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Anti-van der Waerden numbers of graph products with trees

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Abstract

Given a graph G, an exact r-coloring of G is a surjective function $c : V(G) \to [1, \ldots, r]$. An arithmetic progression in G of length j with common difference d is a set of vertices $\{v_1, \ldots, v_j\}$ such that $d(v_i, v_{i+1}) = d$ for $1 \leq i < j$. An arithmetic progression is rainbow if all of the vertices are colored distinctly. The fewest number of colors that guarantees a rainbow arithmetic progression of length three is called the anti-van der Waerden number of G and is denoted $\text{aw}(G, 3)$. It is known that for any connected graphs G and H on two or more vertices we have $3 \leq \text{aw}(G \square H, 3) \leq 4$. A tree is called 3-peripheral if it has three vertices that are pairwise diameter distance from one another. In this work we determine that $\text{aw}(G \square T, 3) = 3$ when T is 3-peripheral and G is any connected graph with $|G| \geq 2$. We also determine that $\text{aw}(T \square T', 3) = 4$ when T and T' are not 3-peripheral and $\text{diam}(T \square T')$ is odd.

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1. Introduction

Ramsey Theory is the study of choosing a set of mathematical elements S, then coloring each element via a surjective coloring function $c : S \to \{1, 2, \ldots, r\}$ and determining if there are substructures that are monochromatic. Despite the name, Ramsey Theory was first studied by Schur in 1917 when he showed that we can always find an N such that if $n \geq N$ and we color $[n] = \{1, 2, \ldots, n\}$ with r colors, there must always be a monochromatic solution to $x + y = z$ (see [18]). Van der Waerden, in 1927, proved that given fixed r and k, there exists an N such that if $n \geq N$ and you color $[n]$ with r-colors, there must a monochromatic k-term arithmetic progression (see [20]). Shortly thereafter, Ramsey showed in 1928 that given a fixed r and k, there exists an N so that if $n \geq N$, then no matter how you color the edges of the complete graph K_n using r many colors, there must be a monochromatic complete subgraph K_k (see [16]).

The scope of Ramsey-type questions has since expanded beyond purely monochromatic structures. In particular, Erdős, Simonovits, and S'os initiated the study of polychromatic (or rainbow) substructures in 1973 (see [10]), and more recently, edge colorings of hypergraphs have also been investigated (see [8]).

Ramsey theory is a central area of graph theory with deep connections to algebra, geometry, number theory, topology, and combinatorics. Broadly speaking, Ramsey-type results assert that sufficiently large or complex systems necessarily contain highly structured subconfigurations. Anti-Ramsey theory formalizes this dual perspective by replacing unavoidable monochromatic structure with unavoidable rainbow structure under constraints on the number of colors. While the guiding philosophy mirrors that of classical Ramsey theory, the resulting behavior differs substantially. As in the classical setting, exact values are rare: most results are asymptotic, with precise determinations typically limited to small graphs or highly structured classes. Anti-van der Waerden problems, which concern rainbow arithmetic progressions in graphs, follow a similar trajectory. Exact values are known primarily for small graphs or highly structured graphs. For example, general upper bounds for trees and exact values for trees with bounded local structure were established in [4]. In particular, a vertex is *bijacent* if it has two degree-two neighbors.

Theorem 1.1. *[4] If T is a tree with ℓ bijacent vertices, then*

$$\mathrm{aw}(T, 3) = \begin{cases} 4, & \text{if } \ell = 0; \\ \log_2(\ell) + 4, & \text{otherwise.} \end{cases}$$

Graph products exhibiting strong symmetry have also been investigated extensively. In particular, the Cartesian product $P_m \square P_n$ provided one of the earliest nontrivial examples, yielding a tight lower bound of three and a tight upper bound of four for all products $G \square H$ (see [17]). These bounds have subsequently been used to determine additional product graphs, including those involving cycles (see [15]).

Since trees form a fundamental class of graphs, Cartesian products involving trees offer a natural setting to investigate anti-van der Waerden numbers. Tree products allow peripheral geometry (vertices realizing the diameter) to interact with branching structure in ways absent

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from more rigid families. Within this class, 3-peripheral trees, defined by the presence of three leaves pairwise at diameter distance, emerge as a critical threshold. Standard contradiction arguments for upper bounds often force rainbow arithmetic progressions whose common difference is the diameter or one less, and the presence of three diameter-separated vertices ensures that many such progressions must occur in $G\square T$. Thus, the 3-peripheral condition isolates the minimal branching required for diameter geometry to play a genuinely structural role, shedding light on how similar mechanisms may govern more complex graph products. We now introduce the necessary definitions, beginning with the notion of a k-term arithmetic progression.

A *k-term arithmetic progression*, k-AP, in $[n]$ is of the form

$$a, a+d, a+2d, \ldots, a+(k-1)d$$

where $k \geq 2$ and $d \geq 1$. An *exact r-coloring* of $[n]$ is a surjective function $c : [n] \to [r]$. The *anti-van der Waerden number of $[n]$ with respect to k-APs*, first defined in [19] in 2013 and denoted $\text{aw}([n], k)$, is the smallest r such that every exact r-coloring of $[n]$ guarantees a rainbow arithmetic k-AP. Note that rainbow 3-APs have been studied previously when balanced colorings are used in [1, 2, 13]. It should also be noted that $\text{aw}([n], 3)$ was studied in [9] and the exact value was determined in [5]. Additionally, rainbow solutions to linear equations in $[n]$ were studied in [7] and [11]. Interestingly, the authors in [7] and [11] produced similar results simultaneously but using differing arguments. The definition of $\text{aw}([n], k)$ was also extended to $\mathbb{Z}_n$ to get $\text{aw}(\mathbb{Z}_n, k)$ in [9] which led to results on arithmetic progression in finite abelian groups by Young in [21]. Rainbow solutions to linear equations in $\mathbb{Z}_n$ have also been studied in [6, 12, 14]. Since $[n]$ behaves like a path graph and $\mathbb{Z}_n$ behaves like a cycle, anti-van der Waerden numbers in graphs, particularly tree graphs, and graph products were studied in [4, 15, 17]. This paper continues in this vein of research.

Graphs $G = (V, E)$ in this paper are simple, undirected and connected, so edge $\{u, v\}$ will be shortened to $uv \in E(G)$ where u, v are vertices in the vertex set $V(G)$. If $uv \in E(G)$, then u and v are *neighbors* of each other. The *distance* between vertex u and v in graph G is denoted $d_G(u, v)$, or just $d(u, v)$ when the context is clear, and is the smallest length of any $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. Graph G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. A subgraph G' of G is an *induced subgraph* if whenever u and v are vertices of G' and uv is an edge of G, then uv is an edge of G'. If S is a nonempty set of vertices of G, then the *subgraph of G induced by S* is the induced subgraph with vertex set S and is denoted $G[S]$. An *isometric subgraph* G' of G is a subgraph such that $d_{G'}(u, v) = d_G(u, v)$ for all $u, v \in V(G')$. If $G = (V, E)$ and $H = (V', E')$ then the *Cartesian product*, written $G\square H$, has vertex set $\{(x, y) : x \in V \text{ and } y \in V'\}$ and (x, y) and (x', y') are adjacent in $G\square H$ if either $x = x'$ and $yy' \in E'$ or $y = y'$ and $xx' \in E$. This paper will use the convention that if

$$V(G) = \{u_1, \ldots, u_{n_1}\} \quad \text{and} \quad V(H) = \{w_1, \ldots, w_{n_2}\},$$

then $V(G\square H) = \{v_{1,1}, \ldots, v_{n_1,n_2}\}$ where $v_{i,j}$ corresponds to the vertices $u_i \in V(G)$ and $w_j \in V(H)$. Also, if $1 \leq i \leq n_2$, then G_i denotes the ith labeled copy of G in $G\square H$.

