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Jordan canonical form of directed Dutch windmill graphs with application to Consensus

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Abstract

In this paper, we consider directed Dutch Windmill graphs, a class of graphs composed of multiple cycles sharing a common vertex. For this topology, the spectral analysis of their Laplacian matrix shows that for some eigenvalues, the algebraic multiplicities exceed their geometric multiplicities. Consequently, these Laplacian matrices are not diagonalizable and admit Jordan chains. We derive analytical expressions for the eigenvalues, eigenvectors, and associated Jordan chains. As an application, we address consensus problems in multi-agent systems. In this context, the network topology is represented by a graph $\mathcal{G}$. The transient responses and final states of the agents are determined by the initial conditions, as well as the eigenvalues and eigenvectors of the associated Laplacian matrix. When considering second-order consensus control, the state matrix derived from the Laplacian may not be diagonalizable due to the graph structure or singular parameter values. We demonstrate that this matrix can be reduced to a Jordan Canonical Form (JCF) and provide explicit analytical expressions for its eigenvalues and generalized eigenvectors.

Keywords:
graph theory, directed Dutch windmill graph, Laplacian matrix spectrum, second-order consensus, Jordan canonical form.
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1. Introduction

In graph theory, the spectral analysis of adjacency and Laplacian matrices allows one to analyze the structure of networks [13, 14]. Most of these works consider undirected graphs. For the latter, the adjacency and Laplacian matrices are symmetric and admit real eigenvalues; moreover, they are always diagonalizable [8].

In this context, windmill graphs are relevant models to depict certain real-world situations. This is the case for citation and social networks [5, 21]. In Ref. [5], Estrada provides analytical expressions for the eigenvalues and eigenvectors of these matrices for undirected windmill graphs. These studies were extended to generalized windmill graphs, wherein this structure is used as a model to describe an octopus neural network [12] and a metro network [9]. It may also be noticed that windmill graphs have received significant attention in chemical graph theory, mainly for computing indices [1].

Dutch Windmill (DWM) graphs are a special case of windmill graphs [5]. The DWM graph $\mathcal{D}_{\kappa}^{\eta}$ consists of η copies of the $\mathcal{C}_{\kappa}$ cyclic graph with a mutual vertex [6]. The well-known Friendship $\mathcal{F}^{\eta}$ graph is an undirected DWM graph with η cycles of length two sharing a mutual vertex. Some recent papers deal with the spectral analysis of directed windmill and friendship graphs. The characteristic polynomial and the eigenvalues of the anti-adjacency matrix of the directed $(4, n)$ windmill graph are computed in Ref. [22]. These elements are also discussed for the directed friendship graph [2].

By their very structure, directed Dutch windmill (DDWM) graphs exhibit spectral properties such that the algebraic and geometric multiplicities of nonzero eigenvalues of the Laplacian matrix differ. Their Laplacian matrix has multiple eigenvalues equal to 1. Consequently, the Laplacian matrix is not diagonalizable, rendering standard spectral analysis methods insufficient. To address this, our first contribution is to derive the characteristic polynomial for the remaining eigenvalues, the Jordan Canonical Form (JCF), the dimensions of the associated Jordan blocks and the generalized eigenvectors. A MATLAB code is provided [3], which computes the JCF of the Laplacian matrix and the associated transformation matrices for a DDWM graph. It has been tested for DDWM graphs containing up to 10,000 vertices. It is worth pointing out that Matlab's jordan function struggles to calculate the JCF of a DDWM graph with more than ten or so vertices.

Consensus methods have received considerable interest over the past two decades. In practice, they enable the control of multi-agent systems in a distributed way, such as swarms of mobile robots [18, 19], and more recently [10, 4]. The information networks wherein the agents operate are described with graphs. Seminal works were tackled in [15, 16]. These works concern linear first-order consensus protocols and address continuous and discrete time networks, time-delays in information exchange, time-varying network topologies, and directed and undirected information flows. Practically, the equations of motion for many vehicles require a second-order dynamic model. In a multi-agent approach, they are controlled through a linear second-order consensus protocol [17]. In Refs. [20, 24, 11], some necessary and sufficient conditions are provided to achieve second-order consensus. In Refs. [24, 11], for directed graphs, the authors provide the eigenvalues and eigenvectors of the second-order consensus state-space matrix from those of the Laplacian matrix. They highlight a Jordan block of size two associated with the zero eigenvalue of the Laplacian matrix. However, for the remainder of the eigenvalues, they only consider the case

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for which the algebraic and geometric multiplicities are equal, meaning the sub-matrix is assumed to be diagonalizable.

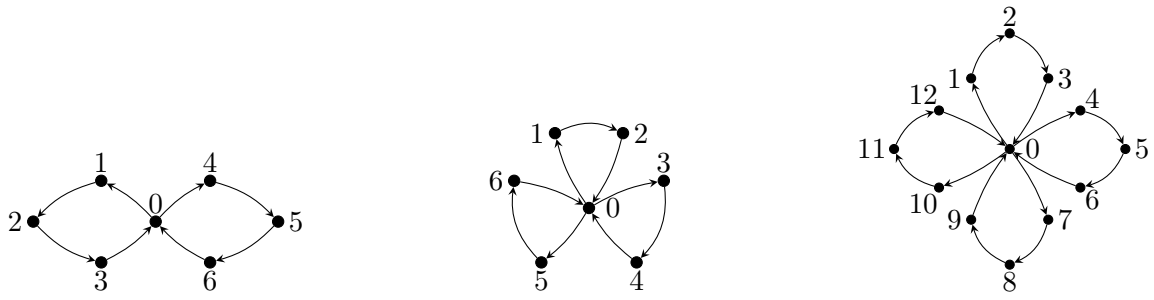
Our second contribution is to provide the systematic analytical derivation of the JCF for the second-order consensus state-space matrix in the general case where this sub-matrix is not diagonalizable. For any graph whose Laplacian matrix exhibits multiple eigenvalues where algebraic and geometric multiplicities differ, we provide the generalized eigenvectors necessary to describe the transient dynamics of the agents. To illustrate this general approach, we compute the JCF of the Laplacian matrix of the $\mathcal{D}_\kappa^\eta$ DDWM graph. Furthermore, we supply a second MATLAB code to compute the JCF of the consensus state matrix from the JCF of the Laplacian matrix. This code has been tested on graphs other than the DDWM.

This paper is organized as follows. In section 2, some elements about graphs and spectral graph theories are reviewed. The Jordan Canonical Form of the Laplacian matrix of the DDWM graph is derived and illustrated in section 3. Section 4 deals with the Jordan Canonical Form of the second-order consensus state matrix and proposes an application.

2. Preliminaries and notations

Graphs enable the description of information flows in networks of agents. A graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$ consisting of a set of vertices $\mathcal{V}$ and a set of edges $\mathcal{E} \subseteq \{\mathcal{V} \times \mathcal{V}\}$. Let $\nu_i \in \mathcal{N}$ be the i-th vertex of $\mathcal{G}$, the pair $(\nu_i, \nu_j) \in \mathcal{E}$ indicates a directed edge from j to i. In other words, i receives information from j and vertex j is adjacent to i. The set of vertices adjacent to vertex i is denoted by $\mathcal{N}_i$.

A cycle C_κ is a directed closed path with $\kappa + 1$ vertices with $\kappa \geq 1$ and a directed Dutch Windmill graph $\mathcal{D}_\kappa^\eta$ consists of $\eta > 1$ copies of the directed cycle graph C_κ with a vertex in common. This vertex is labeled as zero. For this graph, the number of vertices and edges are respectively $n = \eta\kappa + 1$ and $\eta(\kappa + 1)$.



(a) $\mathcal{D}_3^2$ (b) $\mathcal{D}_2^3$ (c) $\mathcal{D}_3^4$

Figure 1: The directed Dutch Windmill graphs $\mathcal{D}_3^2, \mathcal{D}_2^3, \mathcal{D}_3^4$

The adjacency matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is such that $a_{ij} = 1$ if $(\nu_i, \nu_j) \in \mathcal{E}$ and 0 otherwise. The degree matrix $\mathbf{D} \in \mathbb{R}^{n\times n}$ is a diagonal matrix with entries $d_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$ and $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix.

