k$-colorable.*

Corollary 1.1. *For any signed graph (G, σ), $\chi_c(G, \sigma) \leq 2\lceil \chi(G, \sigma)/2 \rceil$.*

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One of the most studied families of graphs in graph coloring is the family of planar graphs, mainly due to the famous Four Color Theorem. A natural question, asked by Máčajová, Raspaud, and Škoviera in [11], is whether the Four Color Theorem extends to signed planar graphs.

By Corollary 1.1, this question is equivalent to asking whether every (simple) signed planar graph has a circular chromatic number at most 4.

Kardŏs and Narboni's paper [10] constructed a non-4-colorable signed planar graph. So, some signed planar graphs have circular chromatic numbers that are strictly more significant than 4. Naserasr, Wang, and Zhu in [12] improved the result of Kardŏs and Narboni and proved the following result.

Theorem 1.1. *[12] There exists a signed planar graph (G, σ) with $\chi_c(G, \sigma) = 14/3$.*

This result is further strengthened by Gu and Zhu in [6], where the following result is proved.

Theorem 1.2. *[6] For any rational number $2 \leq p/q \leq 14/3$, there is a signed planar graph (G, σ) with $\chi_c(G, \sigma) = p/q$.*

For graphs, parallel edges do not affect (circular) chromatic numbers. However, for signed graphs, the presence or absence of parallel edges of different signs influences the circular chromatic number.

Corollary 1.2 (Proposition 3.1.7[19])**.** *Let $(G, \pm)$ be a digon. We have that $\chi_c(G, \pm) = 4$.*

Proof. Think of a and b as the two endpoints in the graph $(G, \pm)$. Finding a circular 4-coloring for $(G, \pm)$ is a simple task. The inequality $\chi_c(G, \pm) \geq 4$ holds because in any circular r-coloring of $(G, \pm)$, the four points $g(a), \overline{g(a)}, g(b), \overline{g(b)}$ need to be positioned at least one unit apart from each other. □

Therefore, if (G, σ) contains a digon (as a subgraph), then $\chi_c(G, \sigma) \geq 4$.

The circular chromatic number, $\chi_c(H, \tau)$, is equal to 1 if and only if the signed graph (H, τ) has no edges. For any value of r less than 2, (H, τ) cannot be circular r-colorable if it has only one edge, whether positive or negative. Therefore, we will always consider $r \geq 2$. Additionally, there exists a characterization of signed graphs for which the circular chromatic number is equal to 2.

Proposition 1.3. *[19] If $\chi_c(H, \tau) = 2$, then a signed graph (H, τ) is switching equal to a signed graph $(H, -)$.*

Basic properties of circular chromatic number of signed graphs, bounding on their circular chromatic number of some special classes of signed graphs are studied in [12].

Frucht and Harary introduced the concept of the corona product of graphs back in 1970 [8]. This involves combining one central graph, denoted as G, with several copies of an outer graph, denoted as H, to create a new graph denoted as $(G \odot H)$. Each vertex in a copy of H is then connected to every vertex in the corresponding vertex of G. For more information, the reader is referred to [17, 16, 14, 2] and the references therein.

For a signed graphs, the corona product, represented by $(G, \sigma) \odot (H, \tau)$, is a technique for creating complex graphs from simple building blocks. (For more details, see [1]). Specifically,

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there are a few types of corona products for graphs: vertex $(G,\sigma)\odot(H,\tau)$, edge $(G,\sigma)\diamond(H,\tau)$, and neighbor$(G,\sigma)\star(H,\tau)$ corona products. These three types of corona products are distinguished from one another based on how the building blocks (represented by signed graphs (G,σ) and (H,τ)) are combined.

Let (G,σ) and (H,τ) be two signed graphs. The *(vertex) corona product* $(G\odot H,\sigma\odot\tau)$ of (G,σ) and (H,τ) is a signed graph with

- the vertex set $V(G\odot H)=V(G)\cup(V(H)\times V(G))$,
- the edge set $E(G\odot H)=E(G)\cup\{(u,v)(u',v):uu'\in E(H),v\in V(G)\}\cup\{v(u,v):v\in V(G),u\in V(H)\}$, and
- the signature $\sigma\odot\tau$ defined as follows:
  1. $(\sigma\odot\tau)(e)=\sigma(e)$, if $e\in E(G)$,
  2. $(\sigma\odot\tau)(e)=\tau(uu')$, if $e=(u,v)(u',v)$ for some $v\in V(G)$ and $uu'\in E(H)$,
  3. $(\sigma\odot\tau)(e)=\sigma(v)\tau(u)$, if $e=v(u,v)$ for some $v\in V(G)$ and $u\in V(H)$,

  where $\sigma(v)=\prod_{e\in E(v)}\sigma(e)$ and $\tau(u)=\prod_{e\in E(u)}\tau(e)$.

In other words, a vertex corona product $(G,\sigma)\odot(H,\tau)$ consists of a copy of (G,σ) and copies of $(H,\tau)=:H_v$ indexed by v with new signed edges added between v and each vertex of H_v.

The edge corona product $(G\diamond H,\sigma\diamond\tau)$ consists of a copy of (G,σ) and $|E(G)|$ copies of (H,τ), joining two end-vertices of e of (G,σ) to each vertex of H_v. The signatures of the edges in G and each H_e are inherited from their originals, while the signature of a new edge is defined as the product of the signatures of its two end vertices.

The neighborhood corona product $(G\star H,\sigma\star\tau)$ is constructed by taking one copy of (G,σ) and $|V(G)|$ copies of H_v of (H,τ), and joining every vertex of the $N[v]:=N(v)\cup\{v\}$ with each vertex of the H_v by a new signed edge, where the sign of the new edge is the product of the signs of its end-vertices (i.e., $\prod_{e\in E_u}\sigma(e)\prod_{e\in E_v}\sigma(e)$.)