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Likewise, if $1 \leq j \leq n_1$, then H_j denotes the jth labeled copy of H in $G\Box H$. In other words, G_i is the induced subgraph $G_i = G\Box H[\{v_{1,i}, \ldots, v_{n_2,i}\}]$, and H_j is the induced subgraph $H_j = G\Box H[\{v_{j,1}, \ldots, v_{j,n_1}\}]$. Notice that i in the subscript in G_i corresponds to the ith vertex of H and j in the subscript in H_j corresponds to the jth vertex of G (see Example 1.2 below).

A *k-term arithmetic progression in graph G* (k-AP) is a set of vertices $\{v_1, \ldots, v_k\}$ such that $d(v_i, v_{i+1}) = d$ for all $1 \leq i \leq k-1$. A k-term arithmetic progression is *degenerate* if $v_i = v_j$ for any $i \neq j$. Note that technically, since a k-AP is a set, the order of the elements does not matter. However, often k-APs will be presented in the order that provides the most intuition. An *exact r-coloring of a graph G* is a surjective function $c : V(G) \to [r]$. A set of vertices S is *rainbow* under coloring c if for every $v_i, v_j \in S$, $c(v_i) \neq c(v_j)$ when $i \neq j$. Given a set $S \subset V(G)$, define $c(S) = \{c(s) \,|\, s \in S\}$. The *anti-van der Waerden number of graph G with respect to k*, denoted $\mathrm{aw}(G, k)$, is the least positive number r such that every exact r-coloring of G contains a rainbow k-term arithmetic progression. If $|V(G)| = n$ and no coloring of the vertices yields a rainbow k-AP, then $\mathrm{aw}(G, k) = n + 1$.

In this paper, P_n denotes the path graph on n vertices and C_n denotes the cycle graph on n vertices. Example 1.2 below discusses the anti-van der Waerden number of $P_3\Box C_6$.

Example 1.2. *Consider the graph $P_3\Box C_6$ where $V(P_3) = \{u_1, u_2, u_3\}$ and $V(C_6) = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. Let $G = P_3$ and $H = C_6$ as in the definition. Now, G_5 is a subgraph of $P_3\Box C_6$ that is isomorphic to P_3 and corresponds to vertex w_5 of C_6. Similarly, H_2 is a subgraph of $P_3\Box C_6$ that is isomorphic to C_6 and corresponds to vertex u_2 of P_3. Note that coloring $c(v_{1,1}) = red$, $c(v_{3,4}) = blue$ and all other vertices white avoids rainbow 3-APs (see Figure 1 below). Since we have a rainbow-free, exact 3-coloring this implies that $4 \leq \mathrm{aw}(P_3\Box C_6, 3)$. Some case analysis, or the use of Theorem 2.9, gives that $\mathrm{aw}(P_3\Box C_6, 3) \leq 4$. Therefore, $\mathrm{aw}(P_3\Box C_6, 3) = 4$.*

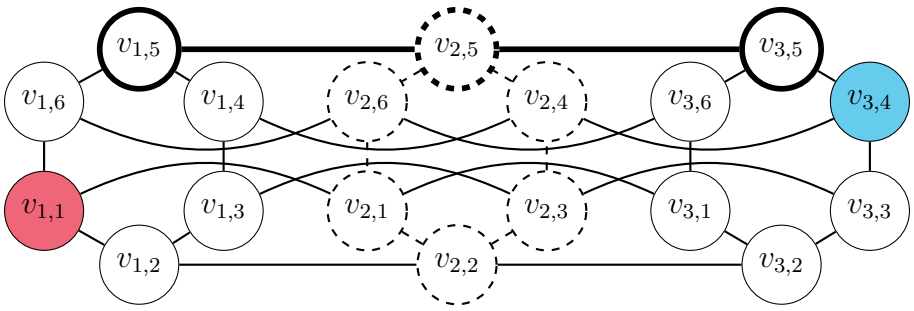


Figure 1. The subgraph G_5 has been bolded and H_2 has been dashed. There is also an exact 3-coloring using the colors *red*, *white* and *blue* that avoids rainbow 3-APs.

As mentioned earlier, many of the conditions in this paper are about peripheral vertices and diameter so some definitions are needed. For a vertex v in a connected graph G the *eccentricity* of v, denoted $\epsilon(v)$, is the distance between v and a vertex furthest from v in G. If a vertex v has $\epsilon(v) = \mathrm{diam}(G)$ we call v a *peripheral* vertex. If a graph G contains vertices $u_1, \ldots, u_n$ such that $d(u_i, u_j) = \mathrm{diam}(G)$ for all distinct $i, j \in \{1, \ldots, n\}$, then we

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call G *n-peripheral*. A graph is *not n-peripheral* if we cannot find n vertices that are pairwise diameter away from each other. Specifically, we focus on graphs that are 3-peripheral and graphs that are not 3-peripheral.

Example 1.3. *In Figure 2 we see that graph H is 3-peripheral since* $\text{diam}(H) = 2$ *and w_1, w_3 and w_4 are each peripheral vertices and are pairwise distance 2 from each other. In contrast, graph G has* $\text{diam}(G) = 3$ *and even though u_4, u_5, and u_6 are peripheral and pairwise equidistant, none of the pairwise distances is* $\text{diam}(G)$*. Thus, G is not 3-peripheral.*

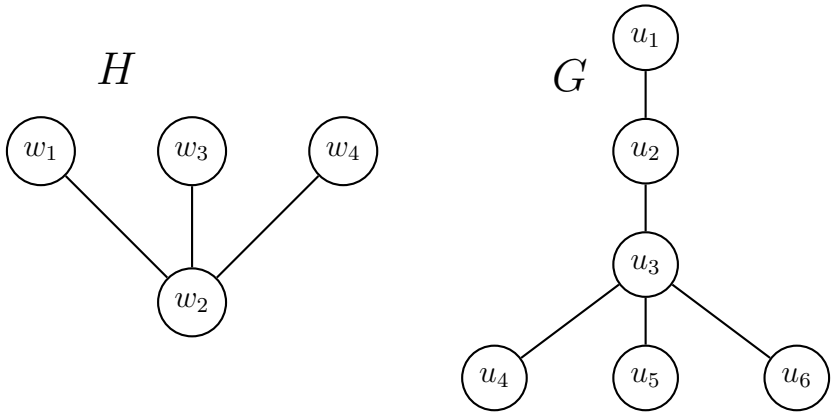


Figure 2. A 3-peripheral tree H and a tree G that is not 3-peripheral.

The paper proceeds with some background results in Section 2. Section 3 focuses on 3-peripheral trees followed by Section 4 where trees that are not 3-peripheral are analyzed. Finally, Section 5 discusses future directions for this research.

2. Preliminary Results

Proposition 2.1 will be used regularly without reference as the result is rather intuitive.

Proposition 2.1. *[15] If $v_{i,j}, v_{h,k} \in V(G\square H)$, then*

$$d_{G\square H}(v_{i,j}, v_{h,k}) = d_G(u_i, u_h) + d_H(w_j, w_k).$$

The diameter of graph G, denoted $\text{diam}(G)$, is the maximum distance between all pairs of vertices in $V(G)$.