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For $\mathcal{C}_\kappa$, the cyclic oriented graph including κ vertices plus the zero vertex, the adjacency matrix reads:

$$\mathbf{A}(\mathcal{C}_\kappa) = \left(\begin{array}{c|cccc} [0] & [1] & \dots & \dots & [\kappa] \\ \hline 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{array}\right) \begin{matrix} [0] \\ [1] \\ \vdots \\ \\ [\kappa] \end{matrix} = \begin{pmatrix} 0 & \mathbf{e}_1^T \\ \mathbf{e}_\kappa & \mathbf{J}_0^\kappa \end{pmatrix}$$

The first row and column are labeled as $[0]$. Let $\mathbf{e}_j$ be the j-th standard basis vector of $\mathbb{R}^\kappa$ where $\mathbf{e}_j^T$ is its transpose. We denote $\mathbf{J}_\lambda^\kappa$ as the $\kappa \times \kappa$ Jordan block with eigenvalue λ. Note that for the eigenvalue 0, the block $\mathbf{J}_0^\kappa$ is a nilpotent matrix.

Let $\mathbf{I}_\kappa$ as the $\kappa \times \kappa$ identity matrix. For the DDWM graph, the adjacency matrix, the degree matrix and the Laplacian matrix read:

$$\mathbf{A}(\mathcal{D}_\kappa^\eta) = \left(\begin{array}{c|ccccc} [0] & [1]\dots[\kappa] & \dots & \dots & [(\eta-1)\kappa+1]\dots[\eta\kappa] \\ \hline 0 & \mathbf{e}_1^T & \dots & \dots & \mathbf{e}_1^T \\ \mathbf{e}_\kappa & \mathbf{J}_0^\kappa & & & \\ \vdots & & \ddots & & \mathbf{0} \\ \vdots & & & \ddots & \\ \vdots & & \mathbf{0} & & \\ \mathbf{e}_\kappa & & & & \mathbf{J}_0^\kappa \end{array}\right) \quad \mathbf{D}(\mathcal{D}_\kappa^\eta) = \left(\begin{array}{c|ccc} \eta & \mathbf{0} & \dots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_\kappa & & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & & \mathbf{I}_\kappa \end{array}\right)$$

$$\mathbf{L}(\mathcal{D}_\kappa^\eta) = \left(\begin{array}{c|ccccc} \eta & -\mathbf{e}_1^T & \dots & \dots & -\mathbf{e}_1^T \\ \hline -\mathbf{e}_\kappa & \mathbf{I}_\kappa - \mathbf{J}_0^\kappa & & & \\ \vdots & & \ddots & & \mathbf{0} \\ \vdots & \mathbf{0} & & \ddots & \\ -\mathbf{e}_\kappa & & & & \mathbf{I}_\kappa - \mathbf{J}_0^\kappa \end{array}\right)$$

$\mathbf{A}(\mathcal{D}_\kappa^\eta)$, $\mathbf{D}(\mathcal{D}_\kappa^\eta)$ and $\mathbf{L}(\mathcal{D}_\kappa^\eta)$ are in $\mathbb{R}^{n\times n}$ with $n = \eta\kappa + 1$.

3. Laplacian eigenvalues and eigenvectors of the DDWM graph

Theorem 3.1. *The Laplacian eigenvalues of the directed Dutch Windmill graph are:*

1. *The roots of the polynomial $(\eta - \lambda)(1-\lambda)^\kappa - \eta$ with $\lambda \in \mathbb{C}$. Moreover, these eigenvalues have an algebraic multiplicity of one, except for the case $\kappa = 2$ and $\eta = 4$, where the root 3 has an algebraic multiplicity of two.*
2. *The eigenvalue 1 with an algebraic multiplicity equal to $(\eta - 1)\kappa$*

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Proof. For a single eigenvalue λ and its associated eigenvector $\mathbf{p}$ or $\mathbf{p}(\lambda)$, we have $\mathbf{Lp} = \lambda\mathbf{p}$ with $\mathbf{p} = \begin{pmatrix} p_0 & p_1 & \dots & p_\kappa & \dots & p_{(\eta-1)\kappa+1} \dots & p_{\eta\kappa} \end{pmatrix}^T$, that is,

$$
\begin{aligned}
&(\eta-\lambda)p_0 = p_1 + p_{(\kappa+1)} + \dots + p_{(\eta-1)\kappa+1} && (1.0)\\
&\begin{cases} (1-\lambda)p_1 = p_2 \\ (1-\lambda)p_2 = p_3 \\ \dots = \\ (1-\lambda)p_\kappa = p_0 \end{cases} && (1.1)\\
&\quad\vdots && \vdots\\
&\begin{cases} (1-\lambda)p_{(\eta-1)\kappa+1} = p_{(\eta-1)\kappa+2} \\ (1-\lambda)p_{(\eta-1)\kappa+2} = p_{(\eta-1)\kappa+3} \\ \dots = \\ (1-\lambda)p_{\eta\kappa} = p_0 \end{cases} && (1.\eta)
\end{aligned}
$$

1. For $\lambda \neq 1$, the η subsystems (1.1)-(1.η) of κ equations are identical and yield the same solutions.

$$
\begin{cases} (\eta-\lambda)p_0 = p_1 + \dots + p_{(\eta-1)\kappa+1} \\ (1-\lambda)^\kappa p_1 = p_0 \\ \dots = \dots \\ (1-\lambda)^\kappa p_{(\eta-1)\kappa+1} = p_0 \end{cases} \tag{2}
$$

$$
\Leftrightarrow \begin{cases} (\eta-\lambda)p_0 = \eta p_1 = \dots = \eta p_{(\eta-1)\kappa+1} \\ p_1 = \dots = p_{(\eta-1)\kappa+1} \end{cases} \tag{3}
$$

$$
\Leftrightarrow \begin{cases} (\eta-\lambda)p_0 = \eta p_{m\kappa+1} \quad \text{with } m \in \{0 \dots \eta-1\} \\ (1-\lambda)^\kappa p_{m\kappa+1} = p_0 \end{cases} \tag{4}
$$

From Eq. (4) it follows that $\kappa+1$ eigenvalues of $\mathbf{L}$ are the roots of:

$$
Q(\lambda) = (\eta-\lambda)(1-\lambda)^\kappa - \eta \tag{5}
$$

From Gershgorin's theorem [8], these eigenvalues have non-negative real parts. Moreover, these roots are simple, if and only if $Q(\lambda)$ and its derivative $Q'(\lambda)$ share no common roots.

$$
Q'(\lambda) = (\kappa+1)(1-\lambda)^{\kappa-1}\left(\lambda - \frac{\eta\kappa+1}{\kappa+1}\right)
$$

The derivative $Q'(\lambda)$ has two roots, $\lambda = 1$ with a multiplicity order $\kappa-1$, and the single root $\frac{\eta\kappa+1}{\kappa+1}$. On the one hand, it is clear that $Q(1) \neq 0$. On the other hand, evaluating $Q(\frac{\eta\kappa+1}{\kappa+1})$ as the second root yields:

$$
Q\left(\frac{\eta\kappa+1}{\kappa+1}\right) = (\eta-1)(1-\eta)^\kappa \frac{\kappa^\kappa}{(\kappa+1)^{\kappa+1}} - \eta
$$

We seek conditions on η and κ for which $Q\left(\frac{\eta\kappa+1}{\kappa+1}\right) = 0$

- κ is odd. Since $\eta > 1$ we have $(1-\eta) < 0$ which implies $Q\left(\frac{\eta\kappa+1}{\kappa+1}\right) < 0$

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- κ is even. Then $Q\left(\frac{\eta\kappa+1}{\kappa+1}\right) = 0 \Leftrightarrow \kappa^{\kappa}(\eta-1)^{\kappa+1} = \eta(\kappa+1)^{\kappa+1}$

  Since $\gcd(\eta, \eta-1) = 1$, it follows that $\gcd(\eta, (\eta-1)^{\kappa+1}) = 1$; thus η must divide κ^{κ}.

  Similarly, since $\gcd(\kappa^{\kappa}, (\kappa+1)^{\kappa+1}) = 1$ we find that κ^{κ} divides η.

