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Upper bounds of the strong 3-rainbow index of corona product graphs

A.N.M. Salman[a,c], Zata Yumni Awanis[b,c], Suhadi Wido Saputro[a]

[a]*Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesa 10, Bandung 40132, Indonesia*
[b]*Research Center for Computing, National Research and Innovation Agency (BRIN), Jalan Raya Jakarta-Bogor KM 46, Cibinong 16911, Indonesia*
[c]*Center for Research Collaboration on Graph Theory and Combinatorics, Indonesia*

msalman@itb.ac.id, zata.yumni.awanis@brin.go.id, suhadi@itb.ac.id

Abstract

A tree in an edge-colored connected graph G of order n, where adjacent edges may be colored the same, is a rainbow tree if no two of its edges have the same color. Let k be an integer with $2 \leq k \leq n$. An edge-coloring of G is called a strong k-rainbow coloring of G if for every $S \subseteq V(G)$ with $|S| = k$, there exists a rainbow tree of minimum size connecting S. The minimum number of colors that are needed such that G admits a strong k-rainbow coloring is called the strong k-rainbow index of G, denoted by $srx_k(G)$. In this paper, we study the srx_3 of corona product between two connected graphs G and H, denoted by $G \odot H$. We first provide an algorithm to define a strong 3-rainbow coloring of $G \odot H$. Hence, we obtain a sharp upper bound for $srx_3(G \odot H)$. We also characterize graphs H with $srx_3(T_n \odot H)$ equals the upper bound.

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1. Introduction

Consider a communication network consisting of several people, where every pair of people is expected to exchange information securely. To achieve this, a secure communication line connect-

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ing them can be established by assigning passwords to the line, possibly involving other people as intermediaries, so that no password is repeated. This is easy to achieve if the number of passwords is unrestricted. However, in practice, economic considerations must also be taken into account. Therefore, the number of passwords should be kept as small as possible. This situation can be modeled by a graph, where the minimum number of passwords required is represented by the *rainbow connection number* rc of a graph, first introduced by Chartrand et al. in 2008 [10].

Throughout this paper, all graphs are simple, finite, and connected. We follow the terminology and notation of Diestel [13]. Let G be an edge-colored graph of order $n \geq 3$, where adjacent edges may receive the same color. The order and size of G are denoted by $|G|$ and $\|G\|$, respectively. The maximum degree of G and the set of neighbors of a vertex v in G are denoted by $\Delta(G)$ and $N_G(v)$, respectively.

In [8], Caro et al. conjectured that computing $rc(G)$ is NP-hard. This conjecture was later proved by Chakraborty et al. [9]. They also showed that deciding whether a given edge-colored graph is rainbow-connected (i.e., whether for every pair of vertices u and v there exists a u–v path whose edges have distinct colors [10]) is NP-complete. Various bounds, algorithms, and computational complexity results for $rc(G)$ have also been studied by other authors; see [18, 19, 20]. For additional results on the rainbow connection number, the readers are referred to [10, 14, 15, 17, 24, 25, 27, 28].

Chartrand et al. [11] introduced a generalization of the rainbow connection number. A tree in G is called a *rainbow tree* if no two of its edges share the same color. Throughout this paper, let k be an integer with $2 \leq k \leq n$ and let $S \subseteq V(G)$ with $|S| = k$. A *rainbow S-tree* is a rainbow tree connecting all vertices of S. When $S = \{u, v\}$, the rainbow S-tree is simply a *rainbow u–v path* [10]. The minimum number of colors required in an edge-coloring of G such that there exists a rainbow S-tree for every k-subset S of $V(G)$ is called the *k-rainbow index* of G, denoted by $rx_k(G)$. In particular, if $S = \{u, v\}$ then $rx_2(G) = rc(G)$. Hence, for every nontrivial connected graph G of order n, it holds that $rc(G) = rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$.

The minimum size of a tree connecting S is called the *Steiner distance* of S, denoted by $d(S)$. The *k-Steiner diameter* of G, denoted by $sdiam_k(G)$, is defined as the maximum Steiner distance among all k-subsets S of $V(G)$. If $S = \{u, v\}$, then $d(S) = d(u, v)$, that is, the length of a shortest u–v path in G, and $sdiam_2(G) = diam(G)$, which is the largest distance between two vertices of G. Hence, $diam(G) = sdiam_2(G) \leq sdiam_3(G) \leq \cdots \leq sdiam_n(G)$. In [11], Chartrand et al. showed that for every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$. They also proved that the rx_k of trees attains the upper bound $n - 1$, while for unicyclic graphs, the value of rx_k is either $n - 1$ or $n - 2$. Consequently, Li et al. [21] characterized graphs whose rx_3 equals $n - 1$ or $n - 2$. Meanwhile, Liu and Hu [22] studied the parameter rx_3 for graphs obtained through several graph operations. Graph operations constitute an important topic in graph theory, as they help reveal structural properties of graphs. Additional results on rx_k can be found in [1, 7, 12, 16, 20, 19, 23].