This paper studies coloring properties of corona products of signed graphs, including vertex corona product, edge corona product, neighbourhood corona product. Assume (G,σ) and (H,τ) are two signed graphs. The vertex corona product, edge corona product and neighborhood corona product of (G,σ) and (H,τ) are denoted by $(G\odot H,\sigma\odot\tau)$, $(G\diamond H,\sigma\diamond\tau)$, and $(G\star H,\sigma\star\tau)$, respectively.

We have the following question:

Question. *What are the chromatic number and circular chromatic number of corona product of two signed graphs?*

Main Results

This paper establishes upper bounds for chromatic and circular chromatic numbers of three types of corona products of signed graphs. Our main contributions are as follows:

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Chromatic number bounds.. For signed graphs (G, σ) and (H, τ), we prove:

- *Vertex corona product* (Theorem 2.1):

$$\chi(G \odot H, \sigma \odot \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 2\}.$$

  This bound can be improved to $\max\{\chi(G, \sigma), \chi(H, \tau) + 1\}$ when $\chi(H, \tau)$ is odd (Theorem 2.2) or when (H, τ) has certain structural properties (Theorem 2.3).

- *Edge corona product* (Theorem 2.4):

$$\chi(G \diamond H, \sigma \diamond \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 4\}.$$

- *Neighborhood corona product* (Theorem 2.5):

$$\chi(G \star H, \sigma \star \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 2(\Delta(G) + 1)\}.$$

Circular chromatic number bounds.. We establish:

- *Vertex corona product* (Theorem 3.1):

$$\chi_c(G \odot H, \sigma \odot \tau) \leq \max\{\chi_c(G, \sigma), \chi_c(H, \tau) + 4\}.$$

- *Edge corona product* (Theorem 3.3):

$$\chi_c(G \diamond H, \sigma \diamond \tau) \leq \max\{\chi_c(G, \sigma), \chi_c(H, \tau) + 8\}.$$

- *Neighborhood corona product* (Theorem 3.4):

$$\chi_c(G \star H, \sigma \star \tau) \leq \max\{\chi_c(G, \sigma), \chi_c(H, \tau) + 4(\Delta(G) + 1)\}.$$

Tightness of bounds.. We prove that the upper bound for the circular chromatic number of vertex corona products is asymptotically tight:

- We construct a family of signed graphs H_{2k+1} (Definition 10) with $\chi_c(H_{2k+1}) = (4k + 2)/(2k - 1)$ (Proposition 3.1 and its tightness proof).
- We show $\chi_c(K_1 \odot H_{2k+1}) = 6$ (Proposition 3.2).
- This demonstrates that as $k \to \infty$, the additive gap approaches exactly 4, proving the bound in Theorem 3.1 is tight.

Several concrete examples (Examples 1–6) further illustrate these bounds for specific graph families.

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2. Chromatic number of corona products of signed graphs

The aim of this section is to determine the chromatic number of corona products of signed graphs. We focus on the vertex, edge, and neighborhood corona product of signed graphs.

Theorem 2.1. *Let (G,σ) and (H,τ) be two signed graphs, then the chromatic number of $(G\odot H,\sigma\odot\tau)$ satisfies $\chi(G\odot H,\sigma\odot\tau)\leq\max\{\chi(G,\sigma),\chi(H,\tau)+2\}$.*

Proof. Let $k=\max\{\chi(G,\sigma),\chi(H,\tau)+2\}$, $M_k=\{\pm1,\pm2,\ldots,\pm q\}$ or $\{0,\pm1,\pm2,\ldots,\pm q\}$ be the color set depending on the parity of k, where q is a natural number. Color the vertices of the copy of (G,σ) with M_k and denote it by ϕ. Such a coloring exists since $k\geq\chi(G,\sigma)$. For each $v\in V(G)$, color the corresponding copy (H_v,τ) of (H,τ) with $M_k\setminus\{\pm q\}$, denoted by ϕ_v, again, since $k-2\geq\chi(H,\tau)$. Now, for each $v\in V(G)$ and each $u\in V(H_v)$, if $\phi_v(u)=\pm\phi(v)\neq0$, then switch the color $\phi_v(u)$ to $\pm q$ respectively; otherwise, if $\phi_v(u)=\phi(v)=0$ recolor u with $+q$. Therefore, it can be readily verified that the coloring ϕ, combined with the adjusted versions of ϕ_v, constitutes a valid coloring of $(G\odot H,\sigma\odot\tau)$ with M_k.

□

Remark 2.1 (Common recoloring strategy). The proofs of Theorems 2.1, 2.2, 2.3, and 2.4 share a common recoloring strategy that we outline here for clarity:

1. *Initial coloring:* Color the base graph (G,σ) with a sufficiently large color set.
2. *Local coloring:* For each copy of (H,τ) (attached to vertices or edges of G), use a restricted color set that reserves some colors.
3. *Conflict resolution:* For each copy, identify vertices whose colors conflict with the attachment points in G, then systematically recolor these vertices using the reserved colors.
4. *Key observation:* The reserved colors are chosen to be at least distance 1 away from all colors in G under the signed graph coloring constraint, ensuring no conflicts arise.

This strategy allows us to "reserve" a small number of colors (2, 4, or $2(\Delta(G)+1)$ depending on the corona product type) to handle all potential conflicts systematically.