  Since η and κ^{κ} are both positive integers. We deduce that $\kappa^{\kappa} = \eta$ and $\eta-1 = \kappa+1$. It follows that $\kappa^{\kappa} = \kappa+2 \Leftrightarrow \kappa^{\kappa-1}(\kappa-1)$ then $\kappa = 2$ (since κ is even). This leads to $\eta = 4$ and 3 is a double root for Eq. (5).

The value $\lambda = 0$ is an obvious root for Eq. (5). It can also be noticed that the directed Dutch Windmill graph contains a directed spanning tree, which confirms that 0 is a simple root [19].

Concerning the eigenvectors $\mathbf{p}(\lambda)$ associated with the eigenvalues that solve Eq. (5), their entries are easily computed using Eq. (1.0) to Eq. (1.η). Let $\mathbf{p}(\lambda) = \begin{pmatrix} p_0(\lambda) & \tilde{\mathbf{p}}(\lambda)^T & \dots & \tilde{\mathbf{p}}(\lambda)^T \end{pmatrix}^T$ with $\tilde{\mathbf{p}}(\lambda)^= \begin{pmatrix} p_1(\lambda) & \dots & p_{\kappa}(\lambda) \end{pmatrix}^T$. In particular, the eigenvector $\mathbf{1}$ is associated with the eigenvalue 0.

Considering Eq. (1.1) to Eq. (1.η), it also appears that $\lambda = 1$ is an eigenvalue of $\mathbf{L}$ with an algebraic multiplicity $(\eta-1)\kappa$. Indeed, since $\mathbf{L}$ is a square matrix of order $\eta\kappa+1$ and $\kappa+1$ of its eigenvalues are solutions of Eq. (5), the remaining eigenvalue $\lambda = 1$ must have an algebraic multiplicity of $(\eta-1)\kappa$.

□

Lemma 3.1. *The geometric multiplicity of the eigenvalue $\lambda = 1$ of the Laplacian matrix L for the directed Dutch Windmill graph $\mathcal{D}_{\kappa}^{\eta}$ is exactly $\eta - 1$.*

Proof. For $l = 1$, the equation $(L - 1I_n)p_1 = 0$ determines the geometric multiplicity of $\lambda = 1$. The matrix J_0^{κ} has a rank of $\kappa - 1$, and its first column is a null vector. Consequently, the block-structured matrix $(L - 1I_n)$ contains η identical column vectors that span a subspace of dimension 1.

$$
\begin{array}{c}
 \\
\left(\mathbf{L} - 1\mathbf{I}_n\right) = \\
 \\
 \\
 \\
 \\
\end{array}
\begin{array}{c}
\begin{array}{ccccc} [0] & [1]\dots[\kappa] & \dots & \dots & [(\eta-1)\kappa+1]\dots[\eta\kappa] \end{array} \\
\left(\begin{array}{c|cccc}
\eta-1 & -\mathbf{e}_1^T & \dots & \dots & -\mathbf{e}_1^T \\
\hline
-\mathbf{e}_{\kappa} & -\mathbf{J}_0^{\kappa} & \mathbf{0} & \dots & \mathbf{0} \\
\vdots & \mathbf{0} & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \mathbf{0} \\
-\mathbf{e}_{\kappa} & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{J}_0^{\kappa}
\end{array}\right)
\end{array}
\begin{array}{l}
[0] \\
[1]\dots[\kappa] \\
\dots \\
\dots \\
[(\eta-1)\kappa+1]\dots[\eta\kappa]
\end{array}
$$

On the other hand, column $[0]$ has nonzero entries on rows $[\kappa] \pmod{\kappa}$, while the last row of J_0^{κ} is a null vector. For columns $[1]$ to $[\eta\kappa]$, the corresponding entries on these specific rows are zero. It follows that column $[0]$ is linearly independent of the other column vectors. Therefore, the

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matrix $(L - 1I_n)$ contains exactly $\eta - 1$ linearly dependent vectors, which implies $\dim(\ker(L - 1I_n)) = \eta - 1$. This establishes that the Jordan Canonical Form of L contains exactly $\eta - 1$ distinct Jordan blocks associated with $\lambda = 1$.

Furthermore, from this column structure, any vector of the form $p_1 = (0, p_1 e_1^T, \ldots, p_{(\eta-1)\kappa+1} e_1^T)^T$ subject to the zero-sum constraint $\sum_{m=0}^{\eta-1} p_{m\kappa+1} = 0$ is a valid eigenvector associated with $\lambda = 1$. □

Theorem 3.2. *The Laplacian matrix L of the directed Dutch Windmill graph $\mathcal{D}_\kappa^\eta$ admits a Jordan Canonical Form where exactly $\eta - 1$ Jordan blocks of size κ are associated with the eigenvalue 1.*

Proof. By Lemma 3.1, there are exactly $\eta-1$ distinct Jordan chains originating from the eigenvalue $\lambda = 1$. To determine the exact size of these blocks, we analyze the local propagation of the Jordan chains $(L - 1I_n)p_l = p_{l-1}$ for $2 \leq l \leq \kappa$.

For each cycle copy $m \in \{0, \ldots, \eta - 1\}$, the global matrix relation decouples into local component-wise scalar equations within each block. Since the matrix $(L - 1I_n)$ acts as a shifting operator within each block via $-J_0^\kappa$, the relation on row $m\kappa + l - 1$ simplifies to:

$$-p_{m\kappa+l} = p_{m\kappa+l-1}$$

where $p_{m\kappa+l}$ denotes the nonzero scalar entry of the generalized eigenvector p_l at position $m\kappa + l$.

By initializing this sequence with the explicit eigenvector p_1 from Lemma 3.1, an immediate induction shows that:

$$p_{m\kappa+l} = (-1)^{l-1} p_{m\kappa+1}$$

while the central coordinate remains $p_0 = 0$ for all $l < \kappa$. This confirms that each of the $\eta - 1$ linearly independent eigenvectors successfully sustains a Jordan chain of length at least κ, yielding generalized eigenvectors of the form $p_l = (0, p_l e_l^T, \ldots, p_{(\eta-1)\kappa+l} e_l^T)^T$ subject to $\sum_{m=0}^{\eta-1} p_{m\kappa+l} = 0$.

Consequently, the size of every Jordan block must be greater than or equal to κ. According to Theorem 3.1, the total algebraic multiplicity of the eigenvalue $\lambda = 1$ is exactly $(\eta - 1)\kappa$. Since there are exactly $\eta - 1$ blocks, the only mathematically possible partition to account for this total dimension is for every single block to have a size strictly equal to κ. □

Let $\{0, \lambda_1, \ldots, \lambda_\kappa\}$ the roots (with multiplicity of 1) of polynomial Eq. (5), then the JCF of the Laplacian matrix $\mathbf{L}$ can be written as:

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & \ldots & 0 & & & \\ 0 & \lambda_1 & & \mathbf{0} & & & \\ \vdots & & \ddots & & & \mathbf{0} & \\ 0 & \mathbf{0} & & \lambda_\kappa & & & \\ & & & & \mathbf{J}_1^\kappa & & \mathbf{0} \\ & \mathbf{0} & & & & \ddots & \\ & & & & \mathbf{0} & & \mathbf{J}_1^\kappa \end{pmatrix} \quad \text{where } \mathbf{J}_1^\kappa = \begin{array}{c} \begin{matrix} 1 & \ldots & \ldots & \kappa \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{matrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ \kappa \end{matrix}$$