In [2], we introduced a new concept called the strong k-rainbow index. A tree of size $d(S)$ connecting the vertices of S is called a *Steiner S-tree*. If $S = \{u, v\}$, then a Steiner S-tree is simply a *u–v geodesic* [10]. An edge-coloring of G is called a *strong k-rainbow coloring* if every k-subset S of $V(G)$ is connected by a rainbow Steiner S-tree. The *strong k-rainbow index* of G, denoted by $srx_k(G)$, is the minimum number of colors required in a strong k-rainbow coloring

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of G. Hence, for every connected graph G, we have $rx_k(G) \leq srx_k(G)$. When $S = \{u, v\}$, the strong 2-rainbow index is called the *strong rainbow connection number* of G, denoted by $src(G)$ [10]. Chartrand et al. [10] showed that $diam(G) \leq rc(G) \leq src(G) \leq \|G\|$. The concept of srx_k is useful in modeling secure communication networks in which every set of k people can exchange information through shortest rainbow trees.

The strong k-rainbow index is well defined for every connected graph, since assigning distinct colors to all edges of a connected graph yields a strong k-rainbow coloring. Thus, it follows that

$$sdiam_k(G) \leq rx_k(G) \leq srx_k(G) \leq \|G\|. \tag{1}$$

In [2], we established Observation 1.1. Based on this observation, we determined the value of srx_k for trees, which attains the upper bound given in (1). This result is stated in Theorem 1.1. We also determined the strong 3-rainbow index of ladders, regular complete bipartite graphs, cycles, and wheels [2], as well as graphs containing certain cycles [6]. In addition, the characterization of graphs with strong 3-rainbow index equal to 2 was obtained in [26].

Observation 1.1. *[2] Let G be a connected graph of order n. Let $e, f \in E(G)$, where e and f are the bridges of G. For each integer k with $2 \leq k \leq n$, every strong k-rainbow coloring must assign distinct colors to e and f.*

Theorem 1.1. *[2] Let T_n be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$, $srx_k(T_n) = \|T_n\| = n - 1$.*

In order to build a larger and more complex communication network, a given network can be extended by applying graph operations. Hence, several studies have investigated the srx_3 under various graph operations, such as amalgamation and comb product ([2, 3, 4, 5]). Although these operations have been studied extensively, the corona product exhibits a different structural behavior. In the corona product $G \odot H$, each vertex of G is connected to all vertices of a corresponding copy of H, producing a hierarchical structure consisting of a central graph G together with several attached copies of H. As a consequence, vertices belonging to different copies of H are typically connected through their corresponding vertex in G, which restricts the possible rainbow Steiner trees. These structural properties introduce additional challenges in constructing strong 3-rainbow colorings and in determining sharp upper bounds for the strong 3-rainbow index.

In this paper, we study the srx_3 of corona product of two connected graphs. Recall that *corona product* of two connected graphs G and H, denoted by $G \odot H$, is defined as the graph obtained by taking one copy of G and $|G|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H. The study of corona product is needed when every people in each division (which is modeled by graph H) in a communication network must pass through their head division (which is represented as a vertex of G) in order to exchange important information with each other.

2. Upper bounds for $srx_3(G \odot H)$

For simplicity, we define $[a, b] = \{x \in \mathbb{Z} \colon a \leq x \leq b\}$. Let G and H be two connected graphs of order $n \geq 2$ and $m \geq 4$, respectively, with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) =$

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$\{w_1, w_2, \ldots, w_m\}$. Consider graphs $G \odot H$. For each $i \in [1, n]$, let H^i denote the i-th copy of H with $V(H^i) = \{v_i^1, v_i^2, \ldots, v_i^m\}$ and $E(H^i) = \{v_i^p v_i^q \colon p, q \in [1, m] \text{ and } w_p w_q \in E(H)\}$. Following the definition of corona product, we define $V(G \odot H) = V(G) \cup \bigcup_{i=1}^{n} V(H^i)$ and $E(G \odot H) = E(G) \cup \bigcup_{i=1}^{n} E(H^i) \cup \{v_i v_i^p \colon i \in [1, n], p \in [1, m]\}$.

Based on (1), we have $\|G \odot H\|$ as the natural upper bound for $srx_3(G \odot H)$. In this paper, we provide a better upper bound for $srx_3(G \odot H)$. To achieve this, we first construct an edge-coloring of $G \odot H$, which is described in Algorithm 2.1.

Before presenting the algorithm, we briefly explain the main idea of the coloring strategy. The edges of $G \odot H$ are divided into three groups: the edges of G, the edges of H^i, and the edges connecting vertices of G with the vertices of H^i. The coloring is constructed so that the edges of G receive distinct colors, while the remaining edges are colored systematically within each copy H^i and its incident edges. This assignment guarantees that for every set of three vertices in $G \odot H$, there exists a rainbow Steiner tree connecting them, which will be proved in Lemma 2.1.

