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Characterizations on the hop Roman domination in trees

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Abstract

A subset S of vertices of G is a hop dominating set if every vertex outside S is at distance two from a vertex of S. The hop domination number of G, denoted by $\gamma_h(G)$, is the minimum cardinality of a hop dominating set of G. A hop Roman dominating function (HRDF) of G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u, v) = 2$. The weight of a HRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of HRDF on G is called the hop Roman domination number of G, and is denoted by $\gamma_{hR}(G)$. In this paper we characterize all graphs G with $\gamma_{hR}(G) = \gamma_h(G)$, all trees T for which $\gamma_{hR}(T) = \gamma_h(T) + 1$ as well as all trees T for which $\gamma_{hR}(T) = \gamma_h(T) + 2$.

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1. Introduction

For notation and graph theory terminology not given here, we refer to [9]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n(G) = |V(G)|$. The *open neighborhood* of a vertex v is $N_G(v) = \{u \in V(G) \,|\, uv \in E(G)\}$. The *degree* of v, denoted by $\deg(v)$, is $|N_G(v)|$. A maximal connected subgraph of G is called a *component* of G. A *rooted tree* distinguishes one vertex r called the root. For each vertex $v \neq r$ of T, the parent of v is the neighbor of v on the unique (r, v)-path, while a child of v is any other neighbor of v. A

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descendant of v is a vertex u such that the unique (r,u)-path contains v. Thus, every child of v is a descendant of v. A vertex of degree one in a tree is referred as a *leaf* and its unique neighbor as a *support vertex*. The set of all leaves of a tree T is denoted by $L(T)$ and the set of all support vertices of T is denoted by $S(T)$. A vertex $u \in N_G(v)$ is a *non-leaf neighbor* of v, if $\deg(u) \geq 2$. A *star* is a tree of order $n \geq 2$ such that either $n = 2$ or $n \geq 3$ and there is precisely one vertex that is not a leaf. If T is a star of order at least three, then we refer the support vertex of T as the center of T, and if T has order two, then we choose one of the leaves as the center of the star. A *double star* is a tree with precisely two vertices that are not leaves (we refer these vertices as the central vertices). The *open neighborhood* of a subset $S \subseteq V$, is $N_G(S) = \bigcup_{v \in S} N_G(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. The *distance* between two vertices u and v in G, denoted by $d(u,v)$, is the minimum length of a (u,v)-path in G. For an integer $k \geq 1$, the set of all vertices at distance k from v is denoted by $N_k(v)$. Also we denote $N_k[v] = N_k(v) \cup \{v\}$. A subset S of vertices of a graph G is an *independent set* of G if no two vertices of S are adjacent in G.

Natarajan and Ayyaswamy [13] introduced the concept of hop domination in graphs. A subset S of vertices of a graph G is a *hop dominating set* (HDS) if every vertex outside S is at distance two from a vertex of S. The *hop domination number*, $\gamma_h(G)$, of G is the minimum cardinality of a hop dominating set of G. A HDS of G of minimum cardinality is referred as a $\gamma_h(G)$-set. The concept of hop domination was further studied, for example, in [2, 8, 10, 11, 13, 14]. We define the *hop-degree* of a vertex v in a graph G, denoted $\deg_h(v)$, to be the number of vertices at distance 2 from v in G. The minimum hop-degree among the vertices of G is denoted by $\delta_h(G)$.

A function $f : V \longrightarrow \{0,1,2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a *Roman dominating function* or just an RDF. The *weight* of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. For an RDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an RDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. The mathematical concept of Roman domination developed by Cockayne et al. [7], and this concept is now well-studied in graph theory, see for example, [3, 4, 5, 6, 12]. A *hop Roman dominating function* (HRDF) is a function $f : V \longrightarrow \{0,1,2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u,v) = 2$. The weight of a HRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a HRDF on G is called the *Hop Roman domination number* of G and is denoted $\gamma_{hR}(G)$. A HRDF with minimum weight is referred as a $\gamma_{hR}(G)$-function. For a HRDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus a HRDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. The concept of hop Roman dominating functions was studied in [1, 15, 16, 17].

It can be seen that the hop domination number of a graph G is bounded above by the hop Roman domination number of G, that is, $\gamma_h(G) \leq \gamma_{hR}(G)$. In this paper we study graphs G with small $\gamma_{hR}(G) - \gamma_h(G)$. We first characterize all graphs G with $\gamma_{hR}(G) = \gamma_h(G)$ and then characterize all trees T for which $\gamma_{hR}(T) = \gamma_h(T) + 1$ as well as all trees T for which $\gamma_{hR}(T) = \gamma_h(T) + 2$. For a subset S of vertices of a graph G, and a vertex $v \in S$, we define the *S-external hop neighborhood* of v to be the set $\text{epn}_2(v,S) = \{w \in V(G) - S \mid N_2(w;G) \cap S = \{v\}\}$. If S is a subset of vertices

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in a graph G, then we denote by $G[S]$ the subgraph of G induced by S.

The following propositions are useful in this paper.

Proposition 1.1 ([13]). *For a path P_n on n vertices,*

$$\gamma_h(P_n) = \begin{cases} 2r, & if\ n = 6r, \\ 2r+1, & if\ n = 6r+1, \\ 2r+2, & if\ n = 6r+s;\ 2 \leq s \leq 5. \end{cases}$$

Proposition 1.2 ([10]). *If G is a graph of order n with $\delta_h(G) \geq 1$, then $\gamma_h(G) \leq \frac{n}{2}$.*

2. Preliminary results

We begin with the following proposition.