Theorem 2.2. *Let (G,σ) and (H,τ) be two signed graphs, and suppose that $\chi(H,\tau)$ is odd. Then, the chromatic number of $(G\odot H,\sigma\odot\tau)$ satisfies $\chi(G\odot H,\sigma\odot\tau)\leq\max\{\chi(G,\sigma),\chi(H,\tau)+1\}$.*

Proof. Let $k=\max\{\chi(G,\sigma),\chi(H,\tau)+1\}$. To show that $\chi(G\odot H,\sigma\odot\tau)\leq k$, it suffices to consider the case when $k=\chi(H,\tau)+1$, since the other case is covered by Theorem 2.1.

Since $\chi(H,\tau)$ is an odd number, we may assume that the color set $M_k=\{\pm1,\pm2,\ldots,\pm q\}$ where q is a natural number. We show that there is a proper coloring of $(G\odot H,\sigma\odot\tau)$ with the colors from M_k.

To start, we color the vertices of the copy of (G,σ) with M_k and denote it by ϕ. Such a coloring exists since $k\geq\chi(G,\sigma)$. Next, for each $v\in V(G)$, color the corresponding copy (H_v,τ) of (H,τ) by $M_{k-1}=\{0,\pm1,\pm2,\ldots,\pm(q-1)\}$ (again, since $k\geq\chi(H,\tau)+1$), denoted by ϕ_v.

Now, for each $v\in V(G)$, if there is no vertex $u\in V(H_v)$, such that $\phi_v(u)=\pm\phi(v)$, then we may recolor the vertices of H_v that are colored by 0 (which form an independent set) with the

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new color q from M_k. Otherwise, for each $u \in V(H_v)$ with $\phi_v(u) = \pm\phi(v) \neq 0$, switch the color $\phi_v(u)$ to $\pm q$ respectively; then for each $u \in V(H_v)$ with $\phi_v(u) = 0$, recolor it with $\phi(v)$ if $\sigma \odot \tau(uv) = -1$, and $-\phi(v)$, if $\sigma \odot \tau(uv) = 1$. With such modifications on ϕ_v's, it is easy to see that each vertex of H_v is properly colored without causing conflicts with ϕ, which completes the proof. □

2.1. Special structures with improved upper bound

It is obvious to see that if $\chi(G, \sigma) \geq \chi(H, \tau) + 2$ then $\chi(G \odot H, \sigma \odot \tau) = \chi(G, \sigma)$ since it takes at least $\chi(G, \sigma)$ signed colors to color the copy of (G, σ) in the corona product. So the only interesting case is when $\chi(G, \sigma) \leq \chi(H, \tau) + 1$. We will show that under certain assumptions on the structure of (H, τ), it suffices to color the vertex corona product with $\chi(H, \tau) + 1$.

Theorem 2.3. *Let (G, σ), (H, τ) be two signed graphs. Suppose $\chi(H, \tau) = k$ is even, and there is a k-coloring of (H, τ) where there exist two colors classes A, B for colors $\pm i$ respectively, such that $A \cup B$ is the disjoint union of two independent sets. Then $\chi(G \odot H, \sigma \odot \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 1\}$.*

Proof. We may assume $\chi(G, \sigma) \leq \chi(H, \tau)+1 = k+1$. Let the color set $M_{k+1} = \{0, \pm 1, \pm 2, \ldots, \pm q\}$ where q is a natural number. We will provide a $(k + 1)$-coloring of $(G \odot H, \sigma \odot \tau)$.

To begin, we color the vertices of the copy of (G, σ) with M_{k+1} and denote it ϕ. Then, for each $v \in V(G)$, define a k-coloring ϕ_v of (H_v, τ) that satisfies the assumption of the theorem. That is, there is an $i \in [q]$ such that $A = \phi^{-1}(i), B = \phi^{-1}(-i)$, and $A \cup B = U \cup U'$, where U and U' are disjoint independent sets.

Now, for each $v \in V(G)$, we may assume $\phi(v) = i$ or $-i$; otherwise, we may permute the coloring ϕ_v to achieve it. Now, recolor the independent set U by 0. For each vertex $u \in U'$, recolor it with $\phi(v)$ if $(\sigma \odot \tau)(uv) = -1$, and $-\phi(v)$ if $(\sigma \odot \tau)(uv) = 1$. With such modifications on the ϕ_v, we get a proper $(k + 1)$-coloring on each H_v, which is not in conflict with ϕ. □

Corollary 2.1. *Let (G, σ), (H, τ) be two signed graphs. Suppose $\chi(H, \tau) = k$ is even and there is a k-coloring of (H, τ) where two of the color classes for colors $\pm i$ are both independent sets, then*

$$\chi(G \odot H, \sigma \odot \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 1\}.$$

Let (H, τ) be a signed graph. The *n-blow-up* of a signed graph (H, τ), denoted by $(H(n), \tau)$, is obtained by replacing every vertex $u \in V(H)$ with n copies $\{u_i : i \in [n]\}$, so that the copies of two vertices $u_i \sim v_j$ if and only if $u \sim v$ and $\tau(u_i v_j) = \tau(uv)$.

Let (H, τ) be the signed graph obtained from a $(K_k(n), +)$ by randomly attaching "negative tails," i.e., a leaf vertex incident to a negative edge, to its vertices with probability 1/2 independently. It can be shown that $\chi(H, \tau) = k$, and it has a unique k-coloring up to color permutations. Then, it is easy to see that for each $2 \leq t \leq k$, we have $\chi(K_t \odot (H, \tau)) = k + 2$.

Moreover, if we choose edges of $K_k(n)$ with density $n^{\epsilon-1}$, we can show the existence of a graph of large girth that achieves the upper bound in Theorem 2.1. We omit the proof here since the proof is of a similar flavor to that of Theorem 61, which we will prove in [4].

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2.2. Bounding χ of other corona products

The next theorem gives the upper bound of the edge corona product $(G \diamond H, \sigma \diamond \tau)$.