Algorithm 2.1.

(1) Input: Two connected graphs G and H of order $n \geq 2$ and $m \geq 4$, respectively, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{w_1, w_2, \ldots, w_m\}$.

(2) Construct one copy of G and n copies of H. For $i \in [1, n]$, join vertex v_i of G to every vertex in the i-th copy of H.

(3) For $i \in [1, n]$, denote the i-th copy of H by H^i, where $V(H^i) = \{v_i^1, v_i^2, \ldots, v_i^m\}$ and $E(H^i) = \{v_i^p v_i^q \colon p, q \in [1, m] \text{ and } w_p w_q \in E(H)\}$.

(4) Define $V(G \odot H) = V(G) \cup \bigcup_{i=1}^{n} V(H^i)$ and $E(G \odot H) = E(G) \cup \bigcup_{i=1}^{n} E(H^i) \cup \{v_i v_i^p \colon i \in [1, n], p \in [1, m]\}$.

(5) Color edges of G by distinct colors $1, 2, \ldots, \|G\|$.

(6) If $H \cong T_m$, proceed to Step (7). Otherwise, continue to Step (8).

(7) (a) For $i \in [1, n]$, let $v_i^m v_i^s \in E(T_m^i)$ for $s \in [1, m-1]$.

(b) For $i \in [1, n]$, color edge $v_i v_i^p$ by color $p + (m-1)(i-1) + \|G\|$ for $p \in [1, m-1]$ and edge $v_i v_i^m$ by the same color as edge $v_i v_i^s$.

(c) For $i \in [1, n]$, color edges of T_m^i by distinct colors in $[1 + (m-1)(i-1) + \|G\|, (m-1)i + \|G\|]$ so that edge $v_i^m v_i^s$ has the same color as edge $v_i v_i^r$, where $v_i^r v_i^m \in E(T_m^i)$ (or $v_i^r v_i^s \in E(T_m^i)$) for $r \in [1, m-1]$ with $r \neq m$ (or $r \neq s$).

(d) This completes the construction.

(8) (a) For $i \in [1, n]$, let h be the girth of H^i with $h \geq 3$, and let C_h^i be a cycle of length h in H^i.

(b) Relabel the vertices of H^i such that $V(C_h^i) = \{v_i^1, v_i^2, \ldots, v_i^h\}$ and $E(C_h^i) = \{v_i^p v_i^{p+1} \colon p \in [1, h] \text{ and } v_i^{h+1} = v_i^1\}$.

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(c) For $h = 3$, let $v_i^4 \in V(H^i) \setminus V(C_3^i)$ so that $v_i^3 v_i^4 \in E(H^i)$ for $i \in [1, n]$.

(d) For $i \in [1, n]$, color edges $v_i v_i^1$ and $v_i v_i^2$ by color $1 + (m-2)(i-1) + \|G\|$, edges $v_i v_i^3$ and $v_i v_i^4$ by color $2 + (m-2)(i-1) + \|G\|$, and edge $v_i v_i^p$ for $p \in [5, m]$ by color $p - 2 + (m-2)(i-1) + \|G\|$.

(e) For $i \in [1, n]$, color edge $v_i^1 v_i^2$ by color $2 + (m-2)(i-1) + \|G\|$ and edge $v_i^3 v_i^4$ by color $1 + (m-2)(i-1) + \|G\|$.

(f) Color the remaining edges of H^i for $i \in [1, n-1]$ with some colors in $[1 + (m-2)i + \|G\|, (m-2)(i+1) + \|G\|]$ and the remaining edges of H^n with some colors in $[1 + \|G\|, m - 2 + \|G\|]$, so that adjacent edges have distinct colors.

(g) This completes the construction.

From the construction above, the coloring uses at most $\|G\| + n(m-1)$ colors when H is a tree and $\|G\| + n(m-2)$ colors otherwise. This observation will be used to derive the upper bounds for $srx_3(G \odot H)$ in the next theorem.

Next, we show that the edge-coloring of $G \odot H$ given in Algorithm 2.1 is a strong 3-rainbow coloring of $G \odot H$.

Lemma 2.1. *An edge-coloring of $G \odot H$ given in Algorithm 2.1 is a strong 3-rainbow coloring of $G \odot H$.*

Proof. Let S be any set of three vertices of $G \odot H$. We have four cases as follows.

Case 1. All vertices of S belong to the same copy H^i for some $i \in [1, n]$

Without loss of generality, let $S = \{v_i^p, v_i^q, v_i^r\}$ for distinct $p, q, r \in [1, m]$. Based on Steps (7)(c), (8)(e), and (8)(f), two adjacent edges of H^i have distinct colors. Hence, if $d(S) = 2$, then there exists a rainbow Steiner S-tree in $G \odot H$. If $d(S) = 3$, then we have the following two subcases.