Proposition 2.1. *For any connected graph G, $\gamma_{hR} \geq \gamma_h(G)$ with equality if and only if G is a complete graph.*

Proof. Let $g = (V_0, V_1, V_2)$ be a $\gamma_{hR}(G)$-function. Clearly, $V_1 \cup V_2$ is a HDS of G. Hence, $\gamma_h(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{hR}(G)$. If $\gamma_h(G) = \gamma_{hR}(G)$, then throughout the above chain we obtain that $|V_2| = 0$ and so $|V_0| = 0$. Therefore, $\gamma_h(G) = \gamma_{hR}(G) = |V_1| = |V(G)| = n$. Now it is clear that G is a complete graph. The converse is obvious. □

Proposition 2.2. *If T is a tree of order $n \geq 3$, then $\gamma_{hR}(T) \geq \gamma_h(T) + 1$, with equality if and only if T is a star of order at least three.*

Proof. We only need to prove the equality according to Proposition 2.1. Let T be a star with center v and u be a leaf of T. It can be seen that $D = \{u, v\}$ is a HDS of G and $\gamma_h(T) = 2$. Let f be a function defined by $f(u) = 2$, $f(v) = 1$ and $f(z) = 0$ for any leaf $z \neq u$ of T. Then f is a HRDF with weight $f(V) = 3 = \gamma_h(T) + 1$. Thus, $\gamma_{hR}(T) \leq \gamma_h(T) + 1$, and thus $\gamma_{hR}(T) = \gamma_h(T) + 1$.

Now assume that $\gamma_{hR}(T) = \gamma_h(T) + 1$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{hR}(T)$-function such that $|V_2|$ is as maximum as possible. Clearly, $|V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{hR}(T) = \gamma_h(T) + 1$. If $|V_2| \geq 2$, then $\gamma_h(T) \leq |V_1| + |V_2| \leq \gamma_h(T) - 1$, a contradiction. Thus $|V_2| \leq 1$. If $|V_2| = 0$, then $V_1 = V(T)$ and by choice of f we obtain that $diam(T) = 1$, a contradiction. Thus, $|V_2| = 1$. Let $V_2 = \{u\}$. Clearly $N(u) \subseteq V_1$. By the choice of f, we obtain that $\deg(u) = 1$. Let $N(u) = \{z\}$. Then every vertex of $N(z)$ is a leaf by the choice of f. Consequently, T is a star. □

3. Trees T with $\gamma_{hR}(T) = \gamma_h(T) + 2$

To characterize trees T with $\gamma_{hR}(T) = \gamma_h(T) + 2$, we need to introduce some families of trees, namely, $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_5$.

We first introduce some families of trees.

- **Family $\mathcal{T}_1$.** Let $\mathcal{T}_1$ be the class of all rooted trees such that the root has degree at least two, any leaf is within distance two from the root, and at least one leaf is at distance two from the root.

  Two trees in the Family $\mathcal{T}_1$ are depicted in Figure 1, where the top vertices are roots.

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- **Family** $\mathcal{T}_2$. Let $\mathcal{T}_2^0$ be the class of all rooted trees such that any leaf is within distance three from the root, at least one child of the root is not leaf, for any child x of the root at most one child of x is not leaf and such child has degree two. Let $\mathcal{T}_2$ be the class of all trees T that T can be obtained from two trees $T_1, T_2 \in \mathcal{T}_2^0$ by joining the root of T_1 to the root of T_2.

  Two trees in the Family $\mathcal{T}_2$ are depicted in Figure 1, where the two top vertices in each tree are roots of trees in $\mathcal{T}_2^0$ that are joined together accoording to the construction of the Family $\mathcal{T}_2$.

- **Family** $\mathcal{T}_3$. Let $\mathcal{T}_3^1$ be the class of all rooted trees such that any leaf is at distance one, two or four from the root; there is at least one leaf at distance four from the root; if x is a child of the root such that x is not leaf, then x has precisely one child of degree more than one and its degree is two; any vertex at distance three from the root has degree two. Let $\mathcal{T}_3^2$ be the class of all rooted trees such that any leaf is within distance two from the root, and at least one leaf is at distance two from the root. Let $\mathcal{T}_3$ be the class of all trees T that T can be obtained from trees $T_1 \in \mathcal{T}_3^1$ and $T_2 \in \mathcal{T}_3^2$ by joining the root of T_1 to the root of T_2.

- **Family** $\mathcal{T}_4$. Let $\mathcal{T}_4^1$ be the class of all rooted trees such that the root has degree at least two; any leaf is within distance four from the root; there is at least one leaf at distance four from the root; there is at least one child x of the root such that x is not leaf and at most one child of x is not leaf and such child is a support vertex of degree two; if x is a leaf at distance four from the root and $xx_1x_2x_3w$ is the path from x to the root w, then $\deg(x_1) = \deg(x_2) = 2$; if x is a leaf at distance three from the root and xx_1x_2w is the path from x to the root w, then $\deg(x_1) = 2$; for any child x of the root at most one child of x has degree more than one and its degree is two. Let $\mathcal{T}_4^2$ be the class of all rooted trees such that any leaf is within distance four from the root; there is at least one child x of the root such that x is not leaf and at most one child of x is not leaf and such child is a support vertex of degree two; if x is a leaf at distance four from the root and $xx_1x_2x_3w$ is the path from x to the root w, then $\deg(x_1) = \deg(x_2) = 2$; if x is a leaf at distance three from the root and xx_1x_2w is the path from x to the root w, then $\deg(x_1) = 2$; for any child x of the root at most one child of x has degree more than one and its degree is two. Let $\mathcal{T}_4$ be the class of all trees T that T can be obtained from trees $T_1 \in \mathcal{T}_4^1$ and $T_2 \in \mathcal{T}_2^2$ by joining the root of T_1 to the root of T_2.

- **Family** $\mathcal{T}_5$. Let $\mathcal{T}_5$ be the class of all trees T that can be obtained from a double star with the central vertices v and w by joining a leaf of v to the central vertex of a star K_{1,t_1}, for some $t_1 \geq 1$, and joining a leaf of w to the central vertex of a star K_{1,t_2}, for some $t_2 \geq 1$.

Two examples of trees $\mathcal{T}_i$ for each $i = 1, 2, ..., 5$ in are demonstrated in Figure 1, where the trees in the first row are in $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, the trees in the second row are in $\mathcal{T}_3$, the trees in the third row are in $\mathcal{T}_4$ and the trees in the fourth row are in $\mathcal{T}_4$.

Finally, let $\mathcal{R} = \bigcup_{i=1}^{5} \mathcal{T}_i$. As the main result of this section, we will prove the following.