Recall that the edge corona product $(G \diamond H, \sigma \diamond \tau)$ consists of a copy of (G, σ) and $|E(G)|$ copies of (H, τ), joining two end-vertices of e of (G, σ) to each vertex of H_v. The signatures of the edges in G and each H_e are inherited from their originals, while the signature of a new edge is defined as the product of the signatures of its two end vertices.

Theorem 2.4. *Let (G, σ) and (H, τ) be two signed graphs. The chromatic number of the edge corona product $(G \diamond H, \sigma \diamond \tau)$ satisfies that*

$$\chi(G \diamond H, \sigma \diamond \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 4\}.$$

Proof. Let $k = \max\{\chi(G, \sigma), \chi(H, \tau) + 4\}$, $M_k = \{\pm 1, \pm 2, \ldots, \pm q\}$ or $\{0, \pm 1, \pm 2, \ldots, \pm q\}$ depending on the parity of k, where q is a natural number. We show that there is a proper coloring of $(G \diamond H, \sigma \diamond \tau)$ with colors from M_k the symmetric signed color set of color k.

To start, we color the vertices of the copy of (G, σ) with M_k and denote it by ϕ. Next, for each $e \in E(G)$, color the corresponding copy H_e of (H, τ), color it with $M_k \setminus \{-q, (-q-1), q-1, q\}$ (again, since $k \geq \chi(H, \tau) + 4$), denoted by ϕ_e.

Now, for each $e = uv \in E(G)$, Without loss of generality, $\phi(u) \neq 0$. For each $x \in V(H_e)$, if $\phi_e(x) = \pm\phi(u)$, then switch the color $\phi_e(x)$ to $\pm(q-1)$ respectively. If $\phi_e(x) = \pm\phi(v) \neq 0$, then switch the color $\phi_e(x)$ to $\pm q$ respectively; otherwise, recolor x with q.

Thus, it is easy to check that the coloring ϕ together with the modified ϕ_e's form a proper coloring of $(G \diamond H, \sigma \diamond \tau)$ with k signed colors. As a consequence, $\chi(G \diamond H, \sigma \diamond \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 4\}$. □

It is known that, neighborhood corona product $(G \star H, \sigma \star \tau)$ is constructed by taking one copy of (G, σ) and $|V(G)|$ copies of H_v of (H, τ), and joining every vertex of the $N[v] := N(v) \cup \{v\}$ with each vertex of the H_v by a new signed edge, where the sign of the new edge is the product of the signs of its end-vertices.

Using the same coloring techniques, one can prove the similar result for the neighborhood corona product. Note that the only difference is that we have to mind the colors of at most $\Delta(G)+1$ vertices. We omit the proof.

Theorem 2.5. *Let (G, σ) and (H, τ) be two signed graphs. The chromatic number of the neighborhood corona product $\chi(G \star H, \sigma \star \tau) \leq \max\{\chi(G, \sigma), \chi(H, \tau) + 2(\Delta(G) + 1)\}$, where $\Delta(G)$ is the maximum degree of G.*

3. Circular chromatic number of corona products of signed graphs

The aim of this section is to determine the circular chromatic number of corona products of signed graphs. We focus on the vertex, edge, and neighborhood corona product of signed graphs.

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Comparison with unsigned graphs

Before presenting our results, we briefly contrast the circular chromatic numbers of corona products in signed versus unsigned graphs to highlight the additional complexity introduced by signatures.

For unsigned graphs, it is known that:

- The chromatic number of the vertex corona product satisfies $\chi(G \odot H) = \max\{\chi(G), \chi(H)+1\}$ for many cases, with the additive increase being exactly 1.
- The circular chromatic number exhibits similar behavior: $\chi_c(G \odot H)$ is often close to $\max\{\chi_c(G), \chi_c(H)+1\}$.

In contrast, for signed graphs:

- The presence of negative edges requires careful handling of *antipodal colors*, which forces us to reserve additional color space.
- Our bounds show additive increases of 2 (chromatic) and 4 (circular chromatic) for vertex corona products exactly double the unsigned case. This reflects the fundamental constraint that negative edges impose: if vertex u has color c, then its neighbor v across a negative edge cannot use either c or $-c$ (the antipodal color).
- The construction of H_{2k+1} (Definition 10) demonstrates that these larger additive constants are unavoidable in the signed setting.

This doubling phenomenon from additive 1 (unsigned) to additive 2 (signed) for chromatic numbers, and from approximately 2 to 4 for circular chromatic numbers is a recurring theme in signed graph coloring and reflects the additional geometric constraints imposed by the circular representation with antipodal points.

Theorem 3.1. *Let (G, σ) and (H, τ) be two signed graphs, then the circular chromatic number $\chi_c(G \odot H, \sigma \odot \tau) \leq \max\{\chi_c(G, \sigma), \chi_c(H, \tau) + 4\}$.*

Proof. Let $r = \max\{\chi_c(G, \sigma), \chi_c(H, \tau)+4\}$. We show that there is a proper coloring of the vertex corona product $(G \odot H, \sigma \odot \tau)$ with colors from C^r.

Firstly, we color the copy of (G, σ) with C^r and denote it by f. Then, we define a coloring of (H, τ) with C^{r-4}, and denote it by g. For each $v \in V(G)$, we cut C^{r-4} into halves and glue both pieces to the larger circle C^r via the map $h_v : C^{r-4} \to C^r$,

$$x \mapsto \begin{cases} x + f(v) + 1, & x \in [0, (r-4)/2), \\ x + f(v) + 3, & x \in [(r-4)/2, r-4), \end{cases}$$

where the operations are taken to be modulo r.