Subcase 1.1. $H \cong T_m$

If $p = s$ and $q = m$, then $T = \{v_i v_i^p, v_i^p v_i^q, v_i v_i^r\}$ is the rainbow Steiner S-tree. For other values of p, q, and r, $T = \{v_i v_i^p, v_i v_i^q, v_i v_i^r\}$ is the rainbow Steiner S-tree.

Subcase 1.2. $H \ncong T_m$

If $h \geq 5$ with $p = 1$, $q = 2$, and $r = 4$ (or $p = 3$, $q = 4$, and $r = 1$), then the rainbow Steiner S-tree is a path of length 3 contained in C_h^i. Based on Steps (8)(e) and (8)(f), all edges of this path have distinct colors. If $p = 1$, $q = 2$, and $r \neq 4$, or $p = 3$, $q = 4$, and $r \neq 1$, then $T = \{v_i v_i^p, v_i^p v_i^q, v_i v_i^r\}$ is the rainbow Steiner S-tree. For other values of p, q, and r, $T = \{v_i v_i^p, v_i v_i^q, v_i v_i^r\}$ is the rainbow Steiner S-tree.

Case 2. Two vertices of S belong to the same copy H^i for some $i \in [1, n]$

Without loss of generality, let $S = \{v_i^p, v_i^q, x\}$ for distinct $p, q \in [1, m]$ and $x \in V(G) \cup V(H^j)$ for $j \in [1, n]$ with $j \neq i$. Based on Steps (5), (7)(b), and (8)(d), there exists a rainbow $v_i - x$ geodesic in $G \odot H$, say path P. If $H \cong T_m$ with $p = s$ and $q = m$ or $H \ncong T_m$ with $p = 1$ and $q = 2$ (or $p = 3$ and $q = 4$), then $T = \{v_i v_i^p, v_i^p v_i^q\} \cup P$ is the rainbow Steiner S-tree. For other values of p and q, $T = \{v_i v_i^p, v_i v_i^q\} \cup P$ is the rainbow Steiner S-tree.

Case 3. Exactly one vertex of S belongs to the copy H^i for some $i \in [1, n]$

Without loss of generality, let $S = \{v_i^p, x_1, x_2\}$ for $p \in [1, m]$, $x_1 \in V(G) \cup V(H^j)$, and $x_2 \in$

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$V(G) \cup V(H^k)$ for distinct $j, k \in [1, n]$, $j \neq i$, and $k \neq i$. Based on Steps (5), (7)(b), and (8)(d), there exists a rainbow Steiner S-tree in $G \odot H$.

Case 4. All vertices of S belong to graph G

Based on Step (5), there exists a rainbow Steiner S-tree in $G \odot H$. □

Using these results, we obtain the upper bounds for $srx_3(G \odot H)$ which are given in the following theorem.

Theorem 2.1. *Let n and m be two integers with $n \geq 2$ and $m \geq 4$. Let G and H be two connected graphs of order n and m, respectively. Then*

$$srx_3(G \odot H) \leq \begin{cases} \|G\| + n\,(m-1), & \textit{if } H \textit{ is a tree}; \\ \|G\| + n\,(m-2), & \textit{otherwise}. \end{cases}$$

Proof. The proof is an immediate consequence of Algorithm 2.1 and Lemma 2.1. □

Next, we prove the sharpness of the upper bounds in Theorem 2.1. These results are given in Theorems 2.2 and 2.3. We first make the following three observations.

Observation 2.1. *Let n and g be two integers with $n \geq 5$ and $g \geq 4$. Let G be a connected graph of order n containing a cycle C_g which admits a strong 3-rainbow coloring. If $e \in E(C_g)$ and f is an arbitrary bridge of G, then e and f have distinct colors.*

Proof. Relabeling vertices of G such that $V(C_g) = \{v_1, v_2, \ldots, v_g\}$ and $E(C_g) = \{v_i v_{i+1} \colon i \in [1, g] \text{ and } v_{g+1} = v_1\}$. Let $e = v_p v_{p+1} \in E(C_g)$ for $p \in [1, g]$ and $f = xy \in E(G)$ be an arbitrary bridge of G. Assume that $d(C_g, x) < d(C_g, y)$. Observe that every rainbow Steiner $\{v_p, v_{p+1}, y\}$-tree must contains edges e and f. This implies these two edges should have distinct colors. □

Observation 2.2. *Let n and m be two integers with $n \geq 2$ and $m \geq 4$. Let G and H be two connected graphs of order n and m, respectively. Let c be a strong 3-rainbow coloring of $G \odot H$.*

(i) If $d_{H^i}(v_i^p, v_i^q) \geq 2$ for each $i \in [1, n]$ and distinct $p, q \in [1, m]$, then $c(v_i v_i^p) \neq c(v_i v_i^q)$.