Theorem 3.1. *If T is a tree of order $n \geq 4$, then $\gamma_{hR}(T) = \gamma_h(T) + 2$ if and only if $T \in \mathcal{R}$.*

Due to the length of the proof, we present the proof in the next section.

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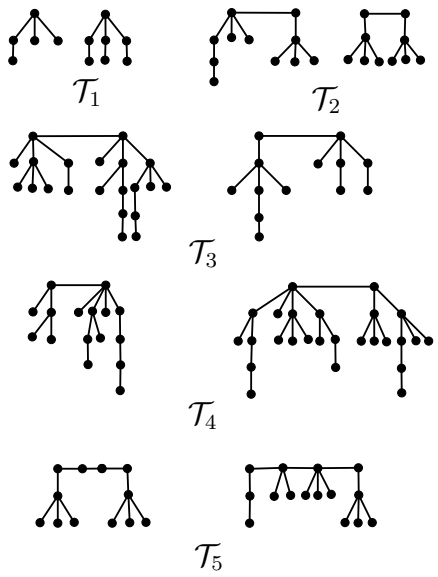


Figure 1. Some examples of trees in $\mathcal{T}_i$ for each $i = 1, 2, ..., 5$.

4. Proof of Theorem 3.1

We first prove the sufficiency.

Lemma 4.1. *If $T \in \mathcal{R}$, then $\gamma_{hR}(T) = \gamma_h(T) + 2$.*

Proof. Let $T \in \mathcal{R}$. Assume that $T \in \mathcal{T}_1$. Let x be the root of T. Let $y \in N(x)$. Clearly $\{x, y\}$ is a HDS for T and so $\gamma_h(T) \leq 2$. Since clearly $\gamma_h(T) \geq 2$, we obtain that $\gamma_h(T) = 2$. On the other hand the function f defined by $f(x) = f(y) = 2$, and $f(u) = 0$ if $u \in V(T) - \{x, y\}$ is a HRDF and so $\gamma_{hR}(T) \leq 4 = 2 + \gamma_h(T)$. Now Propositions 2.1 and 2.2 imply that $\gamma_{hR}(T) = 4$. Consequently, $\gamma_{hR}(T) = \gamma_h(T) + 2$.

Assume next that $T \in \mathcal{T}_2$. Thus T is obtained from two trees $T_1, T_2 \in \mathcal{T}_2^0$ by joining the root of T_1 to the root of T_2. Let v be the root of T_1, w be the root of T_2, and T be obtained by joining v to w. Assume that $(N_3(v) - N_2(w)) \cup (N_3(w) - N_2(v)) \neq \emptyset$. Let $(N_3(v) - N_2(w)) \cup (N_3(w) - N_2(v)) = \{x_1, ..., x_t\}$. For $i = 1, 2, ..., t$, let $y_i \in N(v) \cup N(w)$ with $d(y_i, x_i) = 2$. By construction of the family $T \in \mathcal{T}_2$, for each $i = 1, 2, ..., t$, there is precisely one pair x_i, y_i such that $x_i \in (N_3(v) - N_2(w)) \cup (N_3(w) - N_2(v))$, $y_i \in N(v) \cup N(w)$ and $d(y_i, x_i) = 2$. Observe that $\{x_1, ..., x_t\} \cup \{v, w\}$ is a HDS for T, and so $\gamma_h(T) \leq t + 2$. Let D be a $\gamma_h(T)$-set. For $i = 1, 2, .., t$, D contains at least one vertex of $\{x_i, y_i\}$. Furthermore, D contains at least one vertex of $N_2[v] - N(w)$ and at least one vertex of $N_2[w] - N(v)$. Thus $|D| \geq 2 + t$. Consequently $\gamma_h(T) = 2 + t$. On the other hand, the function f defined with $f(w) = f(v) = 2$, $f(x_i) = 1$ for $i = 1, 2, ..., t$, and $f(u) = 0$ otherwise, is a HRDF for T implying that $\gamma_{hR}(T) \leq 4 + t = 2 + \gamma_h(T)$. Now Propositions 2.1 and 2.2 imply that $\gamma_{hR}(T) = 2 + \gamma_h(T)$. Next assume that $(N_3(v) - N_2(w)) \cup (N_3(w) - N_2(v)) = \emptyset$. Clearly

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$\{v, w\}$ is a HDS for T and so $\gamma_h(T) \leq 2$. Since clearly $\gamma_h(T) \geq 2$, we obtain that $\gamma_h(T) = 2$. On the other hand the function f defined by $f(v) = f(w) = 2$, and $f(u) = 0$ otherwise, is a HRDF for T and so $\gamma_{hR}(T) \leq 4$. Now Propositions 2.1 and 2.2 imply that $\gamma_{hR}(T) = 4$. Consequently, $\gamma_{hR}(T) = \gamma_h(T) + 2$.

Assume next that $T \in \mathcal{T}_3$. Then T is obtained from two trees $T_1 \in \mathcal{T}_3^1$ and $T_2 \in \mathcal{T}_3^2$ by joining the root of T_1 to the root of T_2. Let w be the root of T_1 and v be the root of T_2. Clearly $N_4(w) \neq \emptyset$. Let $N_4(w) = \{x_1, ..., x_t\}$. For each $i = 1, 2, .., t$, let $u_i \in N(x_i)$, $z_i \in N(u_i) - \{x_i\}$, and $y_i \in N(z_i) - \{u_i\}$. Clearly by construction of the Family $\mathcal{T}_3^1$, for each $i = 1, 2, ..., t$, there is precisely one quadruple (x_i, u_i, z_i, y_i). It is evident that $\{v, w\} \cup \{u_i, x_i : i = 1, 2, ..., t\}$ is a HDS for T, and so $\gamma_h(T) \leq 2 + 2t$. Let D be a $\gamma_h(T)$-set. For each $i = 1, 2, ..., t$, D contains at least one vertex of $\{x_i, z_i\}$ and at least one vertex of $\{u_i, y_i\}$. Since $N_2(v) - N(w) \neq \emptyset$, D contains at least one vertex of $N_2[v] - N(w)$. Furthermore, D contains at least one vertex of $N(v)$. Thus, $|D| \geq 2 + 2t$. We conclude that $\gamma_h(T) = 2 + 2t$. On the other hand, the function f defined by $f(w) = f(v) = 2$, $f(x_i) = f(u_i) = 1$ for $i = 1, 2, ..., t$, and $f(u) = 0$ otherwise, is a HRDF for T implying that $\gamma_{hR}(T) \leq 4 + 2t = 2 + \gamma_h(T)$. Thus $\gamma_{hR}(T) \leq 2 + \gamma_h(T)$. Now Propositions 2.1 and 2.2 imply that $\gamma_{hR}(T) = 2 = \gamma_h(T)$.