Observation 2.2. *If G and H are graphs, then a direct consequence of Proposition 2.1 is that*

$$\text{diam}(G\square H) = \text{diam}(G) + \text{diam}(H).$$

Corollary 2.3. *[15] If G' is an isometric subgraph of G and H' is an isometric subgraph of H, then $G'\square H'$ is an isometric subgraph of $G\square H$.*

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Lemma 2.4 is powerful since it guarantees isometric subgraphs. Isometric subgraphs are important when investigating anti-van der Waerden numbers because distance preservation implies k-AP preservation. Lemma 2.6 helps restrict the number of colors each copy of G or H can have within $G\square H$.

Lemma 2.4. *[17] If G is a connected graph on at least three vertices with an exact r-coloring c where $r \geq 3$, then there exists a subgraph G' in G with at least three colors where G' is either an isometric path or $G' = C_3$.*

Using Corollary 2.3 and Lemma 2.4 we obtain Theorem 2.5.

Theorem 2.5. *Let G be a connected graph on at least three vertices. If $diam(G) \leq 2$ and $aw(P_3\square H, 3) = 3$, then $aw(G\square H, 3) = 3$.*

Proof. Let c be an exact 3-coloring of $G\square H$. By Lemma 2.4, there exists a subgraph X of $G\square H$ with 3-colors such that X is either an isometric path or is isomorphic to C_3. If $X \cong C_3$, this forms a rainbow 3-AP and we are done. So suppose that X is an isometric path in $G\square H$. Notice that X is a subgraph of $P_3\square H$, which in turn is a subgraph of $G\square H$.

Notice that $P_3\square H$ is 3-colored under c, because it contains an isometric path. Since $aw(P_3\square H, 3) = 3$, the graph $P_3\square H$ contains a rainbow 3-AP under the coloring c. Since $P_3\square H$ is an isometric subgraph of $G\square H$, by Corollary 2.3, $G\square H$ contains this rainbow 3-AP as well. Therefore, $aw(G\square H, 3) = 3$. □

Examples of graphs G that satisfy Theorem 2.5 are the complete graph on n vertices K_n, the complete bipartite graph on m and n vertices $K_{m,n}$, or graphs with a dominating vertex. Examples of graphs H that satisfy $aw(P_3\square H, 3) = 3$ are P_n with odd n, C_{2k+1}, C_{4k} (see [15]) or 3-peripheral trees (see Section 3 for a full discussion of 3-peripheral trees). Thus, we can cross any graphs G and H above to get an anti-van der Waerden number of three.

Lemma 2.6. *[17] Assume G and H are connected with $|V(H)| \geq 3$. Suppose c is an exact, rainbow-free r-coloring of $G\square H$, such that $r \geq 3$ and $|c(V(G_i))| \leq 2$ for $1 \leq i \leq n$. If $w_iw_j \in E(H)$, then $|c(V(G_i) \cup V(G_j))| \leq 2$.*

Lemmas 2.7 and 2.8 are structural results that significantly limit the way we can color $G\square H$ and be rainbow-free.

Lemma 2.7. *[15] If G and H are connected, $|G|, |H| \geq 2$ and c is an exact r-coloring of $G\square H$, $3 \leq r$, that avoids rainbow 3-APs, then $|c(V(G_i))| \leq 2$ for $1 \leq i \leq |H|$.*

Lemma 2.8. *[4] Let G be a connected graph on m vertices and H be a connected graph on n vertices. Let c be an exact r-coloring of $G\square H$ with no rainbow 3-APs. If $G_1, G_2, \ldots, G_n$ are the labeled copies of G in $G\square H$, then $|c(V(G_j)) \setminus c(V(G_i))| \leq 1$ for all $1 \leq i, j \leq n$.*

Theorem 2.9 was one of the results that initiated the vein of research that this paper is following. In particular, this paper attempts to characterize when the product of two graphs has an anti-van der Waerden number of three and when the product of two graphs has an anti-van der Waerden number of four.

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Theorem 2.9. *[17] If G and H are connected graphs and $|G|, |H| \geq 2$, then* $\mathrm{aw}(G\square H, 3) \leq 4$.

The next result, Theorem 2.10, is used when we find an isometric subgraph $P_m\square P_n$ within $G\square H$ to either show that our coloring must have a rainbow within the isometric subgraph or that we know we can color the isometric subgraph in some way and avoid rainbows.

Theorem 2.10. *[17] For $m, n \geq 2$,*

$$\mathrm{aw}(P_m\square P_n, 3) = \begin{cases} 3, & \text{if } m = 2 \text{ and } n \text{ is even, or } m = 3 \text{ and } n \text{ is odd,} \\ 4, & \text{otherwise.} \end{cases}$$

Some of the results in the paper rely on identifying a spine of a tree T. A *spine* S of a tree T is defined as a path subgraph of T that realizes $\mathrm{diam}(T)$. We say $v \in V(S)$ is the *root* of $u \in V(T) \setminus V(S)$ if v is the closest vertex of S to u. The *branch* B_v of vertex v is the induced subgraph of T containing all vertices whose root is v. Specifically,

$$V(B_v) = \{u : v \text{ is the root of } u\}.$$

Note that $v \in B_v$.

Lemma 2.11. *Suppose T is a tree with a spine S. If $u \in V(T) \setminus V(S)$ such that $u \in V(B_x)$ and $v \in V(T) \setminus V(B_x)$, then there exists a $u' \in V(S)$ such that $d(u, v) = d(u', v)$.*

Proof. Assume $V(S) = \{x_1, \ldots, x_k\}$ with edges x_ix_{i+1} for $1 \leq i \leq k-1$, $u \in V(B_{x_i})$ and $v \in V(B_{x_j}) \subset V(T) \setminus V(B_{x_i})$. For the sake of contradiction, assume there is no $x_\ell \in V(S)$ with $d(x_\ell, v) = d(u, v)$. First, consider the case when $i < j \leq k$ (see Figure 3). This implies that $d(x_1, v) < d(u, v)$, if not then there would be some $\ell \in \{x_1, x_2, \ldots, x_i\}$ with $d(x_\ell, v) = d(u, v)$. However, $d(x_1, v) < d(u, v)$ implies that $d(x_1, x_i) < d(u, x_i)$. Adding $d(x_i, x_k)$ to both sides gives that $d(x_1, x_k) < d(u, x_k)$ which contradicts that S is a spine. A similar contradiction arises if instead it is assumed that $1 \leq j < i$. In this case, $d(x_k, v) < d(u, v)$. This implies that $d(x_k, x_1) < d(u, x_1)$, again contradicting that S is a spine. Thus, there is some $u' \in V(S)$ such that $d(u, v) = d(u', v)$. □

Some of the rainbow-free colorings that are used to get a lower bound on $\mathrm{aw}(G, 3)$ depend on using the path between two peripheral vertices that realize $\mathrm{diam}(G)$. Corollary 2.12 provides a distance result for non-peripheral vertices.