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and $\mathbf{P}$ the associated transformation matrix reads:

$$\begin{array}{c|ccc|ccc|c|c} [0] & [1] & \dots & [\kappa] & \multicolumn{3}{c|}{\text{block of } \kappa \text{ eigenvectors repeated } \eta-1 \text{ times}} & \dots & \eta-1 \\ \left(\begin{array}{cccc|ccc|c|c} 1 & p_0(\lambda_1) & \vdots & p_0(\lambda_k) & 0 & 0 & 0 & \vdots & \vdots \\ \mathbf{1} & \tilde{\mathbf{p}}(\lambda_1) & \vdots & \tilde{\mathbf{p}}(\lambda_k) & p_1\mathbf{e}_1 & p_\ell\mathbf{e}_\ell & p_\kappa\mathbf{e}_\kappa & \vdots & \vdots \\ \mathbf{1} & \vdots & \vdots & \vdots & p_{\kappa+1}\mathbf{e}_1 & p_{\kappa+\ell}\mathbf{e}_\ell & p_{2\kappa}\mathbf{e}_\kappa & \vdots & \vdots \\ \mathbf{1} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{1} & \tilde{\mathbf{p}}(\lambda_1) & \vdots & \tilde{\mathbf{p}}(\lambda_k) & p_{(\eta-1)\kappa+1}\mathbf{e}_1 & p_{(\eta-1)\kappa+\ell}\mathbf{e}_\ell & p_{\eta\kappa}\mathbf{e}_\kappa & \vdots & \vdots \end{array}\right) \end{array} \quad (6)$$

$$\text{subject to constraints} \quad \sum_{m=0}^{\eta-1} p_{m\kappa+\ell} = 0 \quad \text{with } m \in \{0,\dots,\eta-1\}$$

3.1. Example: $\mathcal{D}_2^3$ Laplacian matrix JCF and eigenvectors

Let the $\mathcal{D}_2^3$ directed Dutch Windmill graph depicted in Fig. 1 for which $n = 7$. There are $\kappa + 1 = 3$ complex or real eigenvalues, including zero, plus the eigenvalue 1 with an algebraic multiplicity $n-(\kappa+1) = 4$. Moreover the JCF of its Laplacian matrix $\mathbf{L}$ admits $\eta - 1 = 2$ Jordan blocks of size $\kappa = 2$. We have:

$$\mathbf{L} = \begin{pmatrix} 3 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

According to Eq. (5), the eigenvalues of $\mathbf{L}$ are the roots of the given polynomial

$$0 = \lambda^3 - 5\lambda^2 + 7\lambda$$

and the eigenvalue 1 with an algebraic multiplicity $(\eta-1)\kappa = 4$. Thus, spe$(\mathbf{L}) = \{0, \frac{5\pm i\sqrt{3}}{2}, 1, 1, 1, 1\}$. Computing the associated eigenvectors yields:

- $\lambda_0 = 0$: the sum of the rows of $\mathbf{L}$ is zero, thus $\mathbf{p}_0 = \mathbf{1}^T$.
- $\lambda_1 = \frac{5-i\sqrt{3}}{2}$: setting $p_{21} = 1$ in Eq. (1.1) yields $(1 - \frac{5+i\sqrt{3}}{2}), p_{11} = p_{21}$ and $p_{11} = \frac{-3-i\sqrt{3}}{6}$. Next $p_{01} = 1 - \lambda_1 = \frac{-3+i\sqrt{3}}{2}$. Moreover $p_{31} = p_{51} = p_{11}$ and $p_{41} = p_{61} = p_{21}$.
- $\lambda_2 = \frac{5+i\sqrt{3}}{2}$: setting $p_{22} = 1$ yields $p_{12} = \frac{-3+i\sqrt{3}}{6}$. Next $p_{02} = \frac{-3-i\sqrt{3}}{2}$. Moreover $p_{32} = p_{52} = p_{12}$ and $p_{42} = p_{62} = p_{12}$.
- $\lambda_{3,\dots,6} = 1$: two independent eigenvectors satisfying $(\mathbf{L} - 1\mathbf{I}_n) = \mathbf{0}$ under the constraint Eq. (1.0) can be found. Let $\mathbf{p}_3 = (0, 1, 0, -1, 0, 0, 0)^T$ and $\mathbf{p}_5 = (0, 1, 0, 0, 0, -1, 0)^T$.

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Since the algebraic multiplicity of the eigenvalue 1 is 4 and its geometric multiplicity is 2, for $\ell = 2$, we directly obtain the generalized eigenvectors $\mathbf{p}_4 = (0, 0, 1, 0, -1, 0, 0)^T$ and $\mathbf{p}_6 = (0, 0, -1, 0, 0, 0, 1)^T$.

The JCF and the transformation matrix of $\mathbf{L}$ can now be constructed:

$$\mathbf{J} = \left(\begin{array}{c|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{5-i\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5+i\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad \mathbf{P} = \left(\begin{array}{ccc|cccc} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 & \mathbf{p}_5 & \mathbf{p}_6 \\ \hline 1 & \frac{-3+i\sqrt{3}}{2} & \frac{-3-i\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 1 & \frac{-3-i\sqrt{3}}{6} & \frac{-3+i\sqrt{3}}{6} & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 \\ 1 & \frac{-3-i\sqrt{3}}{6} & \frac{-3+i\sqrt{3}}{6} & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & \frac{-3-i\sqrt{3}}{6} & \frac{-3+i\sqrt{3}}{6} & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

The corresponding MATLAB code is available at [3]. It has been tested with DDWM graphs containing up to 10000 vertices. By way of comparison, the `jordan` function in MATLAB fails to compute the Jordan canonical form of the Laplacian matrix of a $\mathcal{D}_{10}^{10}$ DDWM. Tests were conducted on an Intel Core i7-9750H @ 2.60GHz and 16GB RAM. Eigenvalues and eigenvectors may appear in a different order from the one proposed in this example.

4. JCF of the Second-Order Consensus State Matrix

We now analyze the implications of the structural properties derived in Section 3 on the convergence of a second-order consensus dynamics. We consider a dynamical system defined over a directed graph $\mathcal{G}$ with n agents. The dynamics are governed by the Laplacian matrix $\mathbf{L}$ of graph $\mathcal{G}$. Let $\mathbf{x}$ in $\mathbb{R}^n$ and $\dot{\mathbf{x}} = \mathbf{v}$ be the position and the velocity vectors of the agents.

Let $\mathbf{X}^T = \begin{pmatrix} \mathbf{x} & \dot{\mathbf{x}} \end{pmatrix} \in \mathbb{R}^{2n}$ be the state vector. This dynamical system can be described in a state-space form as:

$$\dot{\mathbf{X}} = -\tilde{\mathbf{L}}\mathbf{X} \text{ with } \tilde{\mathbf{L}} = \begin{pmatrix} \mathbf{0}_n & -\mathbf{I}_n \\ \mathbf{L} & \xi\mathbf{L} \end{pmatrix} \tag{7}$$

where $\tilde{\mathbf{L}} \in \mathbb{R}^{2n \times 2n}$ is the state matrix for the second-order consensus system derived from the Laplacian matrix and $\xi > 0$ is a scalar damping parameter. For a directed graph, to ensure consensus, ξ must be chosen to satisfy the conditions described in [24].

In the following, we compute the spectrum and the generalized eigenvectors of $\tilde{\mathbf{L}}$ when it has eigenvalues with algebraic multiplicities greater than the geometric multiplicities. This case is mentioned in [24, 11] and concerns the zero eigenvalue which appears with an algebraic multiplicity equal to two. To the best of our knowledge, the derivation of the JCF has not yet been established for nonzero eigenvalues with algebraic multiplicities exceeding their corresponding geometric multiplicities. We address this gap by providing explicit analytical expressions for the Jordan chain associated with such eigenvalues.

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Let $\mu_{k1,2}$ with $k \in \{0 \dots n-1\}$ be the eigenvalues of $\tilde{\mathbf{L}}$ with non-negative real part. Their computation is detailed in Appendix 5.1.

$$\mu_{k1,2} = \frac{\xi\lambda_k \pm \sqrt{(\xi\lambda_k)^2 - 4\lambda_k}}{2} \tag{8}$$

Where λ_k is the k-th eigenvalue of $\mathbf{L}$.

Note that the single zero eigenvalue of $\mathbf{L}$ provides the two zero eigenvalues of $\tilde{\mathbf{L}}$. In general terms, for the nonzero eigenvalues λ_k whose algebraic multiplicity is γ, the corresponding eigenvalues $\{\mu_{k1}, \mu_{k2}\}$ have the same algebraic multiplicity. For $\xi = \frac{2}{\sqrt{\lambda_k}}$ and $\lambda_k \in \mathbb{R}^+$, $\mu_{k1} = \mu_{k2}$ so their algebraic multiplicity is twice the algebraic multiplicity of λ_k. This case is not studied here.

