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Maximum boundary independent broadcasts in graphs and trees

J. I. Hoepner, G. MacGillivray, C. M. Mynhardt

Department of Mathematics and Statistics
University of Victoria
Victoria, B.C., Canada V8W 2Y2

julesihoepner@gmail.com, gmacgill@uvic.ca, kieka@uvic.ca

Abstract

A broadcast on a connected graph G is a function $f : V(G) \to \{0, 1, ..., \mathrm{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$. If $d_G(u, v) \geq f(u) + f(v)$ for any pair of vertices u, v with $f(u) > 0$ and $f(v) > 0$, the broadcast is said to be boundary independent.

We show that the maximum weight $\alpha_{bn}(G)$ of a boundary independent broadcast can be bounded in terms of the independence number $\alpha(G)$, and prove that the maximum boundary independent broadcast problem is NP-hard. We investigate bounds on $\alpha_{bn}(T)$ when T is a tree in terms of its order and the number of vertices of degree at least 3, and determine a sharp upper bound on $\alpha_{bn}(T)$ when T is a caterpillar, giving $\alpha_{bn}(T)$ exactly for certain families of caterpillars. We conclude by describing a polynomial-time algorithm to determine $\alpha_{bn}(T)$ for a given tree T.

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1. Introduction

There are several methods by which the concept of independent sets may be generalized to broadcast independence. If we require that no broadcasting vertex hears another, we obtain the definition of *cost independent* broadcasts introduced by Erwin in [6], which we refer to as *hearing independent*, abbreviated h-independent. The definition of boundary independent (or bn-independent)

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broadcasts, in which no broadcasts overlap on edges, was introduced by Neilson [16] and Mynhardt and Neilson [13] as an alternative to hearing independence.

The maximum cost of a boundary independent broadcast on a given graph G is referred to as its *boundary independence number*, denoted $\alpha_{bn}(G)$. For a given integer $k \geq 0$, the problem of determining whether $\alpha_{bn}(G) \geq k$ is called the *maximum bn-independent broadcast problem*. The *hearing independence number* $\alpha_h(G)$ and the *maximum h-independent broadcast problem* are defined similarly.

As any boundary independent broadcast is hearing independent, it follows from the definitions that $\alpha_{bn}(G) \leq \alpha_h(G)$ for all graphs G. In [13], Mynhardt and Neilson showed that $\alpha_h(G)/\alpha_{bn}(G) < 2$, and that this bound is asymptotically best possible. They posed the problem of investigating the ratio $\alpha_{bn}(G)/\alpha(G)$ in [15].

Problem 1.1. *Is it true that $\alpha_{bn}(G) < 2\alpha(G)$ for any graph G?*

It was shown in [13] that $\alpha_{bn}(G) \leq n - 1$ for all graphs G of order n, with equality if and only if G is a path or a *generalized spider*, a tree with exactly one vertex of degree greater than 2. It is easily observed that $\alpha(G) \leq n - \delta(G)$, where $\delta(G)$ denotes the minimum degree among the vertices of G. In [14], Mynhardt and Neilson asked whether a similar inequality existed for the maximum boundary independence number.

Problem 1.2. *Show that $\alpha_{bn}(G) \leq n - \delta(G)$ for any graph G of order n.*

For any tree T, the bound in Problem 1.2 follows immediately from the bound $\alpha_{bn}(T) \leq n-1$ and the fact that $\delta(T) = 1$.

Broadcast definitions and known results are presented in Section 2. In Section 3, we show that $\alpha_{bn}(G) < 2\alpha(G)$ for all G, solving Problem 1.1. We further resolve Problem 1.2 by showing that $\alpha_{bn}(G) \leq n - \delta(G)$ for any graph G.

In Section 4, by considering a transformation from independent sets to boundary independent broadcasts on graphs, we observe that determining whether $\alpha_{bn}(G) \geq k$ for a given integer k is NP-Complete. In Section 5, we investigate the maximum boundary independence number of trees and determine $\alpha_{bn}(G)$ exactly for families of caterpillars.

We continue our study of maximum boundary independence broadcasts in trees in Section 6. Using a method similar to the proof technique employed by Bessy and Rautenbach in [1], we derive an $O(n^9)$ time algorithm to determine $\alpha_{bn}(T)$ for a given tree T.

Open problems and directions for further research are discussed in Section 7.

2. Definitions and Background

Erwin [6] defined a *broadcast* on a nontrivial connected graph G as a function $f : V(G) \to \{0, 1, ..., \text{diam}(G)\}$ such that $f(v)$ is at most the eccentricity $e(v)$ for all vertices v. We say a vertex v is *broadcasting* if $f(v) \geq 1$, and that $f(v)$ is the *strength* of f from v. The *cost* or *weight* of f is $\sigma(f) = \sum_{v \in V(G)} f(v)$.

Given a broadcast f on G and a broadcasting vertex v, a vertex u *hears* f from v if $d_G(u, v) \leq f(v)$. We define the *f-neighbourhood* of v, denoted by $N_f(v)$, as the set of all vertices which hear f from v (including v itself).

The f-*private neighbourhood* of v, denoted by $PN_f(v)$, consists of those vertices that hear f only from v. The f-*boundary* of v is $B_f(v) = \{u \in N_f(v) \,|\, d(u,v) = f(v)\}$. The f-*private boundary* $PB_f(v)$ is defined analogously. In particular, $PB_f(v) = PN_f(v) \cap B_f(v)$. If $u \in N_f(v) \backslash B_f(v)$, v is said to *overdominate* u by k, where $k = f(v) - d_G(u,v)$. A vertex which does not broadcast or hear f from any broadcasting vertex is *undominated*.

Throughout this paper, we partition the set of broadcasting vertices V_f^+ into $V_f^1 = \{v \in V(G) \,|\, f(v) = 1\}$ and $V_f^{++} = \{v \in V(G) \,|\, f(v) > 1\}$. We denote the set of undominated vertices by U_f. A broadcast f is *dominating* if $U_f = \emptyset$. The broadcast domination number, $\gamma_b(G)$, is the minimum weight of such a broadcast. An overview of broadcast domination in graphs is given by Henning, MacGillivray, and Yang in [8].

We say an edge $e = uv$ *hears* f or *is covered by* $w \in V_f^+$ if $u, v \in N_f(w)$ and at least one endpoint does not lie on the f-boundary of w. If no such vertex w exists, then e is $uncovered$. The set of uncovered edges is denoted U_f^E.

An *independent set* on a graph G is a set of pairwise nonadjacent vertices. The minimum cardinality of a maximal independent set, called the *independent domination number* of G, is denoted $i(G)$. A broadcast f is *hearing independent* if $x \notin N_f(v)$ for any $x, v \in V_f^+$. It is *boundary independent* if $N_f(v) \backslash B_f(v) \subseteq PN_f(v)$ for all $v \in V_f^+$.

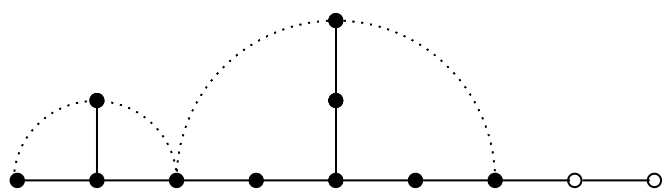

Figure 1: A boundary independent broadcast f on a tree. Vertices v_1 and v_2 broadcast at strengths 1 and 2, respectively. The vertex v_2 overdominates u by 1, whereas w_1 and w_2 are undominated.

Although efficient broadcast domination was shown to be solvable in polynomial time for every graph in [7], the complexity of hearing independence was unknown even for trees until an efficient algorithm was found by Bessy and Rautenbach in [1].

Theorem 2.1. [1] *For any tree T of order n, $\alpha_h(T)$ can be determined in $O(n^9)$ time.*

Hearing independence was further studied by Bessy and Rautenbach [2, 3] and by Dunbar et al. [5]. The more recent study of boundary independent broadcasts was continued by Mynhardt and Neilson in [12, 14, 15] and by Marchessault and Mynhardt in [11]. For terminology and general concepts in graphs theory not defined in this paper, see Chartrand, Lesniak, and Zhang [4].

3. Upper and Lower Bounds on $\alpha_{bn}(G)$

Our focus in this section is to establish bounds on $\alpha_{bn}(G)$ for general graphs, and determine parameters comparable to α_{bn} to place the parameter within a chain of inequalities.