Corollary 2.12. *If u is not a peripheral vertex of T and $v \in V(T)$, then there exists a vertex $w \in V(T)$ such that $d(w, v) = d(u, v) + 1$.*

Proof. Let S be a spine of T, $V(S) = \{x_1, \ldots, x_k\}$ with $x_ix_{i+1} \in E(T)$ for $1 \leq i \leq k-1$. If $u \in V(S)$, then, since u is not a peripheral vertex, $u = x_i$ for some $2 \leq i \leq k-1$. Thus, either $d(x_{i-1}, v) = d(u, v) + 1$ or $d(x_{i+1}, v) = d(u, v) + 1$. If $u \notin V(S)$, Lemma 2.11 indicates that there is an $x_i \in V(S)$ such that $d(x_i, v) = d(u, v)$. Since u is not a peripheral vertex neither is x_i so, again, either $d(x_{i-1}, v) = d(u, v) + 1$ or $d(x_{i+1}, v) = d(u, v) + 1$. □

Lemma 2.11 and Corollary 2.12 are used in Section 3 to prove anti-van der Waerden results on the graph products with 3-peripheral trees.

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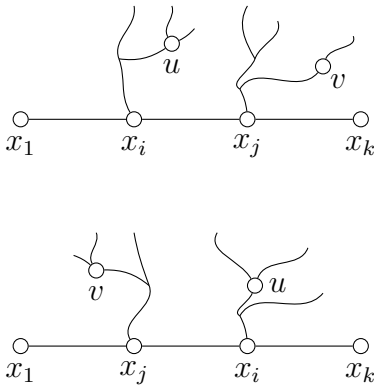


Figure 3. The two cases from the proof of Lemma 2.11. The top figure is when $i < j \leq k$ and the bottom figure is when $1 \leq j < i$.

3. Trees that are 3-peripheral

The idea and motivation for studying 3-peripheral trees arose from noticing that the path P_n maximizes the diameter and the star $K_{1,n-1}$ minimizes the diameter for all trees of order n. Thus, all trees of order n may be considered to be 'between' $K_{1,n-1}$ and P_n. In working on results associated with stars, we realized that $\text{aw}(G\square K_{1,n}, 3) = 3$ for all G and $n \geq 3$. The structure of the star allows for this argument, so we generalized this idea to the class of trees we call 3-peripheral. We showed that $\text{aw}(G\square T, 3) = 3$ for all connected graphs G with $|G| \geq 2$ and all 3-peripheral trees T in the main result of this section, Theorem 3.6. To begin, we present a few Lemmas about peripheral vertices that help establish the main result of this section.

Lemma 3.1. *[3] Suppose T is a tree with $u', v' \in V(T)$ which realize the diameter. If x' is a peripheral vertex of T, then x' is diameter away from u' or diameter away from v'.*

Lemma 3.2. *If T is a 3-peripheral tree, then for any $u, v \in V(T)$ with $d(u, v) = \text{diam}(T)$, there exists $w \in V(T)$ with $d(u, w) = d(v, w) = \text{diam}(T)$.*

Proof. Assume T is a 3-peripheral tree and $u, v \in V(T)$ such that $d(u, v) = \text{diam}(T)$. Since T is 3-peripheral, there exist $x, y, z \in V(T)$ with $d(x, y) = d(y, z) = d(x, z) = \text{diam}(T)$. Apply Lemma 3.1 three times with u playing the role of x' from the Lemma, one for each of $\{x, y\}, \{x, z\}, \{y, z\}$ playing the role of $\{u', v'\}$. This will show that u is diameter away from at least two of x, y, or z. Similarly, we can show that v is diameter away from at least two of x, y, z. This argument shows that one of these vertices is in common, say w. That is, w is one of x, y, z which is diameter away from both u and v, as desired. □

Lemma 3.3. *If T is 3-peripheral, then $\text{diam}(T)$ is even. Furthermore, for any three vertices that are pairwise $\text{diam}(T)$ distance apart, there is some vertex that is equidistant from all three of them.*

Proof. Since T is 3-peripheral, there exist $x, y, z \in V(T)$ with $d(x, y) = d(y, z) = d(x, z) = \text{diam}(T)$. First, assume that the $x - y$ path and $y - z$ path only intersect at y. Note that in a

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tree there is a unique path between any two vertices, thus the $x-z$ path is the path created by traversing the $x-y$ path and then the $y-z$ path. This implies that $d(x,z) = 2\text{diam}(T)$, a contradiction. So now assume that the $x-y$ path and the $y-z$ path intersect at a vertex that is not y. Let y' be the vertex that is closest to z and is on both the $x-y$ path and $y-z$ path. This means

$$d(x,y) = d(x,y') + d(y',y) = \text{diam}(T),$$

$$d(z,y) = d(z,y') + d(y',y) = \text{diam}(T),$$

and

$$d(x,z) = d(x,y') + d(y',z) = \text{diam}(T).$$

These equations yield $d(y,y') = d(x,y') = d(z,y')$ so $\text{diam}(T)$ must be even and y' is equidistant from x, y and z. □

Lemma 3.4 is the base case for Lemma 3.5. Lemma 3.5 is then used to prove the main result of the section, Theorem 3.6.

Lemma 3.4. *If T is a 3-peripheral tree, then* $\text{aw}(P_2\square T, 3) = 3$.

Proof. Let c be an exact 3-coloring of $P_2\square T$ and let T_1 and T_2 denote the two labeled copies of T. Assume c avoids rainbow 3-APs. By Lemma 2.4, there either exists an isometric path or a C_3 containing *red*, *blue*, and *green* in $P_2\square T$. The latter case produces a rainbow 3-AP, thus there exists an isometric path in $P_2\square T$ with *red*, *blue*, and *green*. Let P be the shortest such path, and suppose P is the $v_{j,\alpha} - v_{k,\beta}$ geodesic. Without loss of generality, suppose $c(v_{j,\alpha}) = blue$, $c(v_{k,\beta}) = green$, and $c(v) = red$ for all $v \in V\Big(P \setminus \{v_{j,\alpha}.v_{k,\beta}\}\Big)$. Lemma 2.7 implies that $j \neq k$. Without loss of generality, suppose $j = 1, k = 2$.

Assume $d(v_{1,\alpha}, v_{2,\beta})$ is even. Then there exists a vertex v equidistant from $v_{1,\alpha}$ and $v_{2,\beta}$ on P. So, $\{v_{1,\alpha}, v, v_{2,\beta}\}$ is a rainbow 3-AP of common difference $\frac{d(v_{1,\alpha},v_{2,\beta})}{2}$.

Thus, $d(v_{1,\alpha}, v_{2,\beta})$ is odd. Let S denote a spine of T. Assume $d_T(w_\alpha, w_\beta) < \text{diam}(T)$.

Case 1. Suppose $w_\alpha, w_\beta \in V(S)$.

Since $d_T(w_\alpha, w_\beta) < \text{diam}(T)$, there exists some $w \in V(S)$ such that either

$$d_T(w, w_\beta) = d_T(w_\alpha, w_\beta) + 1$$

or

$$d_T(w, w_\alpha) = d_T(w_\alpha, w_\beta) + 1.$$

Without loss of generality, suppose $d_T(w, w_\beta) = d_T(w_\alpha, w_\beta) + 1$. Let P denote the $w - w_\beta$ geodesic in T. Note that the length of P is odd, and $|c(P_2\square P)| = 3$. Now, Corollary 2.3 and Theorem 2.10 imply that the isometric subgraph $P_2\square P$ in $P_2\square T$ contains a rainbow 3-AP.

Case 2. Suppose $w_\alpha \notin V(S)$ or $w_\beta \notin V(S)$.