Assume next that $T \in \mathcal{T}_4$. Then T is obtained from two trees $T_1 \in \mathcal{T}_4^1$ and $T_2 \in \mathcal{T}_4^2$ by joining the root of T_1 to the root of T_2. Let w be the root of T_1 and v be the root of T_2. Clearly $(N_4(w) - N_3(v)) \cup (N_4(v) - N_3(w)) \neq \emptyset$. Let $(N_4(w) - N_3(v)) \cup (N_4(v) - N_3(w)) = \{x_1, ..., x_t\}$. For each $i = 1, 2, .., t$, let $u_i \in N(x_i)$, $z_i \in N(u_i) - \{x_i\}$, and $y_i \in N(z_i) - \{u_i\}$. Clearly by construction of the Family $\mathcal{T}_4^1$, for each $i = 1, 2, ..., t$, there is precisely one quadruple (x_i, u_i, z_i, y_i). Assume that $((N_3(w) - N_2(v)) \cup (N_3(v) - N_2(w))) \cap L(T) \neq \emptyset$. Let $((N_3(w) - N_2(v)) \cup (N_3(v) - N_2(w))) \cap L(T) = \{x'_1, ..., x'_{t'}\}$. For $i = 1, 2, ..., t'$, let $y'_i \in N(v) \cup N(w)$ with $\deg(y'_i, x'_i) = 2$. By construction of the family $T \in \mathcal{T}_4$, for each $i = 1, 2, ..., t'$, there is precisely one pair x'_i, y'_i such that $x'_i \in ((N_3(w) - N_2(v)) \cup (N_3(v) - N_2(w))) \cap L(T)$, $y'_i \in N(v) \cup N(w)$ and $d(y'_i, x'_i) = 2$. It is evident that $\{v, w\} \cup \{u_i, x_i : i = 1, 2, ..., t\} \cup \{x'_i : i = 1, 2, ..., t'\}$ is a HDS for T, and so $\gamma_h(T) \leq 2 + t' + 2t$. Let D be a $\gamma_h(T)$-set. For each $i = 1, 2, ..., t$, D contains at least one vertex of $\{x_i, z_i\}$ and at least one vertex of $\{u_i, y_i\}$. For $i = 1, 2, ..., t'$, D contains at least one vertex of $\{x'_i, y'_i\}$. Since $N_2(v) - N(w) \neq \emptyset$, D contains at least one vertex of $N_2[v] - N(w)$. Furthermore, since $N_2(w) - N(v) \neq \emptyset$, D contains at least one vertex of $N_2[w] - N(v)$. Thus, $|D| \geq 2 + t' + 2t$. We conclude that $\gamma_h(T) = 2 + t' + 2t$. On the other hand, the function f defined with $f(w) = f(v) = 2$, $f(x_i) = f(u_i) = 1$ for $i = 1, 2, ..., t$, $f(x'_i) = 1$ for $i = 1, 2, ..., t'$, and $f(u) = 0$ otherwise, is a HRDF for T implying that $\gamma_{hR}(T) \leq 4 + t' + 2t = 2 + \gamma_h(T)$. Thus $\gamma_{hR}(T) \leq 2 + \gamma_h(T)$. Now Propositions 2.1 and 2.2 imply that $\gamma_{hR}(T) = 2 + \gamma_h(T)$. Next assume that $((N_3(v) - N_2(w)) \cup (N_3(w) - N_2(v))) \cap L(T) = \emptyset$. It is evident that $\{v, w\} \cup \{u_i, x_i : i = 1, 2, ..., t\}$ is a HDS for T, and so $\gamma_h(T) \leq 2 + 2t$. Let D be a $\gamma_h(T)$-set. For each $i = 1, 2, ..., t$, D contains at least one vertex of $\{x_i, z_i\}$ and at least one vertex of $\{u_i, y_i\}$. Since $N_2(v) - N(w) \neq \emptyset$, D contains at least one vertex of $N_2[v] - N(w)$. Furthermore, Since $N_2(w) - N(v) \neq \emptyset$, D contains at least one vertex of $N_2[w] - N(v)$. Thus, $|D| \geq 2 + 2t$. We conclude that $\gamma_h(T) = 2 + 2t$. On the other hand, the function f defined with $f(w) = f(v) = 2$, $f(x_i) = f(u_i) = 1$ for $i = 1, 2, ..., t$, and $f(u) = 0$ otherwise, is a HRDF for T implying that $\gamma_{hR}(T) \leq 4 + 2t = 2 + \gamma_h(T)$. Thus $\gamma_{hR}(T) \leq 2 + \gamma_h(T)$. Now Propositions 2.1 and 2.2 imply that $\gamma_{hR}(T) = 2 + \gamma_h(T)$

Finally, assume that $T \in \mathcal{T}_5$. Then T is obtained from a double star with the central vertices v

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and w, by joining the leaf u of v to the central vertex x of a star K_{1,t_1}, for some $t_1 \geq 1$, and the leaf z of w to the central vertex y of a star K_{1,t_2}, for some $t_2 \geq 1$. It is evident that $\{x, y, u, z\}$ is a HDS for T and so $\gamma_h(T) \leq 4$. On the other hand, since $diam(T) = 7$, from Proposition 1.1, we have $4 = \gamma_h(P_8) \leq \gamma_h(T)$. Thus $\gamma_h(T) = 4$. On the other hand, the function f defined by $f(u) = f(z) = 2$, $f(x) = f(y) = 1$, and $f(u) = 0$ otherwise, is a HRDF for T implying that $\gamma_{hR}(T) \leq 6 = 2 + \gamma_h(T)$. Thus $\gamma_{hR}(T) \leq 2 + \gamma_h(T)$. Now Propositions 2.1 and 2.2 imply that $\gamma_{hR}(T) = 2 + \gamma_h(T)$. □

We now prove the necessity. Let T be a tree of order $n \geq 4$, and $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_{hR}(T)$-function such that $|V_2^f|$ is as maximum as possible. We proceed with the following Lemma.