In [3], Bessy and Rautenbach found that $\alpha_h(G) < 4\alpha(G)$, adapting a proof technique used by Neilson [16] to show that $\alpha_h(G) < 2\alpha_{bn}(G)$. Mynhardt and Neilson further studied the ratio $\frac{\alpha_{bn}(G)}{\alpha(G)}$ in [15] and asked whether it can be shown that $\frac{\alpha_{bn}(G)}{\alpha(G)} < 2$ for all graphs G.

Theorem 3.1. *For any graph G, $\alpha_{bn}(G) < 2\alpha(G)$.*

Proof. Let f be an α_{bn}-broadcast on G. If $V_f^+ = V_f^1$, V_f^+ is an independent set, so assume there exists v such that $f(v) \geq 2$. Let $u \in B_f(v)$ and consider a subgraph T_u consisting of u and unique geodesics from u to each vertex in $S = \{x \in V_f^+ : u \in B_f(x)\}$, the set of all broadcasting vertices heard by u (if $u \in PB_f(v)$, $S = \{v\}$). Since no edge can be covered by two different broadcasts, T_u is an induced path or spider, hence $|V(T_u)| = 1 + \sum_{w \in S} f(w)$.

Consider a proper two-coloring of T_u and define a new boundary independent broadcast by deleting all broadcasts from the leaves of T_u and adding strength-one broadcasts from each vertex in the color set of highest cardinality (if they have equal cardinality, select one arbitrarily).

Since broadcasts overlap only on boundaries, the broadcasting vertices of the resulting bn-independent broadcast form an independent set of cardinality at least $|V(T_u)|/2$. Repeating the process until no vertices broadcasting at strength greater than 1 remain yields an independent set on G. Since $|V(T_u)| < \sum_{w \in S} f(w)$, it follows that $\alpha_{bn}(G)/2 < \alpha(G)$. $\square$

As $\alpha_{bn}(G) \leq \alpha_h(G)$ for any graph G, Bessy and Rautenbach's bound now follows easily from Neilson's result and Theorem 3.1.

Corollary 3.2. *For any graph G, $\alpha_h(G) < 4\alpha(G)$.*

Let $\delta(G)$ denote the minimum degree of G. It is easy to see that $\alpha(G) \leq n - \delta(G)$, as the inclusion of any vertex to an independent set excludes its neighbours. We determine the analogous result for maximum boundary independence, thereby solving an open problem posed in [14].

Theorem 3.3. *For any graph G of order n, $\alpha_{bn}(G) \leq n - \delta(G)$.*

Proof. The result is clear if $|V(G)| = 1$, so assume the theorem holds for all graphs G such that $|V(G)| \leq n - 1$ and consider a graph G of order n. Let f be an α_{bn}-broadcast on G with $|V_f^1|$ maximum. For any $v \in V_f^{++}$, $PB_f(v) = \emptyset$, otherwise a new boundary independent broadcast of equal weight could be constructed by reducing $f(v)$ by 1, and broadcasting at strength 1 from a vertex in the f-private boundary of v.

For some $v \in V_f^+$, consider the graph $G' = G - PN_f(v)$ of order n', and let $f_{G'}$ be f restricted to G'. By induction, $\sigma(f_{G'}) \leq \alpha_{bn}(G') \leq n' - \delta(G')$, hence $\alpha_{bn}(G) = \sigma(f_{G'}) + f(v) \leq f(v) + n' - \delta(G')$.

Consider a vertex u of minimum degree in G'. First, suppose $u \in B_f(v)$ and let $k = |PN_f(v) \cap N_G(u)|$. Let P be a $u-v$ geodesic in G. Since P has length $f(v)$ and u is adjacent to $k-1$ vertices in $PN_f(v) - V(P)$, we have that $k - 1 + f(v) \leq |PN_f(v)|$. In particular, if $k + f(v) = |PN_f(v)| + 1$, then either $PN_f(v)$ consists of a path on $f(v)$ vertices (and thus $d(v) = 1$), or $f(v) = 2$ and $d_G(v) = k$. In either case, since u hears more than one broadcasting vertex under f, $\delta(G) < d_G(u)$. Thus,

$$\alpha_{bn}(G) \leq f(v) + n' - \delta(G') \leq f(v) + n' - d_G(u) + k \leq n + 1 - d_G(u) \leq n - \delta(G).$$

Otherwise, if $k + f(v) \leq |PN_f(v)|$,

$$\alpha_{bn}(G) \leq f(v) + n' - d_G(u) + k \leq n - \delta(G).$$

Finally, if $u \notin B_f(v)$, then $d_G(u) = d_{G'}(u)$, hence $\delta(G) \leq \delta(G')$ and $\alpha_{bn}(G) \leq n - \delta(G') \leq n - \delta(G)$ as desired. □

4. The Hardness of Determining $\alpha_{bn}(G)$

Given a graph G and a positive integer k, we may verify that a broadcast f on G of cost at least k is boundary independent in polynomial time by checking that $f(u)+f(v) \leq d_G(u, v)$ for every pair of distinct broadcasting vertices $u, v \in V_f^+$. It follows that the maximum bn-independent broadcast problem is in NP. We proceed to show that the problem is NP-complete by a transformation from the independent set problem on G, which was shown to be NP-complete by Karp [10], to the maximum bn-independent broadcast problem on a corresponding graph C_G.

The *corona* $G \odot H$ of graphs G and H is constructed from G and $n = |V(G)|$ copies of H by joining the ith vertex in G by edges to every vertex in the ith copy of H. Let $C_G = G \odot K_1$.

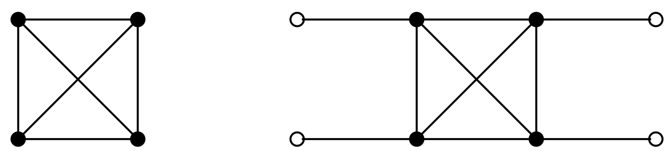

Figure 2: The construction of the corona $K_4 \odot K_1$.

Proposition 4.1. *Let G be a connected graph on $n \geq 2$ vertices and let $C_G = G \odot K_1$. Then $\alpha_{bn}(C_G) = n + \alpha(G)$.*

Proof. Let $l_1, ..., l_n$ be a labeling of the leaves of C_G, and let $v_1, ..., v_n$ be a labeling of the remaining vertices such that l_i is adjacent to v_i for all i. Let $S \subseteq \{v_1, ..., v_n\}$ be a maximum independent set on the subgraph of C_G corresponding to G.

Define a broadcast f on C_G by

$$f(u) = \begin{cases} 2, & \text{if } u = l_i \text{ for some } i \text{ and } v_i \in S, \\ 1, & \text{if } u = l_i \text{ for some } i \text{ and } v_i \notin S, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_u f(u) = n + \alpha(G)$. Since only leaves broadcast, and since $d_{C_G}(l_i, l_j) \leq f(l_i) + f(l_j)$ for any pair of leaves $l_i \neq l_j$, f is boundary independent. Therefore $\alpha_{bn}(C_G) \geq n + \alpha(G)$.

To show that $\alpha_{bn}(C_G) \leq n + \alpha(G)$, let f be an α_{bn}-broadcast on C_G. If a vertex v_i is broadcasting, the bn-independent broadcast f' on C_G defined by $f'(l_i) = f(v_i) + 1$, $f'(v_i) = 0$, and $f'(u) = f(u)$ otherwise has greater cost than f, a contradiction. It follows that $V_f^+ \subseteq \{l_1, ..., l_n\}$.

Suppose some leaf l_i broadcasts at strength $k \geq 3$ under f. Since $f(l_i) \leq e(l_i)$, there exist at least $k - 2$ leaves which hear l_i from a distance greater than 2. Define a new maximum bn-independent broadcast f' by

$$f'(l_i) = \begin{cases} f(l_i), & \text{if } f(l_i) \in \{1,2\}, \\ 2, & \text{if } f(l_i) = k > 2, \\ 1, & \text{if } f(l_i) = 0. \end{cases}$$

and consider $R \subseteq \{v_1, ..., v_n\}$, where $v_i \in R$ if and only if $f'(l_i) = 2$. Two vertices adjacent to leaves broadcasting at strength 2 cannot be adjacent, hence R is an independent set on the subgraph of C_G corresponding to G. Since every leaf broadcasts at strength 1 or 2 under f', we have that $|R| = \alpha_{bn}(C_G) - n$. Therefore $\alpha_{bn}(C_G) \leq n + |R| \leq n + \alpha(G)$. □

Corollary 4.2. *Let G be a graph and k a positive integer. The problem of deciding whether $\alpha_{bn}(G) \geq k$ is NP-complete.*

Proof. We have already observed that the maximum bn-independent broadcast problem is NP. As the transformation described in Proposition 4.1 can be carried out in polynomial time, the problem is NP-complete. □

It follows that for any graph class $\mathcal{C}$ for which $G \in \mathcal{C}$ implies $C_G \in \mathcal{C}$, if the independent set problem is known to be NP-complete for all graphs in $\mathcal{C}$, then so is the maximum bn-independent set problem.