(ii) For distinct $i, j \in [1, n]$ and $p, q \in [1, m]$, $c(v_i v_i^p) \neq c(v_j v_j^q)$.

Proof. We have two cases.

(i) Let $d_{H^i}(v_i^p, v_i^q) \geq 2$ for each $i \in [1, n]$ and distinct $p, q \in [1, m]$. Since $T = \{v_i v_i^p, v_i v_i^q\}$ is the only possible rainbow Steiner $\{v_i, v_i^p, v_i^q\}$-tree, we have $c(v_i v_i^p) \neq c(v_i v_i^q)$.

(ii) Observe that for distinct $i, j \in [1, n]$ and $p, q \in [1, m]$, the rainbow Steiner $\{v_i, v_i^p, v_j^q\}$-tree must contains edges $v_i v_i^p$ and $v_j v_j^q$. Thus, $c(v_i v_i^p) \neq c(v_j v_j^q)$.

□

For further discussion, we always consider G as a tree or a unicyclic graph.

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Observation 2.3. *Let n, m, and g be three integers with $n \geq 2$ and $m, g \geq 4$. Let G and H be two connected graphs of order n and m, respectively, where G is a tree or a unicyclic graph of girth g. Let $e \in E(G)$. If c is a strong 3-rainbow coloring of $G \odot H$, then $c(e) \neq c(v_i v_i^p)$ for each $i \in [1, n]$ and $p \in [1, m]$.*

Proof. Let $e = xy$ be an arbitrary edge of G. Observe that any rainbow Steiner $\{x, y, v_i^p\}$-tree for each $i \in [1, n]$ and $p \in [1, m]$ must contains edges e and $v_i v_i^p$. Thus, $c(e) \neq c(v_i v_i^p)$. □

The following theorem shows that $srx_3(G \odot S_m) = \|G\| + n\,(m-1)$.

Theorem 2.2. *Let n, m, and g be three integers with $n \geq 2$, $m \geq 4$, and $g \geq 9$. Let G be a tree or a unicyclic graph (with girth g) of order n and S_m be a star of order m. Then $srx_3(G \odot S_m) = \|G\| + n\,(m-1)$.*

Proof. Let $V(S_m) = \{w_1, w_2, \ldots, w_m\}$ and $E(S_m) = \{w_1 w_p \colon p \in [2, m]\}$. By using Theorem 2.1, $srx_3(G \odot S_m) \leq \|G\| + n\,(m-1)$.

For the lower bound, suppose that $srx_3(G \odot S_m) \leq \|G\| + n\,(m-1) - 1$. Then there exists a strong 3-rainbow coloring c of $G \odot S_m$ using $\|G\| + n\,(m-1) - 1$ colors. Since $d_{S_m^i}(v_i^p, v_i^q) = 2$ for each $i \in [1, n]$ and distinct $p, q \in [2, m]$, it follows by Observation 2.2 that we need at least $n\,(m-1)$ distinct colors to color edges $v_i v_i^p$ for all $i \in [1, n]$ and $p \in [2, m]$. This implies we have at most $\|G\| - 1$ colors left. Let A be the set of these $\|G\| - 1$ colors. According to Observation 2.3, edges of G should be colored by colors in A. If G is a tree, then $|A| \leq n - 2$, contradicting to Theorem 1.1. If G is a unicylic graph of girth g, then $|A| \leq n - 1$. Let C_g be a cycle of length g in G. Relabeling vertices of G such that $V(C_g) = \{v_1, v_2, \ldots, v_g\}$ and $E(C_g) = \{v_i v_{i+1} \colon i \in [1, g]$ and $v_{g+1} = v_1\}$. Since there are $n-g$ bridges in G, by using Observation 1.1, we need at least $n - g$ colors to color these bridges. It follows by Observation 2.1 that we have at most $g - 1$ colors to color edges of C_g. Since $\|C_g\| = g$, there exist at least two edges of C_g, say e and f, such that $c(e) = c(f)$. Without loss of generality, let $e = v_1 v_2$ and $f = v_p v_{p+1}$ for $p \in [2, g]$, and suppose that these two edges are encountered in the order $v_1 v_2, v_p v_{p+1}$ which is in the clockwise direction. Note that $d_{C_g}(v_1, v_p) \leq \lfloor \frac{g}{2} \rfloor$. Thus, by symmetry, we only prove for $1 \leq p - 1 \leq \lfloor \frac{g}{2} \rfloor$. Observe that the only possible rainbow Steiner $\{v_1, v_{\lceil \frac{p+1}{2} \rceil}, v_{p+1}\}$-tree is a path which contains edges e and f, but $c(e) = c(f)$, a contradiction. Thus, $srx_3(G \odot S_m) = \|G\| + n\,(m-1)$. In particular, $srx_3(G \odot S_m) = nm - 1$ if G is a tree and $srx_3(G \odot S_m) = nm$ if G is a unicyclic graph of girth at least 9. □

The condition $g \geq 9$ is used to avoid short cycles in G that may create alternative Steiner trees connecting the considered vertices and to guarantee that the upper bound in Theorem 2.2 is attained. When the girth of G is smaller, some edges of the cycle C_g may share the same color in a strong 3-rainbow coloring (see [2]), which may lead to a smaller value of $srx_3(G \odot S_m)$. Hence, determining the exact value for other girth values remains an interesting problem.