Assume $w_\alpha, w_\beta \notin V(S)$ where w_α and w_β are in the same branch, say B_x. Without loss of generality, suppose $d_T(w_\beta, x) < d_T(w_\alpha, x)$. Then we can find a $w \in B_x \cup \{x\}$ such that $d_T(w_\alpha, w) = d_T(w_\alpha, w_\beta)$. Define P' to be the $w_\alpha - w$ path in H. Similar to the first case,

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Corollary 2.3 and Theorem 2.10 imply that the isometric subgraph $P_2\square P'$ in $P_2\square T$ contains a rainbow 3-AP.

If $w_\alpha \notin V(S)$ and w_β isn't in the branch containing w_α, then, by Lemma 2.11, there exists some $w_{\alpha'} \in V(S)$ such that $d_T(w_\alpha, w_\beta) = d_T(w_{\alpha'}, w_\beta)$. So, the 3-AP $\{v_{1,\alpha}, v_{2,\beta}, v_{1,\alpha'}\}$ implies that $c(v_{1,\alpha'}) = blue$.

Likewise, if $w_\beta \notin V(S)$ and w_α isn't in the branch containing w_β, then, by Lemma 2.11, there exists some $w_{\beta'} \in V(S)$ such that $d_T(w_\alpha, w_\beta) = d_T(w_\alpha, w_{\beta'})$. The 3-AP $\{v_{2,\beta}, v_{1,\alpha}, v_{2,\beta'}\}$ implies that $c(v_{2,\beta'}) = green$. Case 1 implies that there exists a rainbow 3-AP in $P_3\square H$.

Thus, $d_T(w_\alpha, w_\beta) = \operatorname{diam}(T)$. By Lemma 3.2, there exists some w_γ such that $d_T(w_\alpha, w_\gamma) = d_T(w_\beta, w_\gamma) = \operatorname{diam}(T)$. Define

$$S = \Big\{d(v_{1,\alpha}, v_{1,\beta}), d(v_{1,\beta}, v_{2,\beta}), d(v_{1,\alpha}, v_{2,\alpha}), d(v_{2,\alpha}, v_{2,\beta})\Big\}.$$

Note that $s < d(v_{1,\alpha}, v_{2,\beta})$ for all $s \in S$, so the minimality of $d(v_{1,\alpha}, v_{2,\beta})$ implies that $c(v_{1,\beta}) = c(v_{2,\alpha}) = red$. The 3-APs $\{v_{1,\alpha}, v_{2,\beta}, v_{1,\gamma}\}$ and $\{v_{2,\alpha}, v_{1,\gamma}, v_{2,\beta}\}$ imply that $c(v_{1,\gamma}) = green$, but then $\{v_{1,\alpha}, v_{1,\beta}, v_{1,\gamma}\}$ is a rainbow 3-AP. Thus, $\operatorname{aw}(P_2\square T, 3) = 3$. □

In Lemma 3.5, we generalize the result from $P_2\square T$ to $P_n\square T$ using induction on n where the base case is Lemma 3.4. We assume $\operatorname{aw}(P_\ell\square T, 3) = 3$, and let c be an exact 3-coloring of $P_{\ell+1}\square T$ with no rainbow 3-APs. We apply the induction hypothesis to subgraphs induced by $A = c\left(\bigcup_{i=1}^{\ell} V(T_i)\right)$ and $B = c\left(\bigcup_{i=2}^{\ell+1} V(T_i)\right)$ where each uses at most two colors. By the Inclusion-Exclusion principle, we obtain that the middle layers $\left(\bigcup_{i=2}^{\ell} V(T_i)\right)$ are monochromatic, say red. The remaining two colors appear in T_1 and $T_{\ell+1}$. Choosing the closest vertices colored blue and green in these layers, a distance argument forces the vertices to be diameter distance in T, since otherwise a 3-AP through T_2 yields a contradiction. Using that T is 3-peripheral, we show that there is a vertex at equal distance from the blue and green vertices, which forces additional color constraints that produce a rainbow 3-AP. This contradiction shows $\operatorname{aw}(P_{\ell+1}\square T, 3) = 3$, completing the induction.

Lemma 3.5. *If $n \geq 2$ and T is a 3-peripheral tree, then* $\operatorname{aw}(P_n\square T, 3) = 3$.

Proof. Suppose T is a 3-peripheral tree. By Lemma 3.4, $\operatorname{aw}(P_2\square T, 3) = 3$. Let, $\ell \geq 2$ and suppose $\operatorname{aw}(P_\ell\square T, 3) = 3$. Let c be an exact 3-coloring of $P_{\ell+1}\square T$, without loss of generality, say $red, blue, green \in c(V(P_{\ell+1}\square T))$, and assume c avoids rainbow 3-APs. Let T_i denote the ith labeled copy of T. By hypothesis,

$$\left|c\left(\bigcup_{i=1}^{\ell} V(T_i)\right)\right| \leq 2 \text{ and } \left|c\left(\bigcup_{i=2}^{\ell+1} V(T_i)\right)\right| \leq 2.$$

Define $A = c\left(\bigcup_{i=1}^{\ell} V(T_i)\right)$ and $B = c\left(\bigcup_{i=2}^{\ell+1} V(T_i)\right)$ so that the Inclusion-Exclusion principle immediately gives

$$\left|c\left(\bigcup_{i=2}^{\ell} V(T_i)\right)\right| = |A \cap B| = |A| + |B| - |A \cup B| \leq 2 + 2 - 3 = 1.$$

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So, without loss of generality, suppose

$$c\left(\bigcup_{i=2}^{\ell} V(T_i)\right) = \{red\}.$$

If $c(V(T_1)) = \{red\}$, then $blue, green \in c(V(T_{\ell+1}))$, contradicting Lemma 2.6. If *blue*, *green* $\in c(V(T_1))$, then we contradict Lemma 2.6 again. Thus, without loss of generality, *blue* $\in c(V(T_1))$ and *green* $\in c(V(T_{\ell+1}))$. Say $c(v_{1,j}) =$ *blue* and $c(v_{\ell+1,k}) =$ *green* are the closest blue and green vertices, and without loss of generality, suppose $j \leq k$.

Assume $d_T(w_j, w_k) < \text{diam}(T)$. First, suppose that, without loss of generality, w_j is not a peripheral vertex. Corollary 2.12 implies there exists a $w_i \in V(T)$ such that $d_T(w_i, w_k) = d_T(w_j, w_k) + 1$. Otherwise, if both w_j and w_k are peripheral vertices, then there exists some vertex w_ℓ such that $d_T(w_k, w_\ell) = \text{diam}(T)$. Since $d_T(w_j, w_k) < \text{diam}(T)$, there exists some w_i on the $w_k - w_\ell$ path such that $d_T(w_i, w_k) = d_T(w_j, w_k) + 1$. Now, Lemma 2.6 and the 3-AP $\{v_{1,j}, v_{\ell+1,k}, v_{2,i}\}$ imply that $c(v_{2,i}) =$ *blue*. However, this contradicts that $c(T_2) = \{red\}$. Thus, $d_T(w_j, w_k) = \text{diam}(T)$.