Lemma 4.2. $|V_2^f| = 2$.

Proof. Suppose that $|V_2^f| \geq 3$.Then $|V_1^f| + 6 \leq |V_1^f| + 2|V_2^f| = \gamma_{hR}(T) = \gamma_h(T) + 2$, and we obtain that $|V_1^f| \leq \gamma_h(T) - 4$. Since $|V_2^f| \geq 3$, we have $|V_1^f| + |V_2^f| \leq \gamma_h(T) - 1$, and so we obtain $\gamma_h(T) \leq \gamma_h(T) - 1$, a contradiction. We deduce that $|V_2^f| \leq 2$. Assume that $|V_2^f| = 0$. Then $V_1^f = V(T)$ and $\gamma_{hR}(T) = n$. By Proposition 1.2, $n = \gamma_{hR}(T) = \gamma_h(T) + 2 \leq n/2 + 2$, and so $n \leq 4$. Thus $n = 4$. By Proposition 2.2, T is not a star. Thus $T = P_4$. But then we can assign 2 to any leaf of P_4 and 0 to any support vertex of P_4 to obtain a $\gamma_{hR}(T)$-function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Next assume that $|V_2^f| = 1$. Let $V_2^f = \{v\}$. Then $N(v) \subseteq V_1^f$. If $\deg(v) \geq 2$, then we change the label of a vertex of $N(v)$ to 2 and another vertex to 0 to obtain a $\gamma_{hR}(T)$-function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Thus $\deg(v) = 1$. Let $u \in N(v)$. The any vertex in $N(u) - \{v\}$ is labeled by 0 under f. If there is a vertex u' at distance two from u, then $f(u') = 1$, and we can change $f(u')$ to 0 and $f(u)$ to 2 to obtain a $\gamma_{hR}(T)$-function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Consequently, T is a star. This is a contradiction by Proposition 2.2. We conclude that $|V_2^f| = 2$. □

Thus $|V_2^f| = 2$ by Lemma 4.2. Then $|V_1| = \gamma_h(T) - 2$. Let $V_2^f = \{v, w\}$. If $\text{epn}_2(v, V_2^f) = \emptyset$, then we change $f(v)$ to 1 to obtain a HRDF with weight less than $\gamma_{hR}(T)$, a contradiction. Thus $\text{epn}_2(v, V_2^f) \neq \emptyset$ and similarly $\text{epn}_2(w, V_2^f) \neq \emptyset$. We proceed by the following lemmas namely Lemma 4.3 and Lemma 4.4 according to whether $vw \in E(T)$ or $vw \notin E(T)$.

Lemma 4.3. *If $vw \in E(T)$, then $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$.*

Proof. Let $vw \in E(T)$. Assume first that $V_1^f = \emptyset$.

Assume that $\deg(v) = 1$. Then $\deg(w) > 1$, since $n \geq 4$. Note that $\text{epn}_2(w, V_2^f) \neq \emptyset$. Let $z \in \text{epn}_2(w, V_2^f)$ and $x \in N(z) \cap N(w)$. Clearly $z \in V_0^f$. We show that z is a leaf. Suppose that $\deg(z) \geq 2$. Let $z' \in N(z) - \{x\}$. Then $f(z') = 0$ and so $z\prime \in N_2(v) \cup N_2(w)$. This provide a cycle for T, a contradiction. Thus any vertex at distance two from w is a leaf. Consequently, $T \in \mathcal{T}_1$.

Thus we may assume for the next that $\deg(v) \geq 2$ and $\deg(w) \geq 2$. If $\text{epn}_2(v, V_2^f) = N(w)$ and $\text{epn}_2(w, V_2^f) = N(v)$, then T is a double star and so $T \in \mathcal{T}_1$. Thus assume, without loss of generality, that $\text{epn}_2(v, V_2^f) - N(w) \neq \emptyset$. Assume that $\text{epn}_2(w, V_2^f) - N(v) = \emptyset$. Then any vertex

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in $N(w) - \{v\}$ is a leaf. We can easily see that any vertex in $\text{epn}_2(v, V_2^f) - N(w)$ is a leaf. Thus any leaf of T is within distance two from v, and there is at least one leaf at distance two from v. Consequently $T \in \mathcal{T}_1$. Next assume that $\text{epn}_2(w, V_2^f) - N(v) \neq \emptyset$. As before, we observe that any vertex of $\text{epn}_2(w, V_2^f) - N(v)$ is a leaf. Now let T_1 be the component of $T - vw$ containing v, and T_2 be the component of $T - vw$ containing w. It is evident that $T_1, T_2 \in \mathcal{T}_2^0$, v is the root of T_1 and w is the root of T_2. Therefore, T is obtained from $T_1, T_2 \in \mathcal{T}_2^0$ by joining the root v of T_1 to the root w of T_2. Consequently $T \in \mathcal{T}_2$.