Now let $\mathbf{\Phi}_{k1,2}$ be the eigenvectors associated with $\mu_{k1,2}$. In the following, we recall the analytical expressions of said eigenvectors when associated with the zero eigenvalues and the nonzero eigenvalues with equal algebraic and geometric multiplicities. As our second contribution, we provide an analytical expression of these eigenvectors when the algebraic multiplicity is greater than the geometric multiplicity for a given eigenvector.

4.1. Eigenvectors associated with the zero eigenvalues

These eigenvectors are calculated for a directed or an undirected graph. The calculation is briefly described in [24] and recalled in Appendix 5.2.
Let $\mathbf{\Phi}_{01}^T = \begin{pmatrix} \boldsymbol{\chi}_{01}^T & \mathbf{\Psi}_{01}^T \end{pmatrix} \in \mathbb{R}^{2n}$ the eigenvector of $\tilde{\mathbf{L}}$ associated with $\mu_{01} = 0$. By definition $\tilde{\mathbf{L}}\mathbf{\Phi}_{01} = \mathbf{0}$ and $\mathbf{\Psi}_{01} = \mathbf{0}$. By choosing $\boldsymbol{\chi}_{01} = \frac{1}{\sqrt{n}}\mathbf{1}$, one obtains:

$$\mathbf{\Phi}_{01}^T = \begin{pmatrix} \frac{1}{\sqrt{n}}\mathbf{1}^T & \mathbf{0}^T \end{pmatrix} \tag{9}$$

For the double-zero eigenvalue, the associated geometric multiplicity is 1. For $\mu_{02} = 0$, the associated eigenvector reads:

$$\mathbf{\Phi}_{02}^T = \begin{pmatrix} \mathbf{0}^T & -\frac{1}{\sqrt{n}}\mathbf{1}^T \end{pmatrix} \text{ or } \mathbf{\Phi}_{02}^T = \frac{1}{\sqrt{n}} \begin{pmatrix} \mathbf{1}^t & -\mathbf{1}^T \end{pmatrix} \tag{10}$$

4.2. Eigenvectors associated with the nonzero eigenvalues

Case 1. The algebraic and geometric multiplicities are equal
This condition always holds for undirected graphs but is not necessarily true for directed graphs [23]. The calculation is provided in [11] and detailed in the Appendix 5.3.1. Let λ_k be a nonzero eigenvalue of $\mathbf{L}$ and $\mathbf{p}_k$ its associated eigenvector.

Let $\mathbf{\Phi}_{k1,2}^T = \begin{pmatrix} \boldsymbol{\chi}_{k1,2}^T & \mathbf{\Psi}_{k1,2}^T \end{pmatrix} \in \mathbb{C}^{2n}$ the eigenvector of $\tilde{\mathbf{L}}$ associated with the nonzero eigenvalue $\mu_{k1,2}$.

$$\mathbf{\Phi}_{k1,2}^T = \begin{pmatrix} \mathbf{p}_k^T & -\mu_{k1,2}\mathbf{p}_k^T \end{pmatrix} \tag{11}$$

Case 2. The algebraic multiplicity is greater than the geometric multiplicity and $\mu_{k1} \neq \mu_{k2}$
Consider a directed graph and λ_k a real/complex eigenvalue of its Laplacian matrix $\mathbf{L}$, with an algebraic multiplicity γ. When the algebraic and geometric multiplicities do not coincide, the generalized eigenvectors $\mathbf{p}_{k,j}$ are defined by the recurrence relation:

$$\mathbf{L}\mathbf{p}_{k,j} = \lambda\mathbf{p}_{k,j} + \mathbf{p}_{k,j-1} \tag{12}$$

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From Eq. (8), $\mu_{k1,2}$ also have an algebraic multiplicity γ. Let $\mathbf{\Phi}_{k1,2}$ be their associated generalized eigenvectors. For the sake of clarity, we focus on the derivation for $\mathbf{\Phi}_{k1}$ and omit the index k hereafter. Specifically we denote $\mathbf{\Phi}_{k+j,1}$ as $\mathbf{\Phi}_j$ with $j \in \{0 \dots \gamma - 1\}$, $\mu_{k1,2}$ as $\mu_{1,2}$ and $\mathbf{p}_{k,j}$ as $\mathbf{p}_j$.

Theorem 4.1. *Let $\tilde{\mathbf{L}}$ be a second-order consensus state space matrix such that some of its nonzero eigenvalues μ_{k1} have an algebraic multiplicity γ greater than their corresponding geometric multiplicity. Then, the associated generalized eigenvectors $\mathbf{\Phi}_j$ with $j \in \{0 \dots \gamma - 1\}$ read:*

$$\mathbf{\Phi}_j^T = \begin{pmatrix} \boldsymbol{\chi}_j^T & \mathbf{\Psi}_j^T \end{pmatrix}$$

where

$$\begin{aligned} \boldsymbol{\chi}_j &= \delta\rho\alpha_{j-1}\mathbf{p}_j + \sum_{\ell=0}^{j-2} \rho(\delta\alpha_\ell + \beta_\ell - \xi\alpha_{\ell+1})\mathbf{p}_{\ell+1} \\ \mathbf{\Psi}_j &= -\mu_1\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1} \end{aligned}$$

with $\delta = \mu_1 - \mu_2$, $\rho = \frac{\mu_2}{\mu_1}$ and α_ℓ, $\beta_\ell \in \mathbb{C}$.

Proof. By recurrence:

Expression of the initial eigenvector of the Jordan chain $\mathbf{\Phi}_0$.
From (11):

$$\mathbf{\Phi}_0^T = \begin{pmatrix} \mathbf{p}_0^T & -\mu_1\mathbf{p}_0^T \end{pmatrix} \tag{13}$$

and $\boldsymbol{\chi}_0 = \mathbf{p}_0$. Then $\alpha_0 = 1$ and $\beta_0 = 0$.

Expression of j^{th} generalized eigenvector of the Jordan chain $\mathbf{\Phi}_j$.

$$\tilde{\mathbf{L}}\mathbf{\Phi}_j = \mu_1\mathbf{\Phi}_j + \mathbf{\Phi}_{j-1}$$

$$\begin{pmatrix} \mathbf{0}_n & -\mathbf{I}_n \\ \mathbf{L} & \xi\mathbf{L} \end{pmatrix} \begin{pmatrix} \boldsymbol{\chi}_j \\ \mathbf{\Psi}_j \end{pmatrix} = \mu_1 \begin{pmatrix} \boldsymbol{\chi}_j \\ \mathbf{\Psi}_j \end{pmatrix} + \begin{pmatrix} \boldsymbol{\chi}_{j-1} \\ \mathbf{\Psi}_{j-1} \end{pmatrix}$$

$$-\mathbf{\Psi}_j = \mu_1\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1} \tag{14}$$

$$\mathbf{L}\boldsymbol{\chi}_j + \xi\mathbf{L}\mathbf{\Psi}_j = \mu_1\mathbf{\Psi}_j + \mathbf{\Psi}_{j-1} \tag{15}$$

By substitution, we have:

$$\begin{aligned} & \mathbf{L}\left(\boldsymbol{\chi}_j - \xi(\mu_1\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1})\right) = -\mu_1(\mu_1\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}) - (\mu_1\boldsymbol{\chi}_{j-1} + \boldsymbol{\chi}_{j-2}) \\ \Leftrightarrow\ & \mathbf{L}\left((\xi\mu_1 - 1)\boldsymbol{\chi}_j + \xi\boldsymbol{\chi}_{j-1}\right) = \mu_1^2\boldsymbol{\chi}_j + 2\mu_1\boldsymbol{\chi}_{j-1} + \boldsymbol{\chi}_{j-2} \end{aligned}$$

Let:

$$\begin{aligned} \boldsymbol{\chi}_{j-1} &= \textstyle\sum_{\ell=0}^{j-1} \alpha_\ell\mathbf{p}_\ell \\ \boldsymbol{\chi}_{j-2} &= \textstyle\sum_{\ell=0}^{j-2} \beta_\ell\mathbf{p}_\ell \end{aligned} \tag{16}$$

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With α_ℓ, β_ℓ being complex numbers, not all zero.