By Lemma 3.2, there exists some w_x such that $d_T(w_j, w_x) = d_T(w_k, w_x) = \text{diam}(T)$. Define

$$S = \Big\{d(v_{1,j}, v_{1,k}), d(v_{1,k}, v_{\ell+1,k}), d(v_{1,j}, v_{\ell+1,j}), d(v_{\ell+1,j}, v_{\ell+1,k})\Big\}.$$

Note that $s < d(v_{1,j}, v_{\ell+1,k})$ for all $s \in S$. The minimality of $d(v_{1,j}, v_{\ell+1,k})$ implies that $c(v_{1,k}) = c(v_{\ell+1,j}) =$ *red*. The 3-APs $\{v_{1,j}, v_{\ell+1,k}, v_{1,x}\}$ and $\{v_{\ell+1,j}, v_{1,x}, v_{\ell+1,k}\}$ imply that $c(v_{1,x}) =$ *green*, but then $\{v_{1,j}, v_{1,k}, v_{1,x}\}$ is a rainbow 3-AP. Thus, $\text{aw}(P_{\ell+1}\square T, 3) = 3$. By induction, $\text{aw}(P_n\square T, 3) = 3$ for all $n \geq 2$. □

In Theorem 3.6, we use the technique from the proof of Theorem 4.5 in [17] to extend the result of Lemma 3.5 from $P_n\square T$ to $T\square G$, where G is a connected graph of order at least 2. This result is proved by considering an exact 3-coloring c of $T\square G$ that avoids a rainbow 3-AP and analyze the distribution of colors across the copies T_i of T. Let $\alpha = \max_i |c(T_i)|$, and choose tree T_k attaining α colors. We then argue that for $\alpha \leq 2$, there is a path in T_k of length at most two, and for $\alpha = 3$, either T_k contains an isometric C_3 or T_k contains a path P with 3 colors. If $\alpha = 3$ and T_k contains an isometric C_3, then a rainbow 3-AP in T_k is immediate. Thus, for $\alpha = 1, 2,$ or 3, we consider a shortest path P with α colors and use a case analysis on α and structural arguments to prove a rainbow 3-AP exists, which is a contradiction.

Theorem 3.6. *If T is a 3-peripheral tree and G is connected with $2 \leq |G|$, then*

$$\text{aw}(T\square G, 3) = 3.$$

Proof. Let c be an exact 3-coloring of $T\square G$ where $|G| = \{w_1, \ldots, w_n\}$ and let T_i denote the ith labeled copy of T. Assume c avoids rainbow 3-APs. Let $\alpha = \max_{1\leq i\leq n}\{|c(T_i)|\}$ and choose k such that $|c(T_k)| = \alpha$ (Figure 4 illustrates a coloring c on $T\square G$ with $\alpha = 2$). If $\alpha \leq 2$, there is a path of length at most two in T_k containing α colors. If $\alpha = 3$, Lemma 2.4 implies that T_k contains an isometric C_3 or path containing all three colors. If there exists

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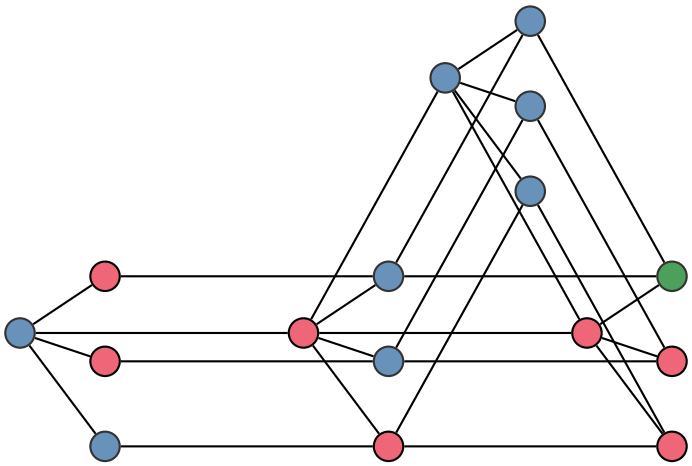


Figure 4. The product of tree H from Figure 2 and the graph G, where G is the cycle graph C_3 with an appended leaf. This coloring is an example where each copy of the tree H has at most two colors.

an isometric C_3 with all three colors, then there exists an immediate rainbow 3-AP in T_k. Thus, there must exist an isometric path in T_k containing α colors. Let P be a shortest such path.

Case 1. Suppose P contains three colors.
First, suppose the length of P is even, and, without loss of generality, suppose the two leaves of P are colored *red* and *blue* and the rest of the vertices are colored *green*. Then there exists a *green* vertex equidistant from the *red* and *blue* ones, creating a rainbow 3-AP.

Second, suppose the length of P is odd, and T_k contains three colors. Note that $w_i \in V(G)$ has a neighbor, say w_j. Then let P_2 denote the $w_i - w_j$ path in G, and let P^* denote the path corresponding to P in T. Then the isometric subgraph $P_2 \square P^*$ of $G \square H$ contains a rainbow 3-AP by Corollary 2.3 and Theorem 2.10.

Case 2. Suppose P contains one color.
This implies that each T_k is monochromatic by the definition of P. Since $T \square G$ has three colors, there exists either an isometric C_3 or an isometric shortest path ρ in a copy of G that has at least three colors by Lemma 2.4. If there is an isometric C_3, then there is an immediate rainbow 3-AP. In the other case, this is just Case 1 with the roles of T and G reversed.

Case 3. Suppose P contains two colors.
Then there must exist some T_k with two colors, say $T_k = \{red, blue\}$. Lemma 2.8 implies that *green* must appear either with *red* or *blue*, without loss of generality, suppose *green* appears with *red*.

Define c' to be the auxiliary coloring of G where

$$c'(w_i) = \begin{cases} red, & \text{if } c(V(T_i)) = \{red\}, \\ \mathcal{C}, & \text{if } \mathcal{C} \in c(V(T_i)). \end{cases}$$

By Lemma 2.4, there either exists an isometric path containing *red*, *blue*, and *green* or a C_3 containing *red*, *blue*, and *green*. The latter case implies an immediate rainbow 3-AP,

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thus there exists an isometric path P' in G with *red*, *blue*, and *green*. Then by Corollary 2.3 and Lemma 3.5, there exists a rainbow 3-AP in the isometric subgraph $T\square P'$ of $T\square G$, contradicting that c avoids rainbow 3-APs. In every case, there exists a rainbow 3-AP in $T\square G$, thus $\text{aw}(T\square G, 3) = 3$. □

4. Trees that are not 3-peripheral

Section 3 culminates with a result about graph products between 3-peripheral trees and arbitrary nontrivial connected graphs. This is due to the strong structure that a graph being 3-peripheral provides. The structure of trees that are not 3-peripheral is much less impactful, and this is discussed in more detail in section 5. That is not to say such trees have no harnessable structure as shown by Lemma 4.2. This section concludes with Theorem 4.3 which provides a class of products of trees which are not 3-peripheral whose anti-van der Waerden number is four.

Lemma 4.1. *If G_1 and G_2 are not 3-peripheral, then $G_1\square G_2$ is not 3-peripheral.*

Proof. We will prove the contrapositive, so assume that $G_1\square G_2$ is 3-peripheral. Then there exist $x, y, z \in V(G_1\square G_2)$ with

$$d_{G_1\square G_2}(x, y) = d_{G_1\square G_2}(y, z) = d_{G_1\square G_2}(x, z) = \text{diam}(G_1) + \text{diam}(G_2)$$

by definition and Observation 2.2. Assume that x, y and z correspond to $x_1, y_1, z_1 \in V(G_1)$ and $x_2, y_2, z_2 \in V(G_2)$, respectively.