Graph classes for which the independent set problem is known to be NP-complete are listed in [17]. In particular, the maximum bn-independent set problem is NP-complete for planar and toroidal graphs, graphs with maximum degree $\Delta \in \{3, 4, 5, 6\}$, triangle-free and C_n-free graphs for $n \in \{4, 5, 6\}$, K_n-free graphs for $n \in \{4, 6, 7\}$, and house-free graphs.

5. Trees

Let T be a tree. Its *branch-leaf representation* $\mathcal{BL}(T)$, also known as the *homeomorphic reduction* of T, is obtained by successively removing a vertex of degree 2 and adding an edge between its two neighbours until no such vertices remain. Deleting all leaves yields the *branch representation* of T, denoted $\mathcal{B}(T)$.

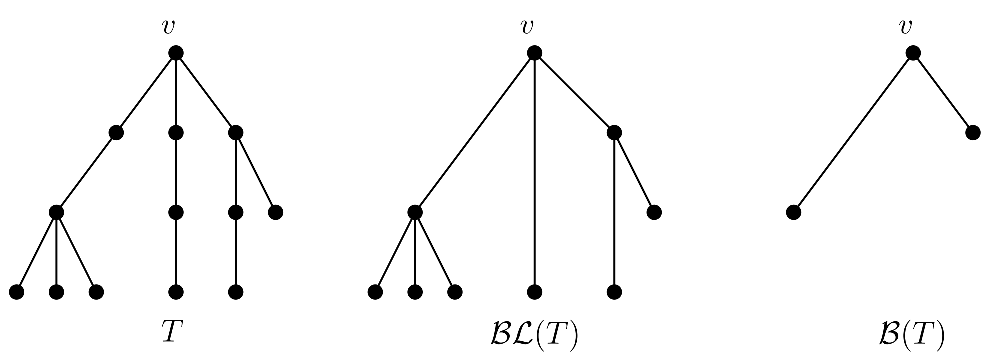

Figure 3: The branch-leaf representation and branch representation of a tree T.

We partition the vertices of T into sets B_T, L_T, and τ_T, where B_T denotes the set of branch vertices of T, L_T the set of leaves, and τ_T the set of vertices of degree 2, called the *trunks* of T.

We further define a subset R_T of B_T consisting of branch vertices adjacent to at most one leaf in $\mathcal{BL}(T)$. In other words, a branch vertex b belongs to R_T if there exists at most one leaf $l \in L_T$ such that for every $b' \in B_T - \{b\}$, the unique $b' - l$ path contains b. In Figure 3 above, $|B_T| = 3$ and $R_T = \{v\}$.

Maximum boundary independent broadcasts on trees were studied by Neilson in [16], who determined that $\alpha_{bn}(T) \leq n - |B_T| + |R_T|$ for any tree T with $B_T \neq \emptyset$. Neilson asked whether this upper bound could be improved to $\alpha_{bn}(T) \leq n - |B_T| + \alpha(T[R_T])$, where $T[R_T]$ is the subgraph of T induced by R_T.

A *caterpillar* is a tree with a diametrical path D such that every vertex lies on D or is adjacent to a vertex on D. We proceed to show that $\alpha_{bn}(C) \leq n - |B_C| + \alpha(C[R_C])$ for all caterpillars C, and determine $\alpha_{bn}(C)$ exactly for certain subclasses of caterpillars. Observe that if C is a caterpillar, $C[R_C]$ is a path or a forest of paths.

The following lemma will be useful throughout this section.

Lemma 5.1. [16] *If f is an α_{bn}-broadcast on a tree T, no leaf of T hears a broadcast from any non-leaf vertex.*

In particular, if f is a maximum boundary independent broadcast on a caterpillar, only leaves and trunks broadcast.

Given a caterpillar C, its *spine* is an (arbitrarily chosen) diametrical path. Label the leaves of C as $l_1, ..., l_m$ such that l_1 and l_m are the endpoints the spine and for all $i \leq j$, $d_C(l_1, l_i) \leq d_C(l_1, l_j)$. Call $l_2, ..., l_{m-1}$ the *inner leaves* of C.

Lemma 5.2. *Let f be an α_{bn}-broadcast on a caterpillar C such that $|V_f^1|$ is maximized. Then $l_1, l_m \in V_f^+$.*

Proof. Suppose not. If $l_1 \in N_f(v)$ for some $v \in V_f^+$, define a new broadcast f' by $f'(v) = f(v-1)$, $f'(l_1) = 1$, and $f'(w) = f(w)$ for all $w \neq v, l_1$. If some vertex u does not hear f', broadcasting at strength 1 from u produces a boundary independent broadcast of greater cost than f, a contradiction. Therefore f' is an α_{bn}-broadcast on C. By Lemma 5.1, v is a leaf, hence $d_C(l_1, v) \geq 2$ and so $v \notin V_f^1$. But then $|V_{f'}^1| > |V_f^1|$, a contradiction.

It follows that $l_1 \in V_f^+$. Similarly, $l_m \in V_f^+$. □

Theorem 5.3. *If C is a caterpillar, then $\alpha_{bn}(C) \leq n - |B_C| + \alpha(C[R_C])$.*

Proof. Let f be an α_{bn}-broadcast on C such that $|V_f^1|$ is maximized, and let l_i be an inner leaf of C adjacent to a branch vertex b_i. If $f(l_i) \geq 3$, then since l_1 and l_m are broadcasting, l_i covers at least $2(f(l_i) - 1)$ edges on the spine.

If $f(l_i)$ is odd, let f' be a broadcast defined by $f'(l_i) = 1$, $f'(v) = 1$ for all v at an even distance from l_i, and $f'(v) = f(v)$ for all $v \notin N_f[l_i]$.

If $f(l_i)$ is even, define f' by $f'(l_i) = 2$, $f'(v) = 1$ for all v at odd distance at least 3 from l_i, and $f'(v) = f(v)$ for all $v \notin N_f[l_i]$.

Figure 4 illustrates the construction of f'.

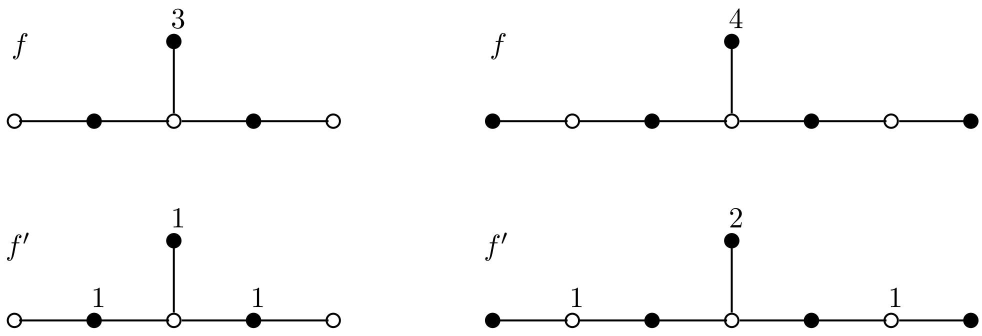

Figure 4: The odd and even cases.

It follows that f' is a boundary independent broadcast on C such that $\sigma(f') = \sigma(f)$ and $|V^1_{f'}| > |V^1_f|$, a contradiction. Thus, $f(l_i) \leq 2$ for each inner leaf l_i. Since f is boundary independent, the set of branch vertices adjacent to leaves that broadcast at strength 2 must form an independent set.

By Lemma 5.1, no branch vertices broadcast under f. Let C' denote the subgraph of C induced by removing all inner leaves and branch vertices, and let $f_{C'}$ denote the restriction of f to C'. Since $f_{C'}$ is boundary independent,

$$\sigma(f_{C'}) \leq \alpha_{bn}(C') \leq |V(C')| - \delta(C') = |V(C')| - 1.$$

Taking the sum over all broadcasting vertices, we find that

$$\alpha_{bn}(C) \leq (|V(C')| - 1) + (L_C - 2) + \alpha(C[R_C]) \leq n - |B_C| + \alpha(C[R_C]).$$

□

Corollary 5.4. *Let C be a caterpillar with $|V(C)| \geq 3$.*

i. If $\tau_C = \emptyset$, then $\alpha_{bn}(C) = |L_C| + \alpha(C[R_C])$.

ii. If C has no two adjacent trunks and no vertices of degree 3, then $\alpha_{bn}(C) = |L_C| + |\tau_C|$.