$$\mathbf{L}\left((\xi\mu_1-1)\boldsymbol{\chi}_j+\xi\textstyle\sum_{\ell=0}^{j-1}\alpha_\ell\mathbf{p}_\ell\right) = \mu_1^2\boldsymbol{\chi}_j+2\mu_1\textstyle\sum_{\ell=0}^{j-1}\alpha_\ell\mathbf{p}_\ell+\sum_{\ell=0}^{j-2}\beta_\ell\mathbf{p}_\ell$$

From Eq. (12),

$$\begin{aligned}\mathbf{L}\left((\xi\mu_1-1)\boldsymbol{\chi}_j\right) &= \mu_1^2\boldsymbol{\chi}_j+(2\mu_1-\xi\lambda)\textstyle\sum_{\ell=0}^{j-1}\alpha_\ell\mathbf{p}_\ell+\sum_{\ell=0}^{j-2}(\beta_\ell-\xi\alpha_{\ell+1})\mathbf{p}_\ell\\ \mathbf{L}\left((\xi\mu_1-1)\boldsymbol{\chi}_j\right) &= \mu_1^2\boldsymbol{\chi}_j+(2\mu_1-\xi\lambda)\alpha_{j-1}\mathbf{p}_{j-1}+\textstyle\sum_{\ell=0}^{j-2}\left[(2\mu_1-\xi\lambda)\alpha_\ell+\beta_\ell-\xi\alpha_{\ell+1}\right]\mathbf{p}_\ell\end{aligned}$$

Next, we divide by $(2\mu_1-\xi\lambda)\alpha_{j-1}$, substitute $\mathbf{p}_\ell=\mathbf{L}\mathbf{p}_{\ell+1}-\lambda\mathbf{p}_{\ell+1}$ and distribute the terms of the sum across the LHS and the RHS.

$$\mathbf{L}\left(\frac{\xi\mu_1-1}{(2\mu_1-\xi\lambda_k)\alpha_{j-1}}\boldsymbol{\chi}_j-\textstyle\sum_{\ell=0}^{j-2}\left[\frac{\alpha_\ell}{\alpha_{j-1}}+\frac{\beta_\ell-\xi\alpha_{\ell+1}}{(2\mu_1-\xi\lambda_k)\alpha_{j-1}}\right]\mathbf{p}_{\ell+1}\right) = \dots$$

$$\mathbf{p}_{j-1}+\frac{\mu_1^2}{(2\mu_1-\xi\lambda_k)\alpha_{j-1}}\boldsymbol{\chi}_j-\lambda_k\textstyle\sum_{\ell=0}^{j-2}\left[\frac{\alpha_\ell}{\alpha_{j-1}}+\frac{\beta_\ell-\xi\alpha_{\ell+1}}{(2\mu_1-\xi\lambda)\alpha_{j-1}}\right]\mathbf{p}_{\ell+1}$$

It is easy to see that $2\mu_1-\xi\lambda=\mu_1-\mu_2=\delta$ and $\dfrac{1}{\xi\mu_1-1}=\dfrac{\mu_2}{\mu_1}=\rho$. It follows that:

$$\mathbf{L}\left[\left(\frac{1}{\delta\rho\alpha_{j-1}}\right)\boldsymbol{\chi}_j-\textstyle\sum_{\ell=0}^{j-2}\left(\frac{\alpha_\ell}{\alpha_{j-1}}+\frac{\beta_\ell-\xi\alpha_{\ell+1}}{\delta\alpha_{j-1}}\right)\mathbf{p}_{\ell+1}\right] = \dots$$

$$\lambda\left[\left(\frac{1}{\delta\rho\alpha_{j-1}}\right)\boldsymbol{\chi}_j-\textstyle\sum_{\ell=0}^{j-2}\left(\frac{\alpha_\ell}{\alpha_{j-1}}+\frac{\beta_\ell-\xi\alpha_{\ell+1}}{\delta\alpha_{j-1}}\right)\mathbf{p}_{\ell+1}\right]+\mathbf{p}_{j-1}$$

By identification terms in Eq. (12):

$$\mathbf{p}_j=\frac{1}{\delta\rho\alpha_{j-1}}\left[\boldsymbol{\chi}_j-\sum_{\ell=0}^{j-2}\rho\left(\delta\alpha_\ell+\beta_\ell-\xi\alpha_{\ell+1}\right)\mathbf{p}_{\ell+1}\right]$$

The summation on the RHS is valid for $j-2\geq 0$ and

$$\boldsymbol{\chi}_j=\delta\rho\alpha_{j-1}\mathbf{p}_j+\sum_{\ell=0}^{j-2}\rho(\delta\alpha_\ell+\beta_\ell-\xi\alpha_{\ell+1})\mathbf{p}_{\ell+1} \tag{17}$$

Finally $\boldsymbol{\Psi}_j$ is obtained from (14). □

Remark 4.1. In standard second order consensus systems, convergence depends solely on the eigenvalues of the state matrix $\tilde{\mathbf{L}}$. In contrast, the presence of Jordan blocks whose effects are explicitly captured by the recurrence coefficients α_j, β_j implies that the evolution of the state vector cannot be expressed as a combination of simple decaying exponentials. Instead, the dynamics present polynomial convergence modes $t^je^{-\mu t}$ [7]. This recurrence relation shows that the convergence of an agent in the network depends not only on its own spectral mode, but also on the

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modes of its neighbors within the Jordan chain. This highlights the sensitivity of the consensus convergence to the graph topology. The longer the Jordan chain , the more significant the transient polynomial growth, which may lead to temporary oscillations or delayed convergence despite a theoretically stable eigenvalue $Re(-\mu) < 0$. Theorem 4.1 provides an analytical expression that captures this coupling, allowing us to predict the system's behavior beyond standard eigenvalue analysis.

4.3. Iterative calculation of the first generalized eigenvectors

From Eq. (13), we have $\boldsymbol{\chi_0} = \mathbf{p}_0$.

- Then, for $j = 1$ and from Eq. (16): $\alpha_0 = 1$, $\beta_0 = 0$

  From Eq. (17),(14), hence: $\boldsymbol{\chi_1} = \delta\rho\mathbf{p}_1$ and $\boldsymbol{\Psi}_1 = -\mu_{k1}\delta\rho\mathbf{p}_1 - \mathbf{p}_0$

- $j = 2$: $\alpha_1 = \delta\rho$, $\alpha_0 = 0$, $\beta_1 = 1$, $\beta_0 = 0$

  $\boldsymbol{\chi_2} = (\delta\rho)^2\mathbf{p}_2 + \rho^3\mathbf{p}_1$ and $\boldsymbol{\Psi}_2 = -\mu_{k1}(\delta\rho)^2\mathbf{p}_2 - (\mu_{k1}\rho^3 + \delta\rho)\mathbf{p}_1$

- $j = 3$: $\alpha_2 = (\delta\rho)^2$, $\alpha_1 = \rho^3$, $\alpha_0 = 0$, $\beta_1 = \delta\rho$, β_0=0

  $\boldsymbol{\chi_3} = (\delta\rho)^3\mathbf{p}_3 + 2\delta\rho^4\mathbf{p}_2 - \xi\rho^4\mathbf{p}_1$

- $j = 4$: $\alpha_3 = (\delta\rho)^3$, $\alpha_2 = 2\delta\rho^4$, $\alpha_1 = -\xi\rho^4$, $\alpha_0 = 0$, $\beta_2 = (\delta\rho)^2$, $\beta_1 = \rho^3$, $\beta_0 = 0$

  $\boldsymbol{\chi_4} = (\delta\rho)^4\mathbf{p}_4 + 3\delta^2\rho^5\mathbf{p}_3 + \rho^4(3\rho^2 - 2)\mathbf{p}_2 + \xi^2\rho^5\mathbf{p}_1$

- ...

For μ_{k2}, the results are identical with $\rho = \dfrac{\mu_{k1}}{\mu_{k2}}$ and $\delta = \mu_{k2} - \mu_{k1}$. The code computing these eigenvectors is made available to the reader at [3].