Now,

$$d_{G_1\square G_2}(x, y) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2), d_{G_1\square G_2}(x, z) = d_{G_1}(x_1, z_1) + d_{G_2}(x_2, z_2)$$

and

$$d_{G_1\square T_2}(y, z) = d_{G_1}(y_1, z_1) + d_{G_2}(y_2, z_2).$$

Thus, $x_1, y_1, z_1 \in V(G_1)$ are pairwise distance $\text{diam}(G_1)$ and $x_2, y_2, z_2 \in V(G_2)$ are pairwise distance $\text{diam}(G_2)$ away from each other so G_1 and G_2 are 3-peripheral. □

Lemma 4.2 is vital in determining when a product involving a tree has anti-van der Waerden number four.

Lemma 4.2. *Suppose T is a tree that is not 3-peripheral and $u_i, u_j \in V(T)$ realize the diameter of T. If there exist $u_x, u_y \in V(T)$ such that $d(u_x, u_j) = \text{diam}(T)$ and $d(u_i, u_y) = \text{diam}(T)$, then $d(u_x, u_y) = \text{diam}(T)$.*

Proof. If $u_i = u_x$ or $u_j = u_y$, then the result is immediate. Define P_x to be the $u_x - u_j$ path, P_y to be the $u_i - u_y$ path, and P to be the $u_i - u_j$ path. Let u_ℓ be the closest vertex in $V(P) \cap V(P_x)$ to u_i, and u_k to be closest vertex in $V(P) \cap V(P_y)$ to u_j. Note that

$$d(u_x, u_\ell) + d(u_\ell, u_j) = d(u_x, u_j) = d(u_i, u_j) = d(u_i, u_\ell) + d(u_\ell, u_j).$$

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Subtracting $d(u_\ell, u_j)$ shows that $d(u_x, u_\ell) = d(u_i, u_\ell)$. Similarly, $d(u_k, u_y) = d(u_k, u_j)$.

Assume $d(u_i, u_k) < d(u_i, u_\ell)$. Then P is the concatenation of the $u_i - u_k$, $u_k - u_\ell$, and $u_\ell - u_j$ paths. Notice that $\operatorname{diam}(T) = d(u_i, u_y) = d(u_i, u_k) + d(u_k, u_y)$ implies that one of $d(u_i, u_k)$ and $d(u_k, u_y)$ is at least $\operatorname{diam}(T)/2$. Similarly, at least one of $d(u_x, u_\ell)$ and $d(u_\ell, u_j)$ is at least $\operatorname{diam}(T)/2$. Concatenating these two paths with the nontrivial $u_k - u_\ell$ path yields a path whose length is greater than $\operatorname{diam}(T)$, a contradiction.

Next, assume $u_k = u_\ell$. If $d(u_i, u_k) \neq \operatorname{diam}(T)/2 \neq d(u_k, u_j)$, then one of the paths $u_i - u_k - u_x$ or $u_y - u_k - u_j$ has length greater than $\operatorname{diam}(T)$, a contradiction. Thus, $d(u_i, u_k) = d(u_k, u_j) = \operatorname{diam}(T)/2$. Since

$$\operatorname{diam}(T) = d(u_i, u_y) = d(u_i, u_k) + d(u_k, u_y) = \frac{\operatorname{diam}(T)}{2} + d(u_k, u_y),$$

it follows that $d(u_k, u_y) = \operatorname{diam}(T)/2$. Since $u_k = u_\ell$ lies on the $u_y - u_j$ path, we have $d(u_y, u_j) = d(u_y, u_k) + d(u_k, u_j) = \operatorname{diam}(T)$. Now u_i, u_j, u_y pairwise realize the diameter, contradicting that T is not 3-peripheral.

Finally, suppose $d(u_i, u_\ell) < d(u_i, u_k)$ so that P is the concatenation of the $u_i - u_\ell$, $u_\ell - u_k$, and $u_k - u_j$ paths. Observe,

$$\begin{aligned} d(u_x, u_y) &= d(u_x, u_\ell) + d(u_\ell, u_k) + d(u_k, u_y) \\ &= d(u_i, u_\ell) + d(u_\ell, u_k) + d(u_k, u_j) \\ &= d(u_i, u_j) \\ &= \operatorname{diam}(T). \end{aligned}$$

□

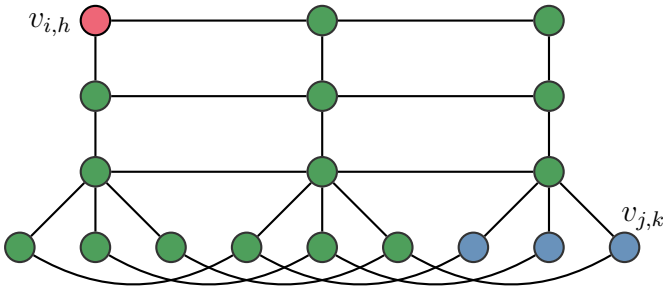


Figure 5. The product of trees G, from Figure 2, and P_3 with vertices labeled as in the proof of Theorem 4.3.

In Theorem 4.3, we show that if two trees T, T' are not 3-peripheral and the diameter of their product is odd, then we can construct an exact 3-coloring of $T \square T'$ with no rainbow 3-APs. Combining this with Theorem 2.9 yields $\operatorname{aw}(T \square T', 3) = 4$. The construction of the coloring is based on the spines of T and T'. It colors vertices *blue* (or *red*) in $T \square T'$ that correspond to vertices in T and T' that are diameter's distance away from the first (or last)

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vertices of the spines, and *green* everywhere else. The key to proving that there is no rainbow 3-AP in the coloring is to take advantage of the structural property of not 3-peripheral trees given by Lemma 4.2. We show that any *blue* and *red* vertices must be distance $\text{diam}(T\square T')$ apart, and prove no *green* vertex can be added to create a rainbow 3-AP.

Theorem 4.3. *If T and T' are trees that are not 3-peripheral with $|T|, |T'| \geq 2$ and $\text{diam}(T\square T')$ is odd, then $\text{aw}(T\square T', 3) = 4$.*

Proof. Suppose $V(T) = \{u_1, \ldots, u_x\}$ and $V(T') = \{w_1, \ldots, w_y\}$. Let S and S' be the spines defined by the $u_i - u_j$ and $w_h - w_k$ geodesics, respectively. Define $c : V(T\square T') \to \{red, blue, green\}$ by

$$c(v_{\alpha,\beta}) = \begin{cases} red, & \text{if } d(v_{\alpha,\beta}, v_{j,k}) = \text{diam}(T\square T'), \\ blue, & \text{if } d(v_{\alpha,\beta}, v_{i,h}) = \text{diam}(T\square T'), \\ green, & \text{otherwise.} \end{cases}$$