Finally, we color the corresponding copies H_v of (H, τ) by the map $g_v = h_v \circ g$. We see that the coloring g_v is proper on H_v, since g is proper and the map h_v only enlarges the distance between the original colors while preserving the antipodal property.

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Further, this coloring is not in conflict with the coloring f on the copy of (G, σ) since the range of g_v is at least 1 away from $f(v)$ or its antipodal $\overline{f(v)}$. Therefore, f together with all the g_v form a proper C^r-coloring of $(G \odot H, \sigma \odot \tau)$. □

Remark 3.1. Since $\chi_c(G \odot H, \sigma \odot \tau) \geq \max\{\chi_c(G, \sigma), \chi_c(H, \tau)\}$, it is immediate to see that when $\chi_c(G, \sigma) \geq \chi_c(H, \tau) + 4$, the upper bound in the previous theorem is tight and the equality holds for all such graphs. Now, it is left to find graphs that achieve the upper bound when $\chi_c(G, \sigma) < \chi_c(H, \tau) + 4$.

3.1. Some examples

Example 1. Let T be an arbitrary signed tree, $C = C_{2k+1}$ be a balanced odd cycle with no negative edge, then $\chi_c(T \odot C) = 4$.

Example 2. Let T be an arbitrary signed tree, $C = C_{2k}$ be a balanced even cycle with no negative edge, then $\chi_c(T \odot C_{2k}) = 3$.

Example 3. Let T be signed tree, then $\chi_c(K_1 \odot T) = 3$ or 2. For example, P is a signed path of length $3k + 2$ where the signature has a pattern $-++-++\ldots++-$, then $\chi_c(K_1 \odot P) = 2$.

Consequently, we have that $\chi_c(T_1 \odot T_2) \leq 3$ for any signed trees T_1 and T_2.

Example 4. Let T be an arbitrary signed tree, $C = C_{2k+1}$ be a balanced odd cycle with no negative edge, then $\chi_c(C \odot T) = \begin{cases} 3, \text{ if } \chi_c(K_1 \odot T) = 3, \\ 2 + 1/k, \text{ if } \chi_c(K_1 \odot T) = 2. \end{cases}$

Example 5. Let G be a signed cycle, $H = C_{2q+1}$ a balanced odd cycles, then $\chi_c(G \odot H) \leq 4$. In particular, if H contains no negative edges, then $\chi_c(G \odot H) = 4$.

Example 6. Let G be a signed cycle, and $C_2 = C_{2q}$ be a balanced, then $\chi_c(G \odot H) \leq 3$. In particular, if H contains no negative edges, then $\chi_c(G \odot H) = 3$.

Here we give a proof of Example 1 and leave other proofs to the readers.

Proof of Example 1. Let $V(C) = \{v_i : i \in \{0, 1, \ldots, 2k\}\}$, and $E(C) = \{v_i v_{i+1 \mod 2k+1} : i \in \{0, 1, \ldots, 2k\}\}$.

To see that $\chi_c(T \odot C) \leq 4$, we first color the local structure of $T \odot C$, namely, for each $v \in V(T)$, we give a 4-coloring for $v \odot C_v \simeq K_1 \odot C$. Since each vertex of C_v has a positive signature, $v \odot C_v$ is a single vertex joining every vertex of C_v with a positive edge.

We define the coloring $f : V(v \odot C_v) \to C^4$ by $f(v) = 0$, and

$$f(v_i) = \begin{cases} 2, & \text{if } i \text{ is odd}, \\ 3, & \text{if } i \neq 0 \text{ and } i \text{ is even}, \\ 1, & \text{if } i = 0. \end{cases}$$

Notice that the color $f(v) = 0$ is at a distance of at least 1 from the colors of vertices of C_v, namely $f(v_i), i \in \{0, 1, \ldots, 2k\}$, so f is a proper coloring of $v \odot C_v$.

Finally, to obtain a coloring of $T \odot C$, we rotate the coloring of the local graphs $v \odot C_v$ to make sure that the local coloring restricted to vertices of T gives a proper coloring. This is possible since $\chi_c(T) = 2$ for every signed tree.

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To prove the lower bound, it suffices to show that $\chi_c(v \odot C) \geq 4$. Assume $\chi_c(v \odot C) = r$ and we have a C^r-coloring $f : V(v \odot C) \to C^r$.

Since v is positively adjacent to each vertex of C, so no color from the interval $(f(v)-1, f(v)+1)$ on C^r is allowed to be assigned to any v_i of C. Thus, f restricted on C is a coloring of C with an arc of C^r.

Since C is an odd cycle, there exists (v_i, v_{i+1}, v_{i+2}) a monotone triple from cycle such that either $f(v_i) \leq f(v_{i+1}) \leq f(v_{i+2})$ or $f(v_i) \geq f(v_{i+1}) \geq f(v_{i+2})$. Since v_iv_{i+1} and $v_{i+1}v_{i+2}$ are positive edges, then $d_f(v_i, v_{i+1}) \geq 1$ and $d_f(v_{i+1}, v_{i+2}) \geq 1$. Adding up the two inequalities, we get $d_f(v_i, v_{i+2}) \geq 2$. This implies that the arc from which C receives its colors has a length at least 2. Therefore, we conclude that $r \geq 4$. □

3.2. Examples that achieve the upper bound: the signed graph H_{2k+1}

Definition 10. Let $k \geq 2$ be an integer. We define the graph H_{2k+1} on the set of vertex $\mathbb{Z}_{2k+1}$ (the cyclic group of order $2k+1$) by joining each pair of elements $i, i+1 \pmod{2k+1}$ with a positive edge and each $i, i+2 \pmod{2k+1}$ with a negative edge.