4.4. Application

We consider a directed network described with the above DDWM $\mathcal{D}_2^3$ graph studied in subsection 3.1. The system dynamics are governed by the second-order consensus model in Eq. (7). The spectrum of $\mathbf{L}$ and $\tilde{\mathbf{L}}$ ($\xi = 1$) are summarized in Tab.1. The eigenvalues λ_k and the eigenvectors $\mathbf{p}_k$ of $\mathbf{L}$ are those obtained in subsection 3.1. The associated generalized eigenvectors are obtained from Eq. (9), (10), (11) and §4.3 case $j = 1$. This gives the transformation matrix $\boldsymbol{\Phi}$ and then the JCF of the state space matrix $\tilde{\mathbf{J}} = \boldsymbol{\Phi}^{-1}\tilde{\mathbf{L}}\boldsymbol{\Phi}$. The structures of these matrices are illustrated on Fig. 2. They are helpful to predict the dynamics of the agents. That of the $\tilde{J}$ matrix reveals the length of the Jordan chains. For the transition matrix $\boldsymbol{\Phi}$, the colors of the markers reflect the magnitudes of the coordinates which weight the modal decay term $t^j e^{-\mu t}$.

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k	0	1	2	3	4	5	6
λ_k	0	$\frac{5-i\sqrt{3}}{2}$	$\frac{5+i\sqrt{3}}{2}$	1 with an algebraic multiplicity of 4			
μ_{k1}	0	$1.351-1.49i$	$1.351+1.49i$	$\frac{1+i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$
$\mathbf{\Phi}_{k1}$	$\frac{1}{\sqrt{n}}\mathbf{1}$ $\mathbf{0}$	$\mathbf{p}_1$ $\mu_{11}\mathbf{p}_1$	$\mathbf{p}_2$ $\mu_{21}\mathbf{p}_2$	$\mathbf{p}_3$ $\mu_{31}\mathbf{p}_3$	$\delta\rho\mathbf{p}_4$ $-\mu_{31}\delta\rho\mathbf{p}_1-\mathbf{p}_3$	$\mathbf{p}_5$ $\mu_{31}\mathbf{p}_5$	$\delta\rho\mathbf{p}_6$ $-\mu_{31}\delta\rho\mathbf{p}_6-\mathbf{p}_5$
μ_{k2}	0	$1.14-0.63i$	$1.14+0.63i$	$\frac{1-i\sqrt{3}}{2}$	$\frac{1-i\sqrt{3}}{2}$	$\frac{1-i\sqrt{3}}{2}$	$\frac{1-i\sqrt{3}}{2}$
$\mathbf{\Phi}_{k2}$	$\frac{1}{\sqrt{n}}\mathbf{1}$ $\frac{-1}{\sqrt{n}}\mathbf{1}$	$\mathbf{p}_1$ $\mu_{12}\mathbf{p}_1$	$\mathbf{p}_2$ $\mu_{22}\mathbf{p}_2$	$\mathbf{p}_3$ $\mu_{32}\mathbf{p}_3$	$\delta\rho\mathbf{p}_4$ $\mu_{32}\frac{\delta}{\rho}\mathbf{p}_4-\mathbf{p}_3$	$\mathbf{p}_5$ $\mu_{32}\mathbf{p}_5$	$\delta\rho\mathbf{p}_6$ $\mu_{32}\frac{\delta}{\rho}\mathbf{p}_6-\mathbf{p}_5$

Table 1: Eigenvalues and eigenvectors of $\tilde{\mathbf{L}}$ for the DDWM $\mathcal{D}_2^3$ graph

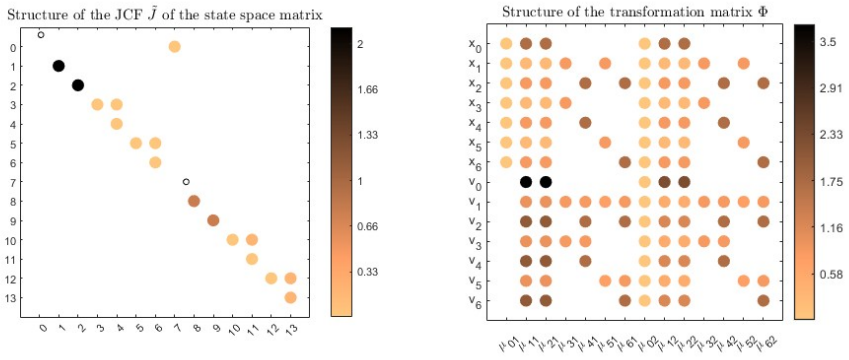


Figure 2: JCF and transformation matrix structures of the 2^{nd} order state space matrix

Regarding the state matrix $\tilde{\mathbf{L}}$, the double-zero eigenvalues $\{\mu_{10}, \mu_{20}\}$ define the consensus manifold. In addition, the spectrum exhibits pairs of complex conjugate eigenvalues characterizing second order modes; their positive real parts guarantee convergence. The structural impact of the DDWM topology is twofold.

On the one hand, it appears that the second order modes defined by $\{\mu_{11}, \mu_{21}\}$, $\{\mu_{12}, \mu_{22}\}$ contribute to the dynamics of the set of agents. These modes are predominant in agent 0 which acts as a hub, followed by agents $\{2, 4, 6\}$ and finally agents $\{1, 3, 5\}$. This scheduling reflects the direction of information flow.

On the other hand, the two sets of four complex eigenvalues have the smallest real parts $\{\mu_{13} \dots \mu_{16}\}$ and $\{\mu_{23} \dots \mu_{36}\}$. They dictate the slowest decay rate, often referred to as the dominant spectral mode. Moreover, the structure of the JCF shows that the length of their Jordan chain is two. This latter provides the polynomial modulation of the exponential decay $t^j e^{-\mu t}$. Logically, the longer

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the chain, the slower the transient responses. The transition matrix $\mathbf{\Phi}$ points out that the dominant mode does not contribute to the dynamic of agent 0 which has the fastest transient response. This is shown in Fig.3.

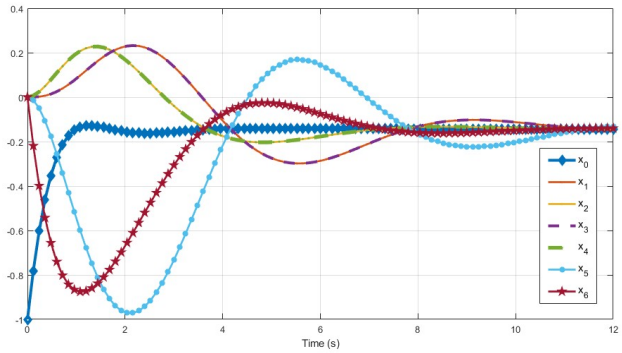


Figure 3: Agent positions under a second-order consensus on DDWM $\mathcal{D}_2^3$ graph.

The proposed method is not limited to DDWM graphs. As it is derived from the computation of the Jordan Chains of the Laplacian matrix, it can be extended to any class of graphs with cascaded structures in which the algebraic multiplicity of the Laplacian exceeds the geometric multiplicity. Numerical validation was performed for several classes of directed cycle-chain graphs exhibiting non-diagonalizable Laplacians [3].