Proof. Suppose $\tau_C = \emptyset$, so that every vertex of C is either a branch vertex or a leaf. By Theorem 5.3, $\alpha_{bn}(C) \leq n - |B_C| + \alpha(C[R_C]) = |L_C| + \alpha(C[R_C])$. Observe that $C[R_C]$ is a forest of paths, and let S be a maximum independent set of $C[R_C]$. Define a broadcast f on C by

$$f(v) = \begin{cases} 2 & \text{if } v \in L_T \text{ and } v \text{ is adjacent to a vertex in } S \\ 1 & \text{if } v \in L_T \text{ and } v \text{ is not adjacent to a vertex in } S \\ 0 & \text{if } v \notin L_T. \end{cases}$$

As only leaves broadcast, two broadcasts from l_i and l_j may overlap only if one or both vertices broadcast at strength 2. Suppose they do. Without loss of generality, assume $f(l_i) = 2$. Let b_i be the neighbour of l_i in S. But then since b_i has only one leaf, $f(l_j) = 2$ and $b_j \in S$, hence $d(l_i, l_j) \geq 4$. It follows that f is a boundary independent broadcast on C. Thus, $\sigma(f) = |L_C| + \alpha(C[R_C]) \leq \alpha_{bn}(C)$.

Suppose instead that C has no two adjacent trunks and no vertices of degree 3. Since $R_C = \emptyset$, $\alpha_{bn}(C) \leq n - |B_C| + \alpha(C[R_C]) = |L_C| + \tau_C$ by Theorem 5.3. Define a broadcast f on C by

$$f(v) = \begin{cases} 1 & \text{if } v \in L_C \text{ or } v \in \tau_C \\ 0 & \text{otherwise.} \end{cases}$$

By definition, no two broadcasting vertices are adjacent. It follows that f is a boundary independent broadcast, hence $\sigma(f) = |L_C| + |\tau_C| \leq \alpha_{bn}(C)$. $\square$

6. A polynomial time algorithm for trees

In Section 4, we found that the problem of determining the cost of a bn-independent broadcast on a general graph G is NP-complete. However, the complexity of boundary independence in trees is unknown.

In [1], Bessy and Rautenbach proved that the hearing independence number $\alpha_h(T)$ can be determined for a tree T in $O(n^9)$ time. In this chapter, we show that their algorithm can be modified to determine the boundary independence number $\alpha_{bn}(T)$ in $O(n^9)$ time.

A *rooted tree* (T, r) is a tree in which a distinguished vertex r serves as a point of reference for all vertices of T. A vertex v is said to be a *descendant* of u if u lies along the unique $r - v$ path, in which case u is an *ancestor* of v. The descendants adjacent to u are known as the *children* of u.

6.1. Definitions and Notation

Let T be a tree of order $n \geq 3$ in which an arbitrary non-leaf vertex r is chosen to be the root. For each $u \in V(T)$, fix an arbitrary linear order on its children. If $v_1, ..., v_k$ are the children of u in this linear order, define $T_{u,i}$ as the subtree of T induced by u and all vertices w such that the unique $u - w$ paths contains one of $v_1, ..., v_i$ (see Figure 5). In addition, define $T_{u,0}$ as the subtree consisting only of the vertex u. If $\mathrm{c}_T(u)$ denotes the number of children of u in T rooted at r, then, in total, there are at most

$$\sum_{u \in V(T)} (\mathrm{c}_T(u) + 1) = deg(r) + 1 + \sum_{u \in V(T) \setminus \{r\}} deg(u) = 2n - 1 = O(n) \tag{1}$$

such subtrees $T_{u,i}$. (Note that if u is a leaf, then $c_T(u) = 0$ and the only subtree counted is $T_{u,0} = \{u\}$).

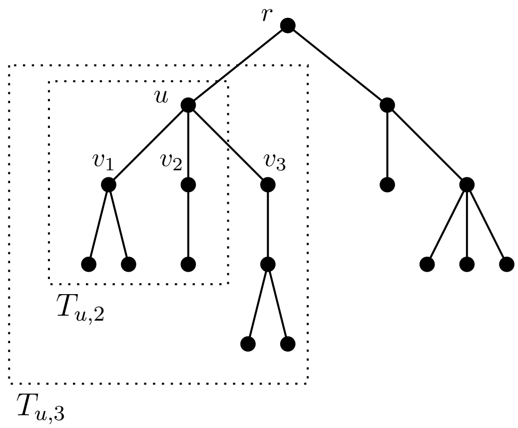

Figure 5: A tree T rooted at r and a vertex u with three children.

Bessy and Rautenbach's algorithm, based on dynamic programming, successively considers subtrees $T_{u,i}$ such that the vertices u are ordered by nonincreasing distance from the root r.

Given a boundary independent broadcast f on T, its restriction to $T_{u,i}$ satisfies the following conditions:

(C1) $f(x) \leq e_T(x)$ for all $x \in V(T_{u,i})$

(C2) $d_T(x,y) \geq f(x) + f(y)$ for every two distinct broadcasting vertices $x, y \in V(T_{u,i})$.

As in [1], we find that if $f(y) > 0$ for some $y \in V(T \backslash T_{u,i})$, then y imposes upper bounds on $f(x)$ for all $x \in V(T_{u,i})$. Specifically, by C2, if $d_T(x,y) \leq f(y)$, then $x \in N_f(y)$ and hence $f(x) = 0$. If $f(y) < d_T(x,y)$, then $f(x) \leq d_T(x,y) - f(y)$.

We may express this upper bound as a function $g_{p,q}(d_T(u,x))$ as follows. First, consider a function $h_t : \mathbb{Z} \to \mathbb{N}_0$ such that $h_t(d) = \max\{0, d-t\}$.

For a vertex $y \in V_f^+$, let $q = f(y)$, $p = f(y) - d_T(u,y)$, and define $g_{p,q} : \mathbb{Z} \to \mathbb{N}_0$ by $g_{p,q}(d) = \max\{0, d-p\}$. Note that $g_{p,q}$ is equivalent to h_t under the restriction $t = f(y) - d_T(u,y)$, and so we have the bound $f(x) \leq g_{p,q}(d_T(u,x))$.

In other words, for each vertex x such that the unique $x-y$ path passes through u, the function $g_{p,q}(d)$ with $d = d_T(u,x)$ establishes an upper bound on $f(x)$ as shown in Figure 6.

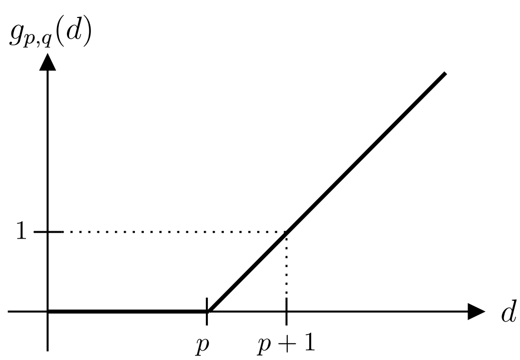

Figure 6: When $d_T(u,x) \geq f(y) - d_T(u,y)$, the upper bound on $f(x)$ imposed by $g_{p,q}(d)$ increases linearly as a function of d.

Define integers $p_{\text{in}},\ q_{\text{in}}$ and y_{in} as follows. If $V_f^+ \backslash V(T_{u,i})$ is empty, let $p_{\text{in}} = -n$ and $q_{\text{in}} = 1$. Otherwise, let $p_{\text{in}} = \max\{p = f(y) - d_T(u,y) : y \in V_f^+ \backslash V(T_{u,i})\}$. Let $y_{\text{in}} \in V_f^+ \backslash V(T_{u,i})$ be a vertex for which this maximum holds. If y_{in} overdominates u, then, by bn-independence, y_{in} is the unique such vertex; otherwise, there may be more than one such vertex and we choose y_{in} arbitrarily. Let $q_{\text{in}} = f(y_{\text{in}})$. Note that

$$(p_{\text{in}}, q_{\text{in}}) = \begin{cases} (-n, 1) & \text{if } V_f^+ \cap V(T \setminus T_{u,i}) = \varnothing \\ (f(y_{\text{in}}) - d_T(u, y_{\text{in}}), f(y_{\text{in}})) & \text{otherwise.} \end{cases} \tag{2}$$

Since $d_T(u, y_{\text{in}}) \geq 0$, it follows that $p_{\text{in}} \leq q_{\text{in}}$ in either case.