The definition above is a generalization of the signed Paley graph of order 5. Moreover, it falls into the more general class of signed circulant graphs with arbitrary sets of positive and negative generators. The star-extremality (i.e. the property that the circular chromatic number of a signed graph is equal to its fractional chromatic number) of general unsigned and signed circulant graphs are discussed in [5].

Proposition 3.1. $\chi_c(H_{2k+1}) \leq (4k+2)/(2k-1)$.

Proof. Let $r = (4k+2)/(2k-1)$. Define the coloring $f : V(H_{2k+1}) = \mathbb{Z}_{2k+1} \to C^r$ as $f(i) = 2ki/(2k-1) \mod r$. This coloring satisfies that $d_f(i, i+1) = 2k/(2k-1) > 1$ and $d_f(i, \overline{i+2}) = 1$, hence is proper. □

The next proposition shows that the circular chromatic number of $K_1 \odot H_{2k+1}$ exceeds $\chi_c(H_{2k+1})$ by approximately 4. By definition, $K_1 \odot H_{2k+1}$ is a single vertex that joins every vertex of H_{2k+1} with a positive edge (note that each vertex of H_{2k+1} has a positive signature). With similar argument as Example 1, we can show its circular chromatic number is 6.

Proposition 3.2. $\chi_c(K_1 \odot H_{2k+1}) = 6$.

Proof. To prove the upper bound, We define the coloring $f : V(K_1 \odot H_{2k+1}) \to C^6$ by $f(v) = 3, f(0) = 0$, where v is the only vertex of K_1 and

$$f(i) = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 5, & \text{if } i \neq 0 \text{ is even.} \end{cases}$$

To see the properness, first notice that the color of v is at a distance of at least 1 away from the colors of H_{2k+1}, satisfying the requirements of the edges incident to v. On the other hand, it is easy to check that f is also appropriate on H_{2k+1}.

To prove the lower bound, let $\chi_c(K_1 \odot H_{2k+1}) = r$. We have C^r-coloring $f : V(K_1 \odot H_{2k+1}) \to C^r$. WLOG, $f(v) = 0$, hence no color from $(r-1, 1)$ can be used for the H_{2k+1}.

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Since H_{2k+1} is colored by an arc from $[1, r-1]$ and $2k+1$ is odd, there exists $(i, i+1, i+2)$ (the indices are mod $2k+1$) a monotone triple such that $f(i) \leq f(i+1) \leq f(i+2)$ or $f(i) \geq f(i+1) \geq f(i+2)$. Since the signature of $i, i+1$ and $i+1, i+2$ are positive, we have $d_f(i, i+2) \geq 2$. Since $i, i+2$ is negative, $d_f(i, \overline{i+2}) \geq 1$, which implies $d_f(i, i+2) \leq \frac{r}{2} - 1$ or $d_f(i, i+2) \geq \frac{r}{2} + 1$. In either case, we get $r \geq 6$, which completes the proof. □

Proposition 3.1 and 3.2 together shows that with $\chi_c(K_1 \odot H_{2k+1}) = 6 \geq \chi_c(H_{2k+1}) + 4 - 6/(2k-1)$. As $k \to \infty$, H_{2k+1} achieves the upper bound in Theorem 3.1.

3.2.1. Circular chromatic number of H_{2k+1}

Here, we show that the upper bound in Proposition 3.1 is in fact tight, hence determining the circular chromatic number of H_{2k+1}.

Theorem 3.2. *For every $k \geq 2$, $\chi_c(H_{2k+1}) = (4k+2)/(2k-1)$.*

To prove the lower bound, we first review a few useful results on the circular chromatic number of signed graphs.

Corollary 3.1 (Corollary 3.3.[12])**.** *If $\chi_c(G, \sigma) = r$, then every circular r-coloring ϕ of (G, σ) has a tight cycle.*

Lemma 3.1 (Lemma 3.4. [12])**.** *Given a signed graph (G, σ), $\chi_c(G, \sigma) = r$ if and only if (G, σ) is circular r-coloring and every circular r-coloring ϕ of (G, σ) has a tight cycle.*

Lemma 3.2 (Proposition 3.5.[12])**.** *Any signed graph (G, σ) which is not a forest has a cycle with s positive edges and t negative edges such that $\chi_c(G, \sigma) = 2(s+t)/(2a+t)$ for some integer a.*

To show that $\chi_c(H_{2k+1}) = (4k+2)/(2k-1)$, we need to find the so-called "best lower approximation" to this number and disprove the possibility of finding a proper coloring with such smaller.

Definition 11 (Best lower approximation). Let p/q be a fraction with $p, q > 1$ and $(p, q) = 1$. The *best candidate* denoted by $BA_l(p/q)$ is the unique fraction a/b with $0 < a < p$, $0 < b < q$, and $bp - aq = 1$.

By taking the continuous fraction expansions, one can find the best lower approximation to $(4k+2)/(2k-1)$.

Proposition 3.3. *For each $k \geq 2$,*

$$BA_l\left(\frac{4k+2}{2k-1}\right) = \begin{cases} 2 + 2/k, & \text{if } k \text{ is even,} \\ 2 + 6/(3k-1), & \text{if } k \text{ is odd.} \end{cases}$$

Case 1. k is even. We first assume k is even and remark that there are no candidates for the circular chromatic number $\chi_c(H_{2k+1})$ between $(2 + 2/k, (4k+2)/(2k-1))$. Indeed, suppose $\chi_c(H_{2k+1}) = a/b$, it must hold that $a \leq 4k+2$ by Lemma 3.2 ($a < 2|V(G)|$). If $a < 4k+2$, we have $a/b < 2+2/k$ since $2+2/k$ is the best lower bound; if $a = 4k+2$ then the best approximation is $(4k+2)/2k = 2 + 1/k < 2 + 2/k$.