Before proceeding to Theorem 2.3, we first introduce several classes of graphs as follows.

Definition 2.1. *Let m, a, b, and c be four integers with $m \geq 4$, $a \geq 1$, and $b, c \geq 0$ such that $m = a + b + c + 2$. For $l \in [1, 2]$, a graph $H_{m,l}(a, b, c)$ is a connected graph of order m with $V(H_{m,l}(a, b, c)) = \{w_1, w_2, \ldots, w_m\}$ and $E(H_{m,l}(a, b, c)) = \{w_{m-1} w_i, w_{m-1} w_j, w_m w_i, w_m w_k \colon i \in [1, a], j \in [a+1, a+b], k \in [a+b+1, a+b+c]\} \cup E_l$, where*

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- $E_l = \{w_{m-1}w_m\}$ *with* $a \geq 2$ *and* $b, c \geq 0$*, or* $a \geq 1$ *and* $b \geq 1$ *(or* $c \geq 1$*) if* $l = 1$*, and*
- $E_l = \emptyset$ *with* $a \geq 2$ *and* $b, c \geq 0$ *if* $l = 2$*.*

Examples of graphs $H_{4,1}$, $H_{4,2}$, $H_{5,1}$, and $H_{5,2}$ are given in Figure 1.

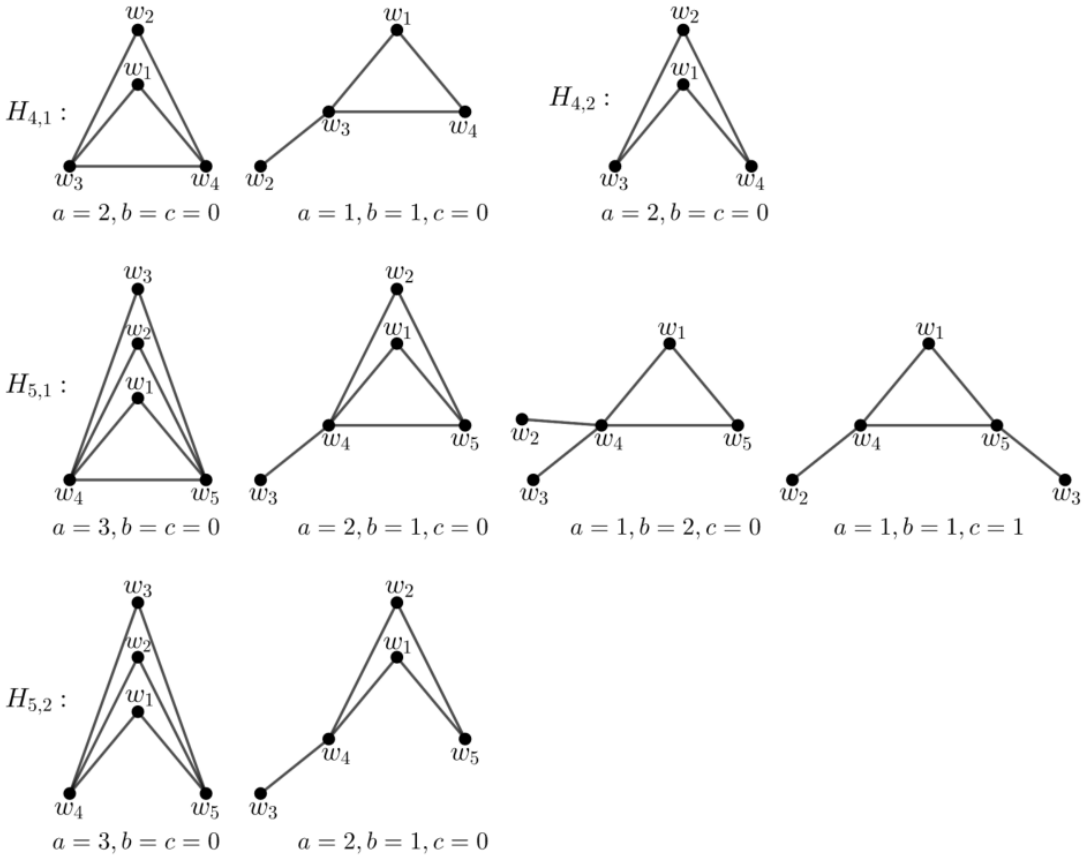


Figure 1. Graphs $H_{4,1}$, $H_{4,2}$, $H_{5,1}$, and $H_{5,2}$

Theorem 2.3. *Let* n*,* m*,* g*,* a*,* b*, and* c *be six integers with* $n \geq 2$*,* $m \geq 4$*,* $g \geq 9$*,* $a \geq 1$*, and* $b, c \geq 0$ *such that* $m = a + b + c + 2$*. Let* G *be a tree or a unicyclic graph (with girth* g*) of order* n *and* $H_{m,l}(a, b, c)$ *for* $l \in [1, 2]$ *be a connected graph of order* m *as defined in Definition 2.1. Then* $srx_3(G \odot H_{m,l}(a, b, c)) = \|G\| + n\,(m - 2)$.

Proof. It follows by Theorem 2.1 that $srx_3(G \odot H_{m,l}(a, b, c)) \leq \|G\| + n\,(m - 2)$. For the lower bound, suppose that $srx_3(G \odot H_{m,l}(a, b, c)) \leq \|G\| + n\,(m - 2) - 1$. Then there exists a strong 3-rainbow coloring of $G \odot H_{m,l}(a, b, c)$ using $\|G\| + n\,(m - 2) - 1$ colors. Observe that for each $i \in [1, n]$ and $p, q \in [1, a + b + c]$, $d_{H_{m,l}(a,b,c)^i}(v_i^p, v_i^q) \geq 2$. Thus, by Observation 2.2, we need at least $n\,(m - 2)$ distinct colors to color edges $v_i v_i^p$ for all $i \in [1, n]$ and $p \in [1, a + b + c]$. This implies we have at most $\|G\| - 1$ colors left. Let A be the set of these $\|G\| - 1$ colors. According to Observation 2.3, edges of G should be colored by colors in A. Hence, by using an argument