Among all broadcasting vertices in $T \backslash T_{u,i}$, y_{in} imposes the strictest upper bound on $f(x)$ for all $x \in T_{u,i}$. That is, for all $x \in V(T_{u,i})$ and all $q = f(y)$, where $y \in V_f^+ \backslash V(T_{u,i})$, we have that $f(x) \leq g_{p_{\text{in}},q_{\text{in}}}(d_T(u,x)) \leq g_{p,q}(d_T(u,x))$.

The existence of p_{in} and q_{in} implies that bounds on $f(x)$ for all $x \in V(T_{u,i})$ can be encoded with just these two values. Symmetrically, the upper bounds on the possible values of f on the vertices of $T \backslash T_{u,i}$ that are imposed by broadcasting vertices in $T_{u,i}$, again expressed as a function of the distance from u in T, can be encoded with two integers p_{out} and q_{out}. That is, if $V_f^+ \cap V(T_{u,i}) = \varnothing$, let $p_{\text{out}} = -n$ and $q_{\text{out}} = 1$; otherwise, let $p_{\text{out}} = \max\{p : y \in V_f^+ \cap V(T_{u,i})\}$, and choose y_{out} similar to y_{in}. Then

$$(p_{\text{out}}, q_{\text{out}}) = \begin{cases} (-n, 1) & \text{if } V_f^+ \cap V(T_{u,i}) = \varnothing \\ (f(y_{\text{out}}) - d_T(u, y_{\text{out}}), f(y_{\text{out}})) & \text{otherwise,} \end{cases} \tag{3}$$

and $p_{\text{out}} \leq q_{\text{out}}$ in either case.

For all $O(n^4)$ possible choices for $p_{\text{in}},\ p_{\text{out}},\ q_{\text{in}},\ q_{\text{out}}$ with $-n \leq p_{\text{in}},\ p_{\text{out}} \leq n$ and $1 \leq q_{\text{in}},\ q_{\text{out}} \leq n$, the algorithm determines the maximum contribution $\sum\limits_{x \in V(T_{u,i})} f(x)$ satisfying:

(C3) $f(x) \leq g_{p_{\text{in}},q_{\text{in}}}(d_T(u,x))$ for every vertex x of $T_{u,i}$.

(C4) If $f(y) > 0$ for some vertex y of $T_{u,i}$, then $g_{p_{\text{out}},q_{\text{out}}}(d) \leq g_{f(y)-d_T(u,y),f(y)}(d)$ for every positive integer d.

6.2. The Algorithm

The following lemma shows we can check C4 in linear time for a given vertex y.

Lemma 6.1. *If* t, f *and* dist *are integers such that* $-n \leq t \leq n$ *and* f, dist $\in [n]$, *then* $h_t(d) \leq h_{f-\text{dist}}(d)$ *for every positive integer* d *if and only if* dist $\geq \max\{f, f-t\}$.

Proof. Suppose $t \leq 0$. Since d is positive, $d \geq t+1$, hence $h_t(d) = d - t$. Moreover,

$$h_{f-\text{dist}}(d) = \begin{cases} 0, & \text{if } d \leq f - \text{dist}, \\ d - f + \text{dist}, & \text{if } d \geq f - \text{dist} + 1. \end{cases}$$

Therefore, $h_t(d) \leq h_{f-\text{dist}}(d)$ if and only if

(i) $d \geq f - \text{dist} + 1$, and
(ii) $d - t \leq d - f + \text{dist}$
for all positive integers d. With $d = 1$, (i) becomes dist $\geq f$. Therefore, (i) and (ii) together are equivalent to dist $\geq \max\{f, f - t\}$.

Suppose instead that $t \geq 1$. When $d \geq t + 1$, $h_t(d)$ is positive, in which case we once again have that $d \geq f - \text{dist} + 1$ and $d - t \leq d - f + \text{dist}$. Since we require that $h_t(d) \leq h_{f-\text{dist}}(d)$ for every positive integer d, we again see that dist $\geq f$ and dist $\geq f - t$, and the lemma follows. □

We can thus check C4 using Lemma 6.1 with $t = p_{\text{out}}$, $f = f(y)$ and dist $= d_T(u, y)$.

Given $u \in V(T)$ and $i \in [k]_0$, let p_{in}, p_{out}, q_{in}, q_{out} be integers with $-n \leq p_{\text{in}}, p_{\text{out}} \leq n$ and $1 \leq q_{\text{in}}, q_{\text{out}} \leq n$. A function

$$f : V(T_{u,i}) \to \mathbb{N}_0 \text{ is } ((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))\text{-}\textit{compatible}$$

if conditions C1, C2, C3, and C4 hold under f. Let $\alpha_{bn}(T_{u,i}, (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$ be the maximum weight of such a function.

Lemma 6.2. *For any non-leaf root r of T, $\alpha_{bn}(T) = \alpha_{bn}(T_{r,c_T(r)}, (-n, 1), (n, 1))$.*

Proof. By definition, $T_{r,c_T(r)} = T$. Since $g_{-n,1}(d) = d + n$ for all nonnegative integers d, $g_{-n,1}(d) \geq n$. Therefore, since $f(x) \leq e_T(x) \leq n$ for all $x \in V(T)$, condition C3 holds by C1. Similarly, since $g_{n,1}(d) = 0$ for all nonnegative integers d, C4 holds by C1 and Lemma 6.1 with $t = p_{\text{out}}$. □

We now show how to determine $\alpha_{bn}(T_{u,i}, (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$ recursively for all $O(n^5)$ choices for (u, i), p_{in}, q_{in}, p_{out}, and q_{out}. Recall that $T_{u,0}$ consists of only the single vertex u.

The following lemma holds for all vertices u of T, but is particularly useful when u is a leaf.

Lemma 6.3. *For any vertex u of T,*

$$\alpha_{bn}(T_{u,0}, (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}})) = \begin{cases} 0, & \textit{if } q_{\text{out}} > p_{\text{out}}, \\ \min\{e_T(u), g_{p_{\text{in}},q_{\text{in}}}(0), p_{\text{out}}\}, & \textit{if } q_{\text{out}} \leq p_{\text{out}}. \end{cases}$$

Proof. First suppose $q_{\text{out}} > p_{\text{out}}$. By (3), either $V_f^+ \cap V(T_{u,0}) = \varnothing$ and so $f(u) = 0$, or y_{out} is a vertex of $T_{u,0}$ such that $d_T(u, y_{\text{out}}) > 0$, which is impossible because $V(T_{u,0}) = \{u\}$. Therefore $f(u) = 0$ and $\alpha_{bn}(T_{u,0}, (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}})) = 0$.

Now suppose that $q_{\text{out}} \leq p_{\text{out}}$. Then (as mentioned above) $q_{\text{out}} = p_{\text{out}}$ and, by (3), $V_f^+ \cap V(T_{u,i}) \neq \varnothing$ and $d_T(u, y_{\text{out}}) = 0$; that is, $u = y_{\text{out}}$ and $f(u) = f(y_{\text{out}}) = q_{\text{out}} > 0$. By C1, $f(u) \leq e_T(u)$, and by C3, $f(u) \leq g_{p_{\text{in}},q_{\text{in}}}(0)$. Hence $\alpha_{bn}(T_{u,0}, (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}})) = \min\{e_T(u), g_{p_{\text{in}},q_{\text{in}}}(0), p_{\text{out}}\}$. □

To illustrate the key recursive step of the algorithm that follows, let $T_{u,i}$ be a nontrivial subtree. Define $p_{\text{in}}^{(0)}$, $q_{\text{in}}^{(0)}$, $p_{\text{out}}^{(0)}$, $q_{\text{out}}^{(0)}$ with respect to $T_{u,i-1}$ as in (2) and (3). Define $p_{\text{in}}^{(1)}$, $q_{\text{in}}^{(1)}$, $p_{\text{out}}^{(1)}$, $q_{\text{out}}^{(1)}$ with respect to T_{v_i,k_i} similarly.