Now, for even k, we show that $2 + 2/k$ is a sufficient number to color H_{2k+1} circularly.

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Claim 1. For each even natural number k, $\chi_c(H_{2k+1}) > 2 + 2/k$.

Proof. Suppose not, there exists $f : V(H_{2k+1}) \to C^{2+2/k}$, a coloring of H_{2k+1}. Since k is even, $2 + 2/k = 2 + 1/(k/2)$ is the circular chromatic number of a balance odd cycle $+C_{k+1}$. Hence, every $BC_{k+1} \subseteq H_{2k+1}$ is colored as a tight cycle by Lemma 3.1. We consider three tight cycles under this coloring.

First, since the cycle C_1 with the vertex set $\{0, 1, 3, 5, \ldots, 2k-1\}$ is colored tightly, then, without loss of generality, the coloring f satisfies that $f(0) = 0$ and $f(2i-1) = 1 - (i-1)/k$, for $i \in [k]$.

Next, the tight cycle C_2 with vertices $\{2k, 1, 3, 5, \ldots, 2k-1\}$ has a vertex $2k$ distinct from C_1. This vertex must receive the color $f(2k) = 1 + 1/k$.

Finally, we observe that the balanced odd cycle C_3 with vertices $\{0, 2, 4, \ldots, 2k\}$ cannot be colored tight since the distance between $f(2k)$ and $f(0)$ is equal to $1 + 1/k$ which is greater than 1, a contradiction. □

Case 2. k is odd. We first remark that although $2 + 6/(3k-1) > 2 + 2/k$, it is not a possible circular chromatic number for H_{2k+1}. Indeed, suppose

$$\chi_c(H_{2k+1}) = 2 + 6/(3k-1) = 3k + 2/(3k-1)/2.$$

It is easy to see that $\gcd(3k+2, (3k-1)/2) = 1$. Since $3k+2$ is an odd number when k is odd, so by Lemma 3.2, $3k+2$ is the length of a cycle in H_{2k+1}. A contradiction.

We now seek for the next candidate x/y with $x \leq 4k+2$ that best approximates $(4k+2)/(2k-1)$.

- If $y > 2k-1$. Since $x \leq 4k+2$, $x/y \leq (4k+2)/2k = 2 + 1/k$.
- if $y \leq 2k-1$. We write $x/y = 2 + a/b$ where a/b is in the lowest term. Since $2 + a/b < 2 + 4/(2k-1)$, we have $b \leq 2k-1$ and $a/b < 4/(2k-1)$, which implies $a < 4$.
  - If $a = 1$, $b > (2k-1)/4$. So the best approximation we get is $2+a/b = 2+2/(k+1)$.
  - If $a = 2$, then $b > (2k-1)/2 = k - 1/2$. Since b is an integer, $b = k$ gives the best approximation $2 + a/b = 2 + 2/k$.
  - If $a = 3$, $b > 3(2k-1)/4$. Similarly, we get the approximation $2 + a/b = 2 + 6/(3k-1)$.

  Since the last candidate $2+6/(3k-1)$ is shown to be impossible for the value of the circular chromatic number of H_{2k+1}, our unique candidate here is $x/y = 2 + 2/k$ which is greater than other ones. We will show that even this candidate is not sufficient.

Claim 2. For each odd natural number k, $\chi_c(H_{2k+1}) > 2 + 2/k$.

Proof. Suppose not, then there exists $g : V(H_{2k+1}) = \{0, 1, \ldots, 2k\} \to C^{2+2/k}$, a coloring of H_{2k+1}. Since k is odd, $2 + 2/k = (2k+2)/k = 4((k+1)/2)/((2(k+1)/2) - 1)$ is the circular chromatic number of the unbalance even cycle UC_{k+1}. Hence, every $UC_{k+1} \subseteq H_{2k+1}$ is colored as a tight cycle by Lemma 3.1.

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We first consider the cycle C_1 with vertices $\{0, 1, 3 \ldots, 2k-1\}$. Because C_1 is a UC_{k+1}, it is colored tightly under g. WLOG, we have $g(0) = 0$ and $g(2i-1) = 1-(i-1)/k$ for each $i \in [k]$.

Next, consider C_2 on $\{1, 3, \ldots, 2k-1, 2k\}$, which has a vertex $2k$ distinct from C_1, so it has to be that $g(2k) = 1 + 1/k$ by tightness of the cycle.

Finally, C_3 with vertices $\{0, 2, \ldots, 2k\}$ is another UC_{k+1}. However, it cannot be colored tight under g, since the distance from $g(2k)$ to $g(0)$ is equal to $1 + 1/k$ which is greater than 1. Contradiction. □

Proof. From Proposition 3.1, Claim 1 and Claim 2, $\chi_c(H_{2k+1})$ belongs to the interval

$$(2 + 2/k, (4k+2)/(2k-1)]$$

. But, by Proposition 3.3 and the discussion in Case 2, no possible values exists in the interior of the interval. This proves that $\chi_c(H_{2k+1}) = (4k+2)/(2k-1)$. □

3.3. Bounding χ_c of other corona products

The next theorem gives the upper bound of the edge corona product $(G \diamond H, \sigma \diamond \tau)$.

Theorem 3.3. *Let (G, σ) and (H, τ) be two signed graphs. The circular chromatic number of the edge corona product*

$$\chi_c(G \diamond H, \sigma \diamond \tau) \leq \max\{\chi_c(G, \sigma), \chi_c(H, \tau) + 8\}.$$

Proof. Let $r = \max\{\chi(G, \sigma), \chi(H, \tau) + 8\}$. We show that there is a proper coloring of the edge corona product $(G \diamond H, \sigma \diamond \tau)$ with colors from C^r, the circle of circumference r obtained by identifying the endpoints of the interval $[0, r]$.