Next assume that $V_1^f \neq \emptyset$. If there are three vertices $x, x_1, x_2 \in V_1^f$ such that $x_1, x_2 \in N(x)$, then we change $f(x_1)$ to 0 and $f(x_2)$ to 2 to obtain a $\gamma_{hR}(T)-$function g with $|V_2^g| > 2$, a contradiction. Therefore, each component of $T[V_1^f]$ is K_2 or K_1. Clearly $V_1^f \cap (N(v) \cup N(w)) = V_1^f \cap (N_2(v) \cup N_2(w)) = \emptyset$. Thus any vertex any K_1-component of $T[V_1^f]$ is adjacent to a vertex in $(N_2(v) - N(w)) \cup (N_2(w) - N(v))$, and a vertex of any K_2-component of $T[V_1^f]$ is adjacent to a vertex in $(N_2(v) - N(w)) \cup (N_2(w) - N(v))$. If a vertex of V_1^f is adjacent to at least two vertices of V_0^f, then T has a cycle, since any vertex of V_0^f is at distance two from v or w, a contradiction. Thus any vertex of V_1^f is adjacent to precisely one vertex of V_0^f. Moreover, precisely one vertex of any K_2-component of $T[V_1^f]$ is adjacent to a vertex of V_0^f. If there are two vertices $x, y \in V_1^f$ with a common neighbor in V_0^f, then we change $f(x)$ to 2 and $f(y)$ to 0 to obtain a $\gamma_{hR}(T)-$function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Thus no pair of vertices of V_1^f have a common neighbor in V_0^f.

The following claim is very useful in the rest of the paper.

Claim 1. $|N_2(z) \cap V_1^f| \leq 1$ for any vertex $z \in V_0^f$.

To see the validity of Claim 1, let $z \in V_0^f$. If there are two vertices $x, y \in V_1^f$ such that $x, y \in N_2(z)$, then we change $f(x)$ and $f(y)$ to 0 and $f(z)$ to 2 to obtain a $\gamma_{hR}(T)-$function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Thus for any vertex $z \in V_0^f$, $|N_2(z) \cap V_1^f| \leq 1$.

Next, we show that $N_2(w) - N(v) \neq \emptyset$ and $N_2(v) - N(w) \neq \emptyset$. It is evident that $(N_2(v) - N(w)) \cup (N_2(w) - N(v)) \neq \emptyset$, since $V_1^f \neq \emptyset$ and any vertex of any K_1-component of $T[V_1^f]$ is adjacent to a vertex of $(N_2(v) - N(w)) \cup (N_2(w) - N(v))$, and precisely a vertex of any vertex of any K_2-component of $T[V_1^f]$ is adjacent to precisely one vertex of $(N_2(v) - N(w)) \cup (N_2(w) - N(v))$. Assume, without loss of generality, that $N_2(w) - N(v) \neq \emptyset$. Suppose that $N_2(v) - N(w) = \emptyset$. Let $x \in V_1^f$. Then x is adjacent to a vertex $y \in N_2(w) - N(v)$. Let $z \in N(w) \cap N(y)$. Then we change $f(v)$ to 0, $f(z) = 2$ and $f(x) = 0$ to obtain a HRDF with weight less than $\gamma_{hR}(T)$, a contradiction. Thus $N_2(v) - N(w) \neq \emptyset$. We consider the following cases.

Assume that V_1^f is an independent set. As before we observe that each vertex of V_1^f is adjacent to precisely one vertex of $(N_2(v) - N(w)) \cup (N_2(w) - N(v))$, thus any vertex of V_1^f is at distance three from v or w. From Claim 1 we obtain that any non-leaf vertex of $N(w) - \{v\}$ has at most one non-leaf neighbor other than w, and also by Claim 1, such non-leaf neighbor has degree two. Similarly by Claim 1 any non-leaf vertex of $N(v) - \{w\}$ has at most one non-leaf neighbor other than v and also by Claim 1, such non-leaf neighbor has degree two. Now let T_1 be the component of $T - vw$ containing v, and T_2 be the component of $T - vw$ containing w. Then T_1 is a rooted tree with root v and T_2 is a rooted tree with root w. Note that any leaf of T_1 is within distance three from v, and since $N_2(w) - N(v) \neq \emptyset$, we conclude that $T_1 \in \mathcal{T}_2^0$. Similarly $T_2 \in \mathcal{T}_2^0$. Therefore, T is obtained from $T_1, T_2 \in \mathcal{T}_2^0$ by joining the root v of T_1 to the root w of T_2. Consequently $T \in \mathcal{T}_2$.

Next assume that V_1^f is not an independent set. Then $T[V_1^f]$ has at least one K_2-component.

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As before we observe that precisely one vertex of any K_2-component of $T[V_1^f]$ is adjacent to a vertex of V_0^f. Thus there exists a non-leaf vertex $x \in (N(w) \cup N(v)) - \{v, w\}$ such that x has a non-leaf neighbor z, where z is adjacent to a vertex of a K_2-component of $T[V_1^f]$. By Claim 1, x has precisely a non-leaf neighbor that is adjacent to a vertex of a K_2-component of $T[V_1^f]$ and also z has degree two. Since $x \in (N(w) \cup N(v)) - \{v, w\}$, without loss of generality, we may assume that $x \in N(w) - \{v\}$. We have the following cases.