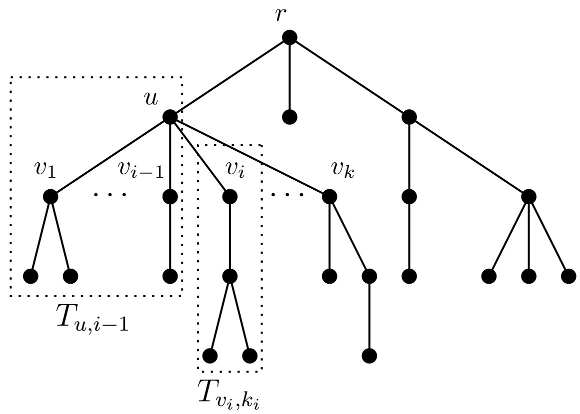

Figure 7: A rooted tree and a vertex u with k children, with $T_{u,i-1}$ and T_{v_i,k_i} as shown.

Let $(p_{\text{in}}, q_{\text{in}}, p_{\text{out}}, q_{\text{out}})$ be given such that $-n \leq p_{\text{in}},\, p_{\text{out}} \leq n$ and $1 \leq q_{\text{in}},\, q_{\text{out}} \leq n$. For example, suppose $(p_{\text{in}}, q_{\text{in}}, p_{\text{out}}, q_{\text{out}}) = (-1, 2, 1, 3)$. The values of p_{out} and q_{out} indicate that at this step, we assume a broadcast of strength 3 from a vertex in $V(T_{u,i})$ overdominates u by 1. We need only consider choices for $p^{(0)}_{\text{out}},\, q^{(0)}_{\text{out}},\, p^{(1)}_{\text{out}},\, q^{(1)}_{\text{out}}$ which do not impose stricter upper bounds on the values of f in $V(T \backslash T_{u,i})$ than p_{out} and q_{out}. For instance, $(p^{(0)}_{\text{out}}, q^{(0)}_{\text{out}}) = (1, 3)$ and $(p^{(1)}_{\text{out}}, q^{(1)}_{\text{out}}) = (-1, 1)$ satisfy this condition as $\max\{p^{(0)}_{\text{out}}, p^{(1)}_{\text{out}} - 1\} \leq p_{\text{out}} = 1$.

Set $(p^{(0)}_{\text{in}},\, q^{(0)}_{\text{in}})$ equal to $(p_{\text{in}},\, q_{\text{in}})$ or $(p^{(1)}_{\text{out}} - 1,\, q^{(1)}_{\text{out}})$ depending on which pair imposes the strictest upper bound on the values of f in $V(T_{u,i})$. Determine $(p^{(1)}_{\text{in}}, q^{(1)}_{\text{in}})$ similarly by comparing $(p_{\text{in}} - 1, q_{\text{in}})$ and $(p^{(0)}_{\text{out}} - 1, q^{(0)}_{\text{out}})$. In our example, $p_{\text{in}} > p^{(1)}_{\text{out}} - 1$ while $p_{\text{in}} - 1 < p^{(0)}_{\text{out}} - 1$, so $(p^{(0)}_{\text{in}},\, q^{(0)}_{\text{in}}) = (-1, 2)$ and $(p^{(1)}_{\text{in}},\, q^{(1)}_{\text{in}}) = (0, 3)$.

Since the algorithm proceeds in order of nonincreasing distance from r, we are given $\alpha_{bn}(T_{u,i-1}, (p^{(0)}_{\text{in}}, q^{(0)}_{\text{in}}), (p^{(0)}_{\text{out}}, q^{(0)}_{\text{out}}))$ and $\alpha_{bn}(T_{v_i,k_i}, (p^{(1)}_{\text{in}}, q^{(1)}_{\text{in}}), (p^{(1)}_{\text{out}}, q^{(1)}_{\text{out}}))$ for all possible values of $p^{(0)}_{\text{out}},\, q^{(0)}_{\text{out}},\, p^{(1)}_{\text{out}},\, q^{(1)}_{\text{out}}$ and corresponding $p^{(0)}_{\text{in}},\, q^{(0)}_{\text{in}},\, p^{(1)}_{\text{in}},\, q^{(1)}_{\text{in}}$. The following lemma and its corollary show that $\alpha_{bn}(T_{u,i}, (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}})) = \max\{\alpha_{bn}(T_{u,i-1}, (p^{(0)}_{\text{in}}, q^{(0)}_{\text{in}}), (p^{(0)}_{\text{out}}, q^{(0)}_{\text{out}})) + \alpha_{bn}(T_{v_i,k_i}, (p^{(1)}_{\text{in}}, q^{(1)}_{\text{in}}), (p^{(1)}_{\text{out}}, q^{(1)}_{\text{out}}))\}$. The recursive process is formalized in Theorem 6.6.

Lemma 6.4. *Let $u \in V(T)$ and $i \in [k]$ be given and let $v_1, ..., v_k$ denote the children of u. Suppose v_i has k_i children. A function $f : V(T_{u,i}) \to \mathbb{N}_0$ is $((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$-compatible if and only if there exist integers*

$$p^{(0)}_{\text{in}},\, p^{(0)}_{\text{out}},\, p^{(1)}_{\text{in}},\, p^{(1)}_{\text{out}},\, q^{(0)}_{\text{in}},\, q^{(0)}_{\text{out}},\, q^{(1)}_{\text{in}},\, q^{(1)}_{\text{out}},$$

with $-n \leq p^{(0)}_{\text{in}},\, p^{(0)}_{\text{out}},\, p^{(1)}_{\text{in}},\, p^{(1)}_{\text{out}} \leq n$ and $1 \leq q^{(0)}_{\text{in}},\, q^{(0)}_{\text{out}},\, q^{(1)}_{\text{in}},\, q^{(1)}_{\text{out}} \leq n$ satisfying the following conditions:

(i) the restriction of f to $V(T_{u,i-1})$ is $((p^{(0)}_{\text{in}}, q^{(0)}_{\text{in}}), (p^{(0)}_{\text{out}}, q^{(0)}_{\text{out}}))$-compatible;

(ii) the restriction of f to $V(T_{v_i,k_i})$ is $((p^{(1)}_{\text{in}}, q^{(1)}_{\text{in}}), (p^{(1)}_{\text{out}}, q^{(1)}_{\text{out}}))$-compatible;

and for every nonnegative integer d:

(iii) $g_{p^{(0)}_{\text{in}}, q^{(0)}_{\text{in}}}(d) = \min\{g_{p_{\text{in}}, q_{\text{in}}}(d),\, g_{p^{(1)}_{\text{out}}, q^{(1)}_{\text{out}}}(d+1)\}$;

(iv) $g_{p^{(1)}_{\text{in}}, q^{(1)}_{\text{in}}}(d) = \min\{g_{p_{\text{in}}, q_{\text{in}}}(d+1),\, g_{p^{(0)}_{\text{out}}, q^{(0)}_{\text{out}}}(d+1)\}$;

(v) $g_{p_{\text{out}}, q_{\text{out}}}(d) \leq \min\{g_{p^{(0)}_{\text{out}}, q^{(0)}_{\text{out}}}(d),\, g_{p^{(1)}_{\text{out}}, q^{(1)}_{\text{out}}}(d+1)\}$.

Proof. Suppose f is $((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$-compatible. Given a vertex $w \in V(T \backslash T_{u,i})$, the distance from w to v_i is one more than the distance from w to u. Thus, (v) holds by C4.

Let $y_{\text{in}}^{(0)}$ be an arbitrarily chosen vertex for which $f(y_{\text{in}}^{(0)}) = q_{\text{in}}^{(0)}$ and $f(y_{\text{in}}) - d_T(u, y_{\text{in}}) = p_{\text{in}}^{(0)}$. If $y_{\text{in}}^{(0)}$ lies in $V(T \backslash T_{u,i})$, the bounds imposed by p_{in} on $f(x)$ for all $x \in V(T_{u,i})$ match those imposed by $p_{\text{in}}^{(0)}$ on the vertices of $T_{u,i-1}$, hence $(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}) = (p_{\text{in}}, q_{\text{in}})$. Otherwise, $y_{\text{in}}^{(0)}$ lies in $V(T_{v_i,k_i})$. Since $g_{p,q}(d+1) = g_{p-1,q}(d)$ when $q \geq 1$, $g_{p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}}(d) = g_{p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)}}(d+1)$. Therefore $g_{p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}}(d) = \min\{g_{p_{\text{in}}, q_{\text{in}}}(d), g_{p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)}}(d+1)\}$ for every nonnegative integer d. Similarly, the minimum in (iv) equals $g_{p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)}}(d)$. Therefore (iii) and (iv) hold.

Since f is $((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$-compatible, $f(x) \leq g_{p_{\text{in}}, q_{\text{in}}}(d_T(u, x))$ for all $x \in V(T_{u,i-1})$. Furthermore, by C2, $f(x) \leq d_T(x, v_i) - f(v_i) \leq g_{p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)}}(d_T(v_i, x)) = g_{p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)}}(d_T(u, x) + 1)$, and so (i) holds for $g_{p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}}(d)$ as in (iii). Similarly, (ii) holds for $g_{p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)}}(d)$ as in (iv).