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similar to that used in Theorem 2.2, we will obtain a contradiction. Thus, $srx_3(G\odot H_{m,l}(a,b,c)) = \|G\| + n\,(m-2)$ for $l \in [1,2]$. In particular, $srx_3(G \odot H_{m,l}(a,b,c)) = nm - n - 1$ if G is a tree and $srx_3(G \odot H_{m,l}(a,b,c)) = nm - n$ if G is a unicyclic graph of girth at least 9. □

3. Characterization of graphs H with certain $srx_3(T_n \odot H)$

Following Theorem 2.1, we obtain that the upper bound for $srx_3(T_n \odot H)$ is $nm-1$ if H is a tree or $nm-n-1$ if H is not a tree. Therefore, in this section, we characterize graphs H with $srx_3(T_n \odot H)$ equals these upper bounds.

3.1. Characterization of graphs H with $srx_3(T_n \odot H) = nm-1$

The following theorem provides characterization of graphs H with $srx_3(T_n \odot H) = nm-1$.

Theorem 3.1. *Let n and m be two integers with $n \geq 2$ and $m \geq 4$. Let T_n be a tree of order n and H be a connected graph of order m. Then $srx_3(T_n \odot H) = nm-1$ if and only if $H \cong S_m$.*

Proof. If $H \cong S_m$, then it follows by Theorem 2.2 that $srx_3(T_n \odot H) = nm-1$.

Conversely, suppose that H is a connected graph with $srx_3(T_n \odot H) = nm-1$ but not a star S_m. Then, H must be a tree with $\triangle(H) \leq m-2$ or a connected graph which contains cycles. If H contains cycles, then $srx_3(T_n \odot H) \leq nm-n-1$ by Theorem 2.1, a contradiction. Thus, we assume that H is a tree with $\triangle(H) \leq m-2$. Without loss of generality, let $v_i^1 \in V(H^i)$ with $d_{H^i}(v_i^1) = \triangle(H)$ and $N_{H^i}(v_i^1) = \{v_i^p \colon p \in [2, \triangle(H)+1]\}$ for each $i \in [1,n]$. Since $|H| = m$, there exists at least a vertex of H^i, say $v_i^{\triangle(H)+2}$, such that $d_{H^i}(v_i^1, v_i^{\triangle(H)+2}) = 2$. It means $v_i^{\triangle(H)+2}v_i^p \in E(H^i)$ for $p \in [2, \triangle(H)+1]$. Without loss of generality, let $v_i^{\triangle(H)+1}v_i^{\triangle(H)+2} \in E(H^i)$. Now, consider the following edge-coloring c of $T_n \odot H$.

1. For each $i \in [1,n]$, define $c(v_iv_i^p) = p-1+(m-2)(i-1)$ for $p \in [2, \triangle(H)+1]$, $c(v_iv_i^1) = c(v_iv_i^2)$, and $c(v_iv_i^{\triangle(H)+2}) = c(v_iv_i^{\triangle(H)+1})$.