Case 1. Every non-leaf vertex $x' \in N(w) - \{v\}$ has a non-leaf neighbor other than w, such as z', such that z' is adjacent to a vertex of a K_2-component of $T[V_1^f]$. By Claim 1, $z\prime$ is the unique vertex with this property, and by Claim 1, z' has degree two. Note that $N_2(v) - N(w) \neq \emptyset$. We show that any vertex of $N_2(v) - N(w)$ is a leaf. Suppose that some vertex of $N_2(v) - N(w)$ is not a leaf. Let $x_1, ..., x_t \in N_2(v) - N(w)$ be the vertices that x_i is not a leaf, for $i = 1, 2, ..., t$. For $i = 1, 2, .., t$, let $y_i \in V_1^f \cap N(x_i)$, and let $u_i \in N(v) \cap N(x_i)$. Note that y_i is the vertex of a K_1-component of $T[V_1^f]$ or y_i is a vertex of a K_2-component of $T[V_1^f]$, for $i = 1, 2, ..., t$, and thus each y_i is a leaf or a support vertex. Let $|\{y_1, y_2, ..., y_t\} \cap L(T)| = t_1$ and $|\{y_1, y_2, ..., y_t\} \cap S(T)| = t_2$, where $t_1 + t_2 = t$. Let $u'_1, ..., u'_{t'} \in N(w) - \{v\}$ be the vertices that are not leaves. For $i = 1, 2, .., t'$, let $x'_i \in N(u'_i) - \{w\}$, $y'_i \in V_1^f \cap N(x'_i)$ and $z'_i \in V_1^f \cap N(y'_i)$. Note that z'_i is a leaf, for $i = 1, 2, ..., t'$. Note that $t_1 + 2t_2 + 2t' = |V_1^f|$ and $\{v\} \cup \{u_i | i = 1, 2, ..., t_1\} \cup \{x_i, u_i | i = 1, 2, ..., t_2\} \cup \{x'_i, u'_i | i = 1, 2, ..., t'\}$ is a HDS for T of cardinality $|D| = 1 + t_1 + 2t_2 + 2t' = 1 + |V_1^f|$, and so $\gamma_h(T) \leq 1 + |V_1^f| = \gamma_{hR}(T) - 3 = \gamma_h(T) - 1$, a contradiction. Thus each vertex of $N_2(v) - N(w)$ is a leaf. Let T_1 be the component of $T - vw$ containing v, and T_2 be the component of $T - vw$ containing w. Clearly any leaf of T_2 is at distance one, two or four from w, and T_2 has at least a leaf at distance four from w. Let $x \in N_4(w)$, and $xyzuw$ is the path between x and w. By Claim 1, $\deg(y) = \deg(z) = 2$. By Claim 1 and the assumption of this case (Case 1), any vertex of $N_3(w) - N_2(v)$ has degree two. Consequently, $T_2 \in \mathcal{T}_3^1$ and w is the root of T_2. Also clearly any leaf of T_1, is within distance two from v, and as before we observe that $N_2(v) - N(w) \neq \emptyset$, thus T_1 has at least a leaf at distance two from v. Consequently, $T_2 \in \mathcal{T}_3^2$ and v is the root of T_1. Therefore, T is obtained from $T_1 \in \mathcal{T}_3^1$ and $T_2 \in \mathcal{T}_3^2$ by joining the root of T_1 to the root of T_2. Hence $T \in \mathcal{T}_3$.

Case 2. There exists a non-leaf vertex $x' \in N(w) - \{v\}$ such that no non-leaf neighbor of x' is adjacent to a vertex of a K_2-component of $T[V_1^f]$. Then for any vertex $z' \in N(x') - \{w\}$, either z' is a leaf or a support vertex of degree two by Claim 1 and there is at most one vertex $z' \in N(x') - \{w\}$ such that z' is a support vertex of degree two.

We show that there exists a non-leaf vertex $a \in N(v) - \{w\}$, such that either any vertex of $N(a) - \{v\}$ is a leaf or a is adjacent to precisely one support vertex of degree two. Suppose that for every non-leaf vertex $a \in N(v) - \{w\}$, there exists a non-leaf vertex $b \in N(a) - \{v\}$ that b is adjacent to a vertex of a K_2-component of $T[V_1^f]$. By Claim 1, b is the unique vertex with this property and by Claim 1, b has degree two. Let $x_1, x_2, ..., x_t \in N(w) - \{v\}$ be the vertices that x_i is not a leaf and has a non-leaf neighbor adjacent to a vertex of a K_2-component of $T[V_1^f]$, for $i = 1, 2, ..., t$. By Claim 1, x_i has precisely one non-leaf neighbor adjacent to a vertex of a K_2-component of $T[V_1^f]$, for $i = 1, 2, ..., t$. For $i = 1, 2, ..., t$, let y_i be the non-leaf neighbor of x_i other than w, $z_i \in V_1^f \cap N(y_i)$, and $u_i \in V_1^f \cap N(z_i)$. Let $x'_1, x'_2, ..., x'_{t_1} \in N(w) - \{v\}$ be the vertices that x'_i is not a leaf and a non-leaf vertex $y'_i \in N(x'_i) - \{w\}$ is adjacent to a K_1-component of $T[V_1^f]$. Let $z'_i \in V_1^f$ be the K_1-component of $T[V_1^f]$ that is adjacent to y'_i, for

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$i = 1, 2, ..., t_1$. By Claim 1, y_i' is unique and y_i' has degree two, for $i = 1, 2, ..., t_1$. Let $a_1, ..., a_{t'}$ be the non-leaf neighbors of v other than w such that a_i has a non-leaf neighbor other than v, for $i = 1, 2, ..., t'$. Let $b_i \in N(a_i) - \{v\}$ be a non-leaf neighbor of a_i that is adjacent to a vertex of a K_2-component of $T[V_1^f]$, for $i = 1, 2, ..., t'$. By Claim 1, b_i is unique for each $i = 1, 2, ..., t'$. Note that $2t+2t'+t_1 = |V_1^f|$. Then $\{w\} \cup \{y_i, x_i | i = 1, 2, ..., t\} \cup \{x_i' \mid i = 1, 2, ..., t_2\} \cup \{a_i, b_i | i = 1, 2, ..., t'\}$ is a HDS for T, and so $\gamma_h(T) \leq 1 + t_1 + 2t + 2t' = 1 + |V_1^f| = \gamma_{hR}(T) - 3 = \gamma_h(T) - 1$, a contradiction. Thus v has a non-leaf neighbor v' such that either any vertex in $N(v') - \{w\}$ is a leaf or v' has a non-leaf neighbor that is adjacent to the vertex of a K_1-component of $T[V_1^f]$. Let T_1 be the component of $T - vw$ containing v, and T_2 be the component of $T - vw$ containing w. Clearly any leaf of T_2 is within distance four from w, and $(N_4(w) - N_3(v)) \neq \emptyset$, and thus we can conclude that $T_2 \in \mathcal{T}_4^1$. Note that w is the root of T_2. Also clearly any leaf of T_2 is within distance four from v, and thus we can conclude that $T_1 \in \mathcal{T}_4^2$. Note that v is the root of T_1. Therefore, T is obtained from $T_1 \in \mathcal{T}_4^1$ and $T_2 \in \mathcal{T}_4^2$ by joining v to w. Consequently $T \in \mathcal{T}_4$. □

To complete the proof of Theorem 3.1, it remains to discuss on the case where $vw \notin E(T)$. We do this in the next lemma.