Conversely, suppose there exist integers

$$p_{\text{in}}^{(0)},\ p_{\text{out}}^{(0)},\ p_{\text{in}}^{(1)},\ p_{\text{out}}^{(1)},\ q_{\text{in}}^{(0)},\ q_{\text{out}}^{(0)},\ q_{\text{in}}^{(1)},\ q_{\text{out}}^{(1)},$$

with $-n \leq p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)} \leq n$ and $1 \leq q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}, q_{\text{out}}^{(1)} \leq n$ satisfying $(i) - (v)$.

By (i) and (ii), f satisfies C1; that is, $f(x) \leq e_T(x)$ for all $x \in V(T_{u,i-1})$ and $x \in V(T_{v_i,k_i})$.

Conditions (i) and (ii) further imply that C2 holds for any $x, y \in V(T_{u,i-1})$ and for any $x, y \in V(T_{v_i,k_i})$. Thus, to prove f satisfies C2, it remains to show that $f(x) + f(y) \leq d_T(x, y)$ for all $x \in V(T_{u,i-1})$ and $y \in V(T_{v_i,k_i})$ such that $f(x), f(y) > 0$.

By (i), the restriction of f to $V(T_{u,i-1})$ satisfies C3; that is,

$$f(x) \leq g_{p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}}(d_T(u, x)).$$

By (iii),

$$f(x) \leq g_{p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)}}(d_T(u, x) + 1).$$

Thus, by (ii) and C4,

$$\begin{aligned} f(x) &\leq g_{f(y) - d_T(u,y), f(y)}(d_T(u, x) + 1) \\ &= g_{f(y) - d_T(v_i,y), f(y)}(d_T(v_i, x)) \\ &\leq \min\{0, d_T(x, y) - f(y)\}. \end{aligned}$$

Since x is broadcasting, $f(x) > 0$ and so $f(x) \leq d_T(x, y) - f(y)$, satisfying C2.

By (iii), $g_{p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}}(d) \leq g_{p_{\text{in}}, q_{\text{in}}}(d)$, and by (iv), $g_{p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)}}(d) \leq g_{p_{\text{in}}, q_{\text{in}}}(d+1)$ for every nonnegative integer d. Thus f satisfies (C3).

Finally, consider $y \in V(T_{u,i})$ such that $f(y) > 0$. If $y \in V(T_{u,i-1})$, then by (v),

$$g_{p_{\text{out}}, q_{\text{out}}}(d) \leq g_{p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)}}(d)$$

for all positive integers d. Again by (i), we may apply C4 to the restriction of f to $T_{u,i-1}$, obtaining

$$g_{p_{\text{out}}, q_{\text{out}}}(d) \leq g_{f(y) - d_T(u,y), f(y)}(d).$$

Similarly, if $f(y) > 0$ for some $y \in V(T_{v_i,k_i})$, then

$$\begin{aligned} g_{p_{\text{out}},q_{\text{out}}}(d) &\leq g_{p^{(1)}_{\text{out}},q^{(1)}_{\text{out}}}(d+1) \\ &\leq g_{f(y)-d_T(v_i,y),f(y)}(d+1). \\ &= g_{f(y)-d_T(u,y),f(y)}(d) \end{aligned}$$

for every positive integer d. Therefore f satisfies $C1 - C4$. □

Observe that for any $i \in [k]$, $V(T_{u,i-1}) \cup V(T_{v_i,k_i}) = V(T_{u,i})$. The following is a consequence of Lemma 6.4.

Corollary 6.5. *Suppose $u \in V(T)$ has k children. Let $i \in [k]$ be given and let v_i have k_i children. Then*

$$\begin{aligned} &\alpha_{bn}(T_{u,i},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}})) \\ &= \max\{\alpha_{bn}(T_{u,i-1},(p^{(0)}_{\text{in}},q^{(0)}_{\text{in}}),(p^{(0)}_{\text{out}},q^{(0)}_{\text{out}})) + \alpha_{bn}(T_{v_i,k_i},(p^{(1)}_{\text{in}},q^{(1)}_{\text{in}}),(p^{(1)}_{\text{out}},q^{(1)}_{\text{out}}))\} \end{aligned} \tag{4}$$

over all choices of $p^{(0)}_{in}$, $p^{(0)}_{\text{out}}$, $p^{(1)}_{in}$, $p^{(1)}_{\text{out}}$, $q^{(0)}_{in}$, $q^{(0)}_{\text{out}}$, $q^{(1)}_{in}$, $q^{(1)}_{\text{out}}$ with $-n \leq p^{(0)}_{\text{in}}, p^{(0)}_{\text{out}}, p^{(1)}_{\text{in}}, p^{(1)}_{\text{out}} \leq n$ and $1 \leq q^{(0)}_{\text{in}}, q^{(0)}_{\text{out}}, q^{(1)}_{\text{in}}, q^{(1)}_{\text{out}} \leq n$ satisfying conditions (iii), (iv), and (v) of Lemma 6.4.

Proof. Let $p^{(0)}_{\text{in}}$, $p^{(0)}_{\text{out}}$, $p^{(1)}_{\text{in}}$, $p^{(1)}_{\text{out}}$, $q^{(0)}_{\text{in}}$, $q^{(0)}_{\text{out}}$, $q^{(1)}_{\text{in}}$, $q^{(1)}_{\text{out}}$ be integers satisfying the conditions of the lemma such that the maximum in (4) is attained. Let f be a broadcast such that the restrictions of f to $T_{u,i-1}$ and T_{v_i,k_i} correspond to $\alpha_{bn}(T_{u,i-1},(p^{(0)}_{\text{in}},q^{(0)}_{\text{in}}),(p^{(0)}_{\text{out}},q^{(0)}_{\text{out}}))$ and $\alpha_{bn}(T_{v_i,k_i},(p^{(1)}_{\text{in}},q^{(1)}_{\text{in}}),(p^{(1)}_{\text{out}},q^{(1)}_{\text{out}}))$, respectively. By definition, the restrictions are $((p^{(0)}_{\text{in}},q^{(0)}_{\text{in}}),(p^{(0)}_{\text{out}},q^{(0)}_{\text{out}}))$-compatible and $((p^{(1)}_{\text{in}},q^{(1)}_{\text{in}}),(p^{(1)}_{\text{out}},q^{(1)}_{\text{out}}))$-compatible, hence conditions (i) and (ii) in the statement of Lemma 6.4 are satisfied. Thus, by Lemma 6.4, the restriction of f to $T_{u,i}$ is boundary independent. □

We may now prove the main theorem of this section.

Theorem 6.6. *Given a tree T with $n \geq 3$ vertices, its boundary independent broadcast number $\alpha_{bn}(T)$ can be determined in $O(n^9)$ time.*

Proof. Select a non-leaf vertex r as the root and process the vertices of T in order of non-increasing distance from r. Recall that by (1), there are $O(n)$ choices for (u,i). Suppose u and i are given. For each of the $O(n^4)$ choices for p_{in}, q_{in}, p_{out}, and q_{out}, the value $\alpha_{bn}(T_{u,i},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}}))$ can be determined in $O(n^4)$ time as follows. If $i=0$, apply Lemma 6.3 to find $\alpha_{bn}(T_{u,i},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}}))$ in linear time. Otherwise:

- For each of the $O(n^4)$ possible choices for the four integers $p^{(0)}_{\text{out}}$, $p^{(1)}_{\text{out}}$, $q^{(0)}_{\text{out}}$, and $q^{(1)}_{\text{out}}$, check condition (v) from Lemma 6.4 in constant time.
- Given p_{in}, q_{in}, $p^{(0)}_{\text{out}}$, $p^{(1)}_{\text{out}}$, $q^{(0)}_{\text{out}}$, and $q^{(1)}_{\text{out}}$, apply conditions (iii) and (iv) of Lemma 6.4 to determine $p^{(0)}_{\text{in}}$, $p^{(1)}_{\text{in}}$, $q^{(0)}_{\text{in}}$, and $q^{(1)}_{\text{in}}$ in constant time.

- Finally, add $\alpha_{bn}(T_{u,i-1},(p_{\text{in}}^{(0)},q_{\text{in}}^{(0)}),(p_{\text{out}}^{(0)},q_{\text{out}}^{(0)}))$ and $\alpha_{bn}(T_{v_i,k_i},(p_{\text{in}}^{(1)},q_{\text{in}}^{(1)}),(p_{\text{out}}^{(1)},q_{\text{out}}^{(1)}))$ and apply Corollary 6.5.