Figure 5 illustrates this coloring c on $G\square P_3$, where G is the tree from Figure 2 and $P_3 = (w_1, w_2, w_3)$. By Lemma 4.1, $T\square T'$ is not 3-peripheral and so c is a well-defined coloring. Note that if a rainbow 3-AP exists, it must contain a *red* and a *blue* vertex. Let v_{r_1,r_2} and v_{b_1,b_2} denote arbitrary *red* and *blue* vertices, respectively. Proposition 2.1 and Observation 2.2 imply that $d_T(u_{r_1}, u_j) = \text{diam}(T) = d_T(u_{b_1}, u_i)$ and $d_{T'}(w_{r_2}, w_k) = \text{diam}(T') = d_{T'}(w_{b_2}, w_h)$. Now, Lemma 4.2 implies that $d_T(u_{r_1}, u_{b_1}) = \text{diam}(T)$ and $d_{T'}(w_{r_2}, w_{b_2}) = \text{diam}(T')$, so

$$d(v_{r_1,r_2}, v_{b_1,b_2}) = \text{diam}(T\square T').$$

Now, for the sake of contradiction, assume there exists some $v_{\alpha,\beta} \in V(T\square T')$ such that $\{v_{b_1,b_2}, v_{r_1,r_2}, v_{\alpha,\beta}\}$ or $\{v_{r_1,r_2}, v_{b_1,b_2}, v_{\alpha,\beta}\}$ is a 3-AP. If $\{v_{b_1,b_2}, v_{r_1,r_2}, v_{\alpha,\beta}\}$ is a 3-AP, then $d(v_{r_1,r_2}, v_{\alpha,\beta}) = \text{diam}(T\square T')$. Thus, Proposition 2.1 and Observation 2.2 imply that

$$d_T(u_{r_1}, u_\alpha) = \text{diam}(T) \text{ and } d_{T'}(w_{r_2}, w_\beta) = \text{diam}(T').$$

The first equality, along with $d_T(u_i, u_{b_1}) = \text{diam}(T)$ and Lemma 4.2, imply that $d_T(u_i, u_\alpha) = \text{diam}(T)$. Similarly, the second equality, along with $d_{T'}(w_h, u_{b_2}) = \text{diam}(T')$ and Lemma 4.2, imply that $d_{T'}(w_h, w_\beta) = \text{diam}(T')$. Thus, $c(v_{\alpha,\beta}) = blue$, and $\{v_{b_1,b_2}, v_{r_1,r_2}, v_{\alpha,\beta}\}$ is not a rainbow 3-AP. A similar argument shows that $\{v_{r_1,r_2}, v_{b_1,b_2}, v_{\alpha,\beta}\}$ is not a rainbow 3-AP.

Thus, any rainbow 3-AP must be of the form $\{v_{r_1,r_2}, v_{\alpha,\beta}, v_{b_1,b_2}\}$ for some $v_{\alpha,\beta} \in V(T\square T')$ meaning $d(v_{r_1,r_2}, v_{\alpha,\beta}) = d(v_{\alpha,\beta}, v_{b_1,b_2})$.

Since $T\square T'$ is bipartite, we can partition $V(T\square T')$ into partite sets A and B. Because $d(v_{r_1,r_2}, v_{b_1,b_2})$ is odd, the vertices v_{r_1,r_2} and v_{b_1,b_2} are in different parts, say $v_{r_1,r_2} \in A$ and $v_{b_1,b_2} \in B$. If $d(v_{r_1,r_2}, v_{\alpha,\beta})$ is even, then $v_{\alpha,\beta} \in A$ and therefore $d(v_{b_1,b_2}, v_{\alpha,\beta})$ is odd. Similarly, if $d(v_{r_1,r_2}, v_{\alpha,\beta})$ is odd, then $d(v_{b_1,b_2}, v_{\alpha,\beta})$ is even. So, no such $v_{\alpha,\beta}$ exists, and c contains no rainbow 3-APs. Therefore, $\text{aw}(T\square T', 3) \geq 4$, and Theorem 2.9 implies that $\text{aw}(T\square T', 3) = 4$. □

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5. Future Work

In Section 3 we determined $\text{aw}(T\square G, 3)$ when T is 3-peripheral. In Section 4 we determined $\text{aw}(T\square T', 3)$ when T and T' are not 3-peripheral and $\text{diam}(T\square T')$ is odd. The remaining case to consider is when T and T' are not 3-peripheral and $\text{diam}(T\square T')$ is even.

To illustrate why this case is more difficult, recall that Theorem 2.10 implies that if $\text{diam}(P_m\square P_n)$ is even, then $\text{aw}(P_m\square P_n, 3) = 3$ only if $m \leq 3$ or $n \leq 3$. When $\text{diam}(P_m\square P_n)$ is odd the coloring which colors $v_{1,1}$ red, $v_{m,n}$ blue, and the remaining vertices green avoids rainbow 3-APs. This was generalized in Section 3 to the coloring found in the proof of Theorem 4.3. Alternatively, when $\text{diam}(P_m\square P_n)$ is even with $m, n \geq 4$ the coloring which colors $v_{1,1}$ red, $v_{m-1,n}$ and $v_{m,n-1}$ blue, and the remaining vertices green avoids rainbow 3-APs. When attempting to generalize this coloring, problems arise. Following the set-up in the proof of Theorem 4.3, in particular $v_{j,k}$ and $v_{i,h}$ are diameter vertices in $T\square T'$, the generalized coloring $c : V(T\square T') \to \{red, blue, green\}$ can be defined by

$$c(v_{\alpha,\beta}) = \begin{cases} red, & \text{if } d(v_{\alpha,\beta}, v_{j,k}) = \text{diam}(T\square T'), \\ blue, & \text{if } d(v_{\alpha,\beta}, v_{i,h}) = \text{diam}(T\square T') - 1, \\ green, & \text{otherwise.} \end{cases}$$

In Figure 6, we consider two different products of trees. Note that each tree in each product is not 3-peripheral and the product has even diameter. The figure on the left has two different colorings based on our choices of $v_{i,h}$ and $v_{j,k}$. One coloring uses $\{red, yellow, \mathbb{white}\}$ and is not rainbow-free and the other coloring uses $\{green, cyan, black\}$ and is rainbow-free. However, the product on the right has more symmetry so our choice of $v_{i,h}$ and $v_{j,k}$ does not matter and this coloring does have a rainbow.

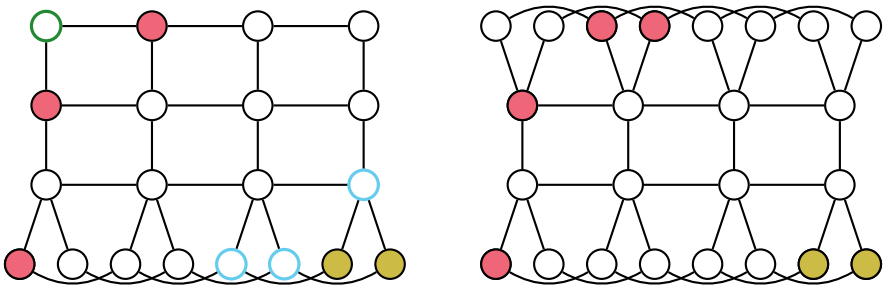


Figure 6. When we apply the coloring to the product on the left we get two different colorings depending on our choice of $v_{j,k}$ and $v_{i,h}$. Due to symmetry, there is only one coloring for the graph on the right.

These examples suggest two things. First, 3-peripheral or not 3-peripheral alone will not) fully characterize which tree products have anti-van der Waerden number three and which have anti-van der Waerden number four. Second, the product in Figure 6 on the left was rainbow-free with one coloring and had a rainbow with another coloring. This means there might be a 'good choice' required when trying to find a rainbow using a generalized coloring when the diameter of the product is even.

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In studying peripheral trees, Theorem 3.6 found that the $\mathrm{aw}(G\square T, 3) = 3$ when $|G| \geq 2$ and T is a 3-peripheral tree. This led to the following conjecture.

Conjecture 5.1. *If T is a k-peripheral tree and G is a nontrivial connected graph, then*

$$\mathrm{aw}(T\square G, k) = k.$$

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