To start, we color the copy of (G, σ) with C^r and denote it by f. Such a coloring exists since $r \geq \chi_c(G, \sigma)$. Next, we get a coloring g of (H, τ) with C^{r-8}, again, since $r \geq \chi_c(H, \tau) + 8$.

Now, for each $e = uv \in E(G)$, suppose, WLOG, $f(u)$, $f(v)$, $\overline{f(u)}$ and $\overline{f(v)}$ are in cyclic order on C^r.

We further assume that the intervals $[f(u)-1, f(u)+1]$, $[f(v)-1, f(v)+1]$, $[\overline{f(u)}-1, \overline{f(u)}+1]$, and $[\overline{f(v)} - 1, \overline{f(v)} + 1]$ are mutually disjoint. Let map $h_e : C^{r-8} \to C^r$, be defined as

$$x \mapsto \begin{cases} x + f(u) + 1, & x \in [0, f(v) - f(u) - 2), \\ x + f(u) + 3, & x \in [f(v) - f(u) - 2, (r-8)/2), \\ x + f(u) + 5, & x \in [(r-8)/2, f(v) - f(u) + r/2 - 6), \\ x + f(u) + 7, & x \in [f(v) - f(u) + r/2 - 6, (r-8)/2, r - 8), \end{cases}$$

where the operations are taken to be modulo r. If, to the contrary, these intervals are not mutually disjoint, WLOG, we may assume $[f(u) - 1, f(u) + 1] \cap [f(v) - 1, f(v) + 1] \neq \emptyset$. Then, define h_e as

$$x \mapsto \begin{cases} x + f(v) + 1 & x \in [0, (r-8)/2), \\ x + f(v) + 5, & x \in [(r-8)/2, r - 8). \end{cases}$$

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Finally, we color the corresponding copy H_e of (H, τ) by the map $g_e = h_e \circ g$.

We see that the coloring g_e is proper on H_e, since g is proper and the map h_e only enlarges the distance between the original colors while preserving the antipodal property. Further, this coloring is not in conflict with the coloring f on the copy of (G, σ) since the range of g_e is at least 1 away from $f(u)$, $f(v)$ or their antipodals $\overline{f(u)}$, $\overline{f(v)}$. Therefore, f together with all the g_e form a proper coloring of $(G \diamond H, \sigma \diamond \tau)$ with C^r. □

Using the same coloring techniques, one can prove the similar result for the neighborhood corona product. Note that the only difference is that we have to mind the colors of at most $\Delta(G)+1$ vertices. We omit the proof.

Theorem 3.4. *Let (G, σ) and (H, τ) be two signed graphs. The Circular chromatic number of the neighborhood corona product $(G \star H, \sigma \star \tau)$ is given by*

$$\chi_c(G \star H, \sigma \star \tau) \leq \max\{\chi_c(G, \sigma), \chi_c(H, \tau) + 4(\Delta(G) + 1)\}$$

4. Conclusion and Open Problems

In this paper, we established upper bounds for the chromatic and circular chromatic numbers of vertex, edge, and neighborhood corona products of signed graphs. We demonstrated that the bound for circular chromatic number of vertex corona products is asymptotically tight through our construction of the signed graph family H_{2k+1}.

Several natural questions remain open:

Problem 1. *Determine the exact chromatic number of $\chi(G \odot H, \sigma \odot \tau)$ for general signed graphs. Specifically:*

- *Is there a characterization of pairs (G, σ), (H, τ) for which $\chi(G \odot H, \sigma \odot \tau) = \max\{\chi(G, \sigma), \chi(H, \tau) + 2\}$?*
- *Can the bound be improved to $\max\{\chi(G, \sigma), \chi(H, \tau) + 1\}$ under weaker conditions than those in Theorem 2.3?*

Problem 2. *Find families of signed graphs that achieve the upper bounds for edge and neighborhood corona products. In particular:*

- *Does there exist a family of signed graphs (G_n, σ_n) and (H_n, τ_n) such that*

$$\chi_c(G_n \diamond H_n, \sigma_n \diamond \tau_n) = \chi_c(H_n, \tau_n) + 8 - o(1)?$$

- *Can the bound $\chi_c(G \star H, \sigma \star \tau) \leq \chi_c(H, \tau) + 4(\Delta(G) + 1)$ be improved when G has special structure (e.g., when G is a tree or planar)?*

Problem 3. *Establish lower bounds for corona products. While our upper bounds are constructive, matching lower bounds would determine exact values. For instance:*

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- *What is the minimum value of* $\chi_c(G \odot H, \sigma \odot \tau) - \max\{\chi_c(G, \sigma), \chi_c(H, \tau)\}$ *over all non-trivial signed graphs?*
- *Can we characterize when this difference equals 0, 2, or 4?*

Problem 4. *Extend the results to other graph products. The natural next step is to investigate:*

- *Cartesian products:* $\chi_c(G \square H, \sigma \square \tau)$
- *Lexicographic products:* $\chi_c(G[H], \sigma[\tau])$
- *Strong products:* $\chi_c(G \boxtimes H, \sigma \boxtimes \tau)$

Understanding how signatures interact with different product structures could reveal deeper principles of signed graph coloring.

Problem 5. *Investigate the computational complexity. Determine the complexity of:*

- *Computing* $\chi(G \odot H, \sigma \odot \tau)$ *given* $\chi(G, \sigma)$ *and* $\chi(H, \tau)$
- *Deciding whether* $\chi_c(G \odot H, \sigma \odot \tau) \leq k$ *for a given rational* k

We hope these questions stimulate further research in the coloring theory of signed graphs and their products.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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