Since there are $O(n)$ choices for (u,i), it follows that $\alpha_{bn}(T_{r,c_T(r)},(-n,1),(n,1))$ can be determined in $O(n^9)$ time. By Lemma 6.2, this value equals the boundary independent broadcast number $\alpha_{bn}(T)$. □

We illustrate the algorithm with a simple example. Consider the tree T below, in which one of the two non-leaf vertices is arbitrarily defined to be the root. The remaining vertices are labelled u_1 through u_4 such that $d_T(u_i,r) \le d_T(u_j,r)$ whenever $i > j$.

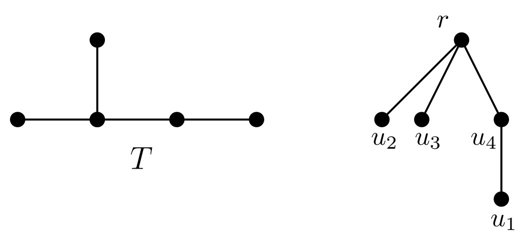

Figure 8: A tree T and its rooted structure.

Note that $e_T(u_i) = 3$ for $i \in \{1,2,3\}$. By Lemma 6.3, for each $1 \le i \le 3$, we have that

$$\alpha_{bn}(T_{u_i,0},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}})) = \begin{cases} 0, \text{ if } q_{\text{out}} > p_{\text{out}}, \\ \min\{3, g_{p_{\text{in}},q_{\text{in}}}(0), p_{\text{out}}\}, \quad \text{if } q_{\text{out}} \le p_{\text{out}} \end{cases}$$

over all choices for p_{in}, q_{in}, p_{out}, and q_{out} such that $-5 \le p_{\text{in}}, p_{\text{out}} \le 5$ and $1 \le q_{\text{in}}, q_{\text{out}} \le 5$. Observe that since $T_{u_i,0}$ consists of only a single vertex, u_i is broadcasting if and only if $q_{\text{out}} = p_{\text{out}}$. Then $\alpha_{bn}(T_{u_i,0},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}})) = 0$ if $p_{\text{in}} \ge 0$, and $\alpha_{bn}(T_{u_i,0},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}})) \in \{1,2,3\}$ if $p_{\text{in}} < 0$.

Otherwise, $q_{\text{out}} > p_{\text{out}}$, in which case $\alpha_{bn}(T_{u_i,0},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}})) = 0$ as expected.

Recall that by Lemma 5.1, given a tree T and an α_{bn}-broadcast f on T, no leaf of T hears f from a non-leaf. In particular, no vertex adjacent to a leaf belongs to V_f^+, hence the restriction of an α_{bn}-broadcast f to $T_{u_4,0} = \{u_4\}$ has weight 0. By Lemma 6.5, we have that

$$\begin{aligned} &\alpha_{bn}(T_{u_4,1},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}})) \\ &= \max\{\alpha_{bn}(T_{u_4,0},(p_{\text{in}}^{(0)},q_{\text{in}}^{(0)}),(p_{\text{out}}^{(0)},q_{\text{out}}^{(0)})) + \alpha_{bn}(T_{u_1,0},(p_{\text{in}}^{(1)},q_{\text{in}}^{(1)}),(p_{\text{out}}^{(1)},q_{\text{out}}^{(1)}))\} \\ &= \max\{\alpha_{bn}(T_{u_1,0},(p_{\text{in}}^{(1)},q_{\text{in}}^{(1)}),(p_{\text{out}}^{(1)},q_{\text{out}}^{(1)}))\} \end{aligned}$$

over all choices of $p_{\text{in}}^{(0)}$, $p_{\text{out}}^{(0)}$, $p_{\text{in}}^{(1)}$, $p_{\text{out}}^{(1)}$, $q_{\text{in}}^{(0)}$, $q_{\text{out}}^{(0)}$, $q_{\text{in}}^{(1)}$, $q_{\text{out}}^{(1)}$ with $-5 \le p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)} \le 5$ and $1 \le q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}, q_{\text{out}}^{(1)} \le 5$ satisfying conditions (iii), (iv), and (v) of Lemma 6.4.

We may calculate $\alpha_{bn}(T_{r,1},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}}))$ as $\max\{\alpha_{bn}(T_{u_2,0},(p_{\text{in}}^{(1)},q_{\text{in}}^{(1)}),(p_{\text{out}}^{(1)},q_{\text{out}}^{(1)}))\}$ similarly. Thus, to determine $\alpha_{bn}(T_{r,2},(p_{\text{in}},q_{\text{in}}),(p_{\text{out}},q_{\text{out}})$, we need only consider the subtrees $T_{u_2,0}$ and $T_{u_3,0}$.

Finally, by Lemmas 6.2 and 6.5, we may calculate $\alpha_{bn}(T)$ as

$$\begin{aligned}&\alpha_{bn}(T_{r,3},(-n,1),(n,1))\\&=\max\{\alpha_{bn}(T_{r,2},(p^{(0)}_{\text{in}},q^{(0)}_{\text{in}}),(p^{(0)}_{\text{out}},q^{(0)}_{\text{out}}))+\alpha_{bn}(T_{u_4,1},(p^{(1)}_{\text{in}},q^{(1)}_{\text{in}}),(p^{(1)}_{\text{out}},q^{(1)}_{\text{out}}))\}\end{aligned} \tag{5}$$

again over all $p^{(0)}_{\text{in}}$, $p^{(0)}_{\text{out}}$, $p^{(1)}_{\text{in}}$, $p^{(1)}_{\text{out}}$, $q^{(0)}_{\text{in}}$, $q^{(0)}_{\text{out}}$, $q^{(1)}_{\text{in}}$, $q^{(1)}_{\text{out}}$ with $-5 \leq p^{(0)}_{\text{in}}, p^{(0)}_{\text{out}}, p^{(1)}_{\text{in}}, p^{(1)}_{\text{out}} \leq 5$ and $1 \leq q^{(0)}_{\text{in}}, q^{(0)}_{\text{out}}, q^{(1)}_{\text{in}}, q^{(1)}_{\text{out}} \leq 5$ satisfying Lemma 6.4.

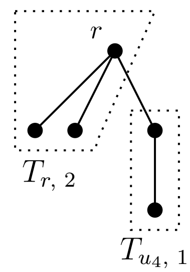

Figure 9: The partition of T considered in (5).

Repetitive calculations show that this maximum is achieved when $\alpha_{bn}(T_{r,2},(p^{(0)}_{\text{in}},q^{(0)}_{\text{in}}),(p^{(0)}_{\text{out}},q^{(0)}_{\text{out}})) = 2$ and $\alpha_{bn}(T_{u_4,1},(p^{(1)}_{\text{in}},q^{(1)}_{\text{in}}),(p^{(1)}_{\text{out}},q^{(1)}_{\text{out}})) = 2$. Since $f(r) = f(u_4) = 0$ under an α_{bn}-broadcast f, we find that maximum boundary independence is achieved when $f(u_1) = 2$ and $f(u_2) = f(u_3) = 1$. Hence $\alpha_{bn}(T) = 4$.

7. Open problems

In [16], Neilson determined that $\alpha_{bn}(T) \leq n-|B_T|+|R_T|$ for all trees with at least one branch vertex, which we improved to $\alpha_{bn}(C) \leq n-|B_C|+\alpha(C[R_C])$ for caterpillars. It may be possible to extend this result to all trees.

Conjecture 7.1. *For any tree T of order n with at least one branch vertex, $\alpha_{bn}(T) \leq n-|B_T|+\alpha(T[R_T])$.*

In particular, we found that equality holds for caterpillars C with no trunks, as well as caterpillars with no branches of degree 3 and no two adjacent trunks, when $|V(C)| \geq 3$. It would be of interest to further classify graphs for which equality holds.

Problem 7.2. *Characterize trees T such that $\alpha_{bn}(T) = n-|B_T|+\alpha(T[R_T])$.*

Problem 7.3. *Find a closed formula to determine $\alpha_{bn}(C)$ exactly for caterpillars or other classes of trees.*

In Section 6, we showed that the maximum bn-independence problem is solvable in polynomial time on all trees, which we illustrated with a simple example. For further research, more efficient algorithms may be possible when considering the additional constraints of boundary independence compared to hearing independence.

Problem 7.4. *Improve the running time in Theorem 6.6, or show it is best possible.*

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