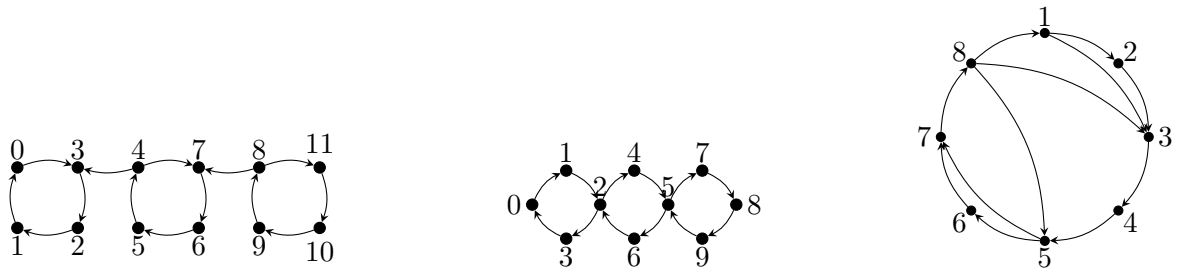


Figure 4: Directed cycle chains and circular graph with non-trivial Laplacian Jordan structure

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5. Appendice

5.1. Eigenvalues of $\tilde{\mathbf{L}}$

The following results are provided in [24], [11]. They are relevant for directed and undirected graphs. The details of the calculation are not explicitly described in these papers. From [8],

$$\det\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} = \det(\mathbf{D})\det(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})$$

$$\det(\mathbf{A}\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$$

Thus:

$$\begin{aligned}\det(\mu\mathbf{I}_{2n}-\tilde{\mathbf{L}}) &= \det\begin{pmatrix}\mu\mathbf{I}_n & \mathbf{I}_n\\ -\mathbf{L} & \mu\mathbf{I}_n-\xi\mathbf{L}\end{pmatrix}\\ &= \det(\mu\mathbf{I}_n-\xi\mathbf{L})\det\left(\mu\mathbf{I}_n+(\mu\mathbf{I}_n-\xi\mathbf{L})^{-1}\mathbf{L}\right)\\ &= \det\left((\mu\mathbf{I}_n-\xi\mathbf{L})\mu\mathbf{I}_n+\mathbf{L}\right)\\ &= \det\left(\mu^2\mathbf{I}_n-(\mu\xi-1)\mathbf{L}\right)\end{aligned} \tag{18}$$

On the one hand, $\mathbf{L}$ is a real square matrix thus it admits a JCF, let $\mathbf{J}=\mathbf{P}^{-1}\mathbf{L}\mathbf{P}$. On the other hand, the determinant is invariant under a change of basis then $\det(\mathbf{J})=\det(\mathbf{L})$.

$$\begin{aligned}\det\left(\mu^2\mathbf{I}_n-(\mu\xi-1)\mathbf{L}\right) &= \det\left(\mathbf{P}^{-1}\left(\mu^2\mathbf{I}_n-(\mu\xi-1)\mathbf{L}\right)\mathbf{P}\right)\\ &= \det\left(\mu^2\mathbf{I}_n-(\mu\xi-1)\mathbf{P}^{-1}\mathbf{L}\mathbf{P}\right)\\ &= \det\left(\mu^2\mathbf{I}_n-(\mu\xi-1)\mathbf{J}\right)\end{aligned} \tag{19}$$

$\mu^2\mathbf{I}_n-(\mu\xi-1)\mathbf{J}$ is an upper triangular matrix, then:

$$\det\left(\mu^2\mathbf{I}_n-(\mu\xi-1)\mathbf{L}\right)=\prod_{k=1}^{n}(\mu^2-(\mu\xi-1)\lambda_k) \tag{20}$$

The eigenvalues of $\tilde{\mathbf{L}}$ are:

$$\begin{aligned}\mu_{k1} &= \frac{\xi\lambda_k+\sqrt{(\xi\lambda_k)^2-4\lambda_k}}{2}\\ \mu_{k2} &= \frac{\xi\lambda_k-\sqrt{(\xi\lambda_k)^2-4\lambda_k}}{2}\end{aligned}$$

Note that $\xi\lambda_k$ is the sum of the roots $\mu_{k1}+\mu_{k2}$ and λ_k their product $\mu_{k1}\mu_{k2}$.

From Gershgorin's theorem λ_k has a non-negative real part. Consequently, as $\xi>0$ the same applies to μ_{ki}.

5.2. Eigenvectors associated with the zero eigenvalues

The calculation of these eigenvectors is proposed in [24] for a directed or an undirected graph. Let $\mathbf{\Phi}^T(\mu_{01})=\begin{pmatrix}\boldsymbol{\chi}^T & \mathbf{\Psi}^T\end{pmatrix}\in\mathbb{R}^{2n}$ the eigenvector of $\tilde{\mathbf{L}}$ associated with the eigenvalue $\mu_{01}=0$.

$$\begin{pmatrix}\mathbf{0}_n & -\mathbf{I}_n\\ \mathbf{L} & \xi\mathbf{L}\end{pmatrix}\begin{pmatrix}\boldsymbol{\chi}\\ \mathbf{\Psi}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}$$

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Then

$$\begin{array}{rcl} \mathbf{\Psi} & = & \mathbf{0} \\ \mathbf{L}\boldsymbol{\chi} + \xi\mathbf{L}\mathbf{\Psi} & = & \mathbf{0} \end{array}$$

$\mathbf{1}$ is the eigenvector of $\mathbf{L}$ associated with $\lambda_0 = 0$, then by choosing $\chi = \frac{1}{\sqrt{n}}\mathbf{1}$, it comes:

$$\mathbf{\Phi}^T(\mu_{01}) = \begin{pmatrix} \frac{1}{\sqrt{n}}\mathbf{1}^T & \mathbf{0}^T \end{pmatrix} \tag{21}$$

The second eigenvector $\mathbf{\Phi}(\mu_{02})$ associated with μ_{02} is obtained from the Jordan decomposition:

$$\begin{array}{rcl} \tilde{\mathbf{L}}\mathbf{\Phi}(\mu_{02}) & = & \mathbf{\Phi}(\mu_{01}) + \mu_{02}\mathbf{\Phi}(\mu_{02}) \\ \tilde{\mathbf{L}}\mathbf{\Phi}(\mu_{02}) & = & \mathbf{\Phi}(\mu_{01}) \end{array}$$

$$\begin{pmatrix} \mathbf{0}_n & -\mathbf{I}_n \\ \mathbf{L} & \xi\mathbf{L} \end{pmatrix} \begin{pmatrix} \boldsymbol{\chi} \\ \mathbf{\Psi} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}}\mathbf{1} \\ \mathbf{0} \end{pmatrix}$$

This leads to: $\mathbf{\Psi} = -\frac{1}{\sqrt{n}}\mathbf{1}$ and $\boldsymbol{\chi} = \mathbf{0}$ (or $\boldsymbol{\chi} = \mathbf{1}$)

$$\mathbf{\Phi}(\mu_{02})^T = \begin{pmatrix} \mathbf{0}^T & -\frac{1}{\sqrt{n}}\mathbf{1}^T \end{pmatrix} \text{ ou } \mathbf{\Phi}^T(\mu_{02}) = \frac{1}{\sqrt{n}}\begin{pmatrix} \mathbf{1}^T & -\mathbf{1}^T \end{pmatrix} \tag{22}$$

5.3. Eigenvectors associated with the nonzero eigenvalues

5.3.1. Algebraic and geometric multiplicities of the eigenvalue are equal

Let λ a nonzero eigenvalue of $\mathbf{L}$ and $\mathbf{p}$ its associated eigenvector.

Let $\mathbf{\Phi}^T = \begin{pmatrix} \boldsymbol{\chi}^T & \mathbf{\Psi}^T \end{pmatrix} \in \mathbb{C}^{2n}$ the eigenvector of $\tilde{\mathbf{L}}$ associated with the nonzero eigenvalue μ:

$$\begin{pmatrix} \mathbf{0}_n & -\mathbf{I}_n \\ \mathbf{L} & \xi\mathbf{L} \end{pmatrix} \begin{pmatrix} \boldsymbol{\chi} \\ \mathbf{\Psi} \end{pmatrix} = \mu \begin{pmatrix} \boldsymbol{\chi} \\ \mathbf{\Psi} \end{pmatrix} \tag{23}$$

$$\begin{array}{rcl} -\mathbf{\Psi} & = & \mu\boldsymbol{\chi} \\ \mathbf{L}\boldsymbol{\chi} + \xi\mathbf{L}\mathbf{\Psi} & = & \mu\mathbf{\Psi} \end{array} \tag{24}$$

Straightforward calculations yield:

$$\mathbf{L}\boldsymbol{\chi} = \frac{-\mu^2}{(1-\xi\mu)}\boldsymbol{\chi} \tag{25}$$

By identifying Eq. (25) to $\mathbf{L}\mathbf{p} = \lambda\mathbf{p}$, it results $\boldsymbol{\chi} = \mathbf{p}$ and $\lambda = \dfrac{\mu^2}{\xi\mu - 1}$. Next $\mathbf{\Psi} = -\mu\mathbf{p}$ Finally:

$$\mathbf{\Phi}^T = \begin{pmatrix} \mathbf{p}^T & -\mu\mathbf{p}^T \end{pmatrix} \tag{26}$$

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