Lemma 4.4. *If $vw \notin E(T)$, then $T \in \mathcal{T}_1 \cup \mathcal{T}_5$.*

Proof. Assume that $vw \notin E(T)$. Clearly $d(v, w) \geq 2$. We first show that $d(v, w) = 3$. Suppose that $d(v, w) = 2$ and $P : vuw$ be a path of length 2 between v and w. Observe that $N(v) \subseteq V_1^f$. If $\deg(v) \geq 2$, then we change the label of a vertex of $N(v)$ to 2 and another vertex to 0 to obtain a $\gamma_{hR}(T)$-function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Thus $\deg(v) = 1$. Similarly we have $\deg(w) = 1$. Thus any vertex of $epn_2(v, V_2^f)$ is adjacent to u, thus $epn_2(v, V_2^f) = N(u) - \{v, w\}$ and similarly $epn_2(w, V_2^f) = N(u) - \{v, w\}$. If there is a vertex u' at distance two from u, then $f(u') = 1$, and we can change $f(u')$ to 0 and $f(u)$ to 2 to obtain a $\gamma_{hR}(T)$-function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Thus each vertex of $N(u)$ is a leaf. Consequently, T is a star. This is a contradiction by Proposition 2.2. Thus $d(v, w) \geq 3$. Now suppose that $d(v, w) \geq 6$ and $P : v = v_1v_2v_3...w$ be a path of length at least 6 between v and w. Then $f(v_2) = 1$, $f(v_3) = 0$, $f(v_4) = 1$, and we can change $f(v_4)$ to 0 and $f(v_2)$ to 2 to obtain a $\gamma_{hR}(T)$-function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Hence we have $d(v, w) \leq 5$. Suppose that $d(v, w) = 4$. Let $P : v = v_1v_2v_3v_4v_5 = w$ be a path of length 4 between v and w. Then $f(v_2) = f(v_4) = 1$, $f(v_3) = 0$, and we can change $f(v_4)$ to 0 and $f(v_2)$ to 2 to obtain a $\gamma_{hR}(T)$-function g with $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Thus $d(v, w) \neq 4$. Finally, suppose that $d(v, w) = 5$ and $P : v = v_1v_2v_3v_4v_5v_6 = w$ be a path of length 5 between v and w. Then $f(v_2) = f(v_5) = 1, f(v_3) = f(v_4) = 0$. If $\deg(v) \geq 2$ and $x \in N(v) - \{v_2\}$, then $f(x) = 1$, and we can change $f(x)$ to 0 and $f(v_2)$ to 2 to obtain a $\gamma_{hR}(T)$-function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Thus, $\deg(v) = 1$. Similarly $\deg(w) = 1$. Then we change $f(v)$, $f(w)$, $f(v_2)$ and $f(v_5)$ to 0 and $f(v_3)$ and $f(v_4)$ to 2 to obtain a HRDF with weight less than $\gamma_{hR}(T)$, a contradiction by the choice of f. Thus, $d(v, w) \neq 5$. Consequently, $d(v, w) = 3$.

Let $P : vuzw$ the path between v and w. Since $v \in V_2^f$, we have $N(v) - \{u\} \subseteq V_1^f$. If $\deg(v) \geq 3$, then we change the label of a vertex of $N(v) - \{u\}$ to 2 and another vertex to 0

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to obtain a $\gamma_{hR}(T)$-function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Thus $\deg(v) \leq 2$. Similarly we have $\deg(w) \leq 2$.

Assume that $\deg(w) = 2$. Let $N(w) - \{z\} = \{y\}$. We show that $\deg(y) > 1$. Suppose that y is a leaf. If $\deg(v) = 1$, then $D = \{u, z\}$ is a HDS in T and so $|D| < |V_1^f| + 2 = \gamma_h(T)$, a contradiction. Thus assume that $\deg(v) = 2$. Then $D = \{v, u, z\}$ is a HDS in T and so $|D| < |V_1^f| + 2 = \gamma_h(T)$, a contradiction. Thus y is not a leaf. We next show that each vertex of $N(y) - \{w\}$ is a leaf. Suppose that there is a vertex $y' \in N(y) - \{w\}$ such that y' is not a leaf, and let z be adjacent to y'. Then $f(z) = 1$. Since $f(y) = 1$, we can change the label of z to 0 and y to 2 to obtain a $\gamma_{hR}(T)$-function g such that $|V_2^g| > |V_2^f|$, a contradiction by the choice of f. Thus each vertex of $N(y) - \{w\}$ is a leaf. Suppose that $\deg(v) = 1$. Then each vertex of $N(u) - \{z, v\}$ is a leaf, since otherwise there exist a HDS D with $|D| < |V_1^f| + 2 = \gamma_h(T)$, a contradiction. Since $f(z) = 0$, by Claim 1, each vertex of $N(z) - \{u, w\}$ is a leaf. Now $\{z, w\}$ is a HDS of cardinality less than $\gamma_h(T)$, a contradiction. Thus $\deg(v) = 2$. Let $N(v) - \{u\} = \{y'\}$. As before, we obtain that each vertex of $N(y') - \{v\}$ is a leaf. From Claim 1 we can easily see that each vertex of $(N(u) - \{z, v\}) \cup (N(z) - \{u, w\})$ is a leaf. Then $T \in \mathcal{T}_5$.

Next assume that $\deg(w) = 1$. If $\deg(v) = 2$, then as before we have a contradiction. Thus $\deg(v) = 1$. If $N_2(u) - N(z) \neq \emptyset$, and $u' \in N_2(u) - N(z)$, then $f(u') = 1$, and we can change $f(v)$ to 0, $f(u)$ to 2, $f(z)$ to 2, $f(w)$ to 0 and $f(u')$ to 0, to obtain a HRDF with weight less than $\gamma_{hR}(T)$, a contradiction. Thus $N_2(u) - N(z) = \emptyset$. Similarly, $N_2(z) - N(u) = \emptyset$. Thus T is a double star and so $T \in \mathcal{T}_1$. □

We suggest the problem of characterization of graphs G other than trees with $\gamma_{hR}(G) = \gamma_h(G) + 1$ or $\gamma_{hR}(G) = \gamma_h(G) + 2$ for readers.

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