2. If $\triangle(H) = m-2$, then directly go to step 3. Otherwise, define $c(v_iv_i^p) = p-2+(m-2)(i-1)$ for each $i \in [1,n]$ and $p \in [\triangle(H)+3, m]$.

3. For each $i \in [1,n]$, define $c(v_i^1v_i^p) = c(v_iv_i^{p+1})$ for each $p \in [2, \triangle(H)]$, $c(v_i^1v_i^{\triangle(H)+1}) = c(v_iv_i^2)$, and $c(v_i^{\triangle(H)+1}v_i^{\triangle(H)+2}) = c(v_i^1v_i^2)$.

4. If $\triangle(H) = m-2$, then directly go to step 5. Otherwise, color the remaining $m-\triangle(H)-2$ edges of H^i by distinct colors in $[\triangle(H)+1+(m-2)(i-1), (m-2)i]$ for each $i \in [1,n]$.

5. Define $c(e_i) = i + n\,(m-2)$ for $e_i \in E(T_n)$ with $i \in [1, n-1]$.

Based on the edge-coloring c above, we can check that there exists a rainbow Steiner tree connecting every three vertices of $T_n \odot H$. Thus, $srx_3(T_n \odot H) \leq nm-n-1$, a contradiction. □

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3.2. Characterization of graphs H with $srx_3(T_n \odot H) = nm - n - 1$

In this subsection, we consider H as a connected graph that is not a tree. Let $h \geq 3$ be the girth of H. For each $i \in [1, n]$, let C_h^i be a cycle of length h in H^i. Relabeling vertices of H^i such that $V(C_h^i) = \{v_i^1, v_i^2, \ldots, v_i^h\}$ and $E(C_h^i) = \{v_i^p v_i^{p+1} : p \in [1, h] \text{ and } v_i^{h+1} = v_i^1\}$.

The characterization of graphs H with $srx_3(T_n \odot H) = nm - n - 1$ is given in Theorem 3.2. Before we proceed to this result, we first provide some lemmas which will be used to prove Theorem 3.2.

Lemma 3.1. *Let T_n be a tree of order $n \geq 2$, then $srx_3(T_n \odot K_4) = 2n + 1$.*

Proof. Suppose that $srx_3(T_n \odot K_4) \leq 2n$. Then there exists a strong 3-rainbow coloring c of $T_n \odot K_4$ using $2n$ colors. Since any rainbow Steiner $\{v_i, v_j, v_k\}$-tree for distinct $i, j, k \in [1, n]$ should be contained in T_n, it follows by Theorem 1.1 that we need at least $n - 1$ distinct colors to color edges of T_n. Let $c(e_i) = i$ for $e_i \in E(T_n)$ with $i \in [1, n - 1]$. By using Observation 2.3, this forces $c(v_i v_i^p) \in [n, 2n]$ for all $i \in [1, n]$ and $p \in [1, 4]$. Now, we consider the following two cases.

Case 1. There exists $i \in [1, n]$ such that $c(v_i v_i^p) \neq c(v_i v_i^q)$ for distinct $p, q \in [1, 4]$

Without loss of generality, let $c(v_1 v_1^p) \neq c(v_1 v_1^q)$ for distinct $p, q \in [1, 4]$. This forces $c(v_j v_j^p) = c(v_j v_j^q)$ for all $j \in [2, n]$ and distinct $p, q \in [1, 4]$ by Observation 2.2. Now, consider $\{v_2^p, v_2^q, x\}$ where $x \in V(T_n) \cup V(K_4^j)$ for all $j \in [1, n]$ with $j \neq 2$. Observe that edge $v_2^p v_2^q$ should be contained in any rainbow Steiner tree connecting these three vertices. Thus, $c(v_2^p v_2^q) \notin [1, 2n]$. This implies we need one new distinct color to color edge $v_2^p v_2^q$, which is impossible.

Case 2. $c(v_i v_i^p) = c(v_i v_i^q)$ for all $i \in [1, n]$ and distinct $p, q \in [1, 4]$

Without loss of generality, let $c(v_i v_i^p) = i + n - 1$ for each $i \in [1, n]$ and $p \in [1, 4]$. Now, consider $\{v_1^p, v_1^q, x\}$ where $x \in V(T_n) \cup V(K_4^j)$ for all $j \in [2, n]$. By using an argument similar to that used in the proof of Case 1, we obtain that $c(v_1^p v_1^q) = 2n$ for all distinct $p, q \in [1, 4]$. However, there is no rainbow Steiner $\{v_1^1, v_1^2, v_1^3\}$-tree, a contradiction.

Now, we show that $srx_3(T_n \odot K_4) \leq 2n + 1$ by defining a strong 3-rainbow coloring $c : E(T_n \odot K_4) \to [1, 2n + 1]$. First, assign distinct colors $1, 2, \ldots, n - 1$ to the edges of T_n. For each $i \in [1, n]$, assign colors $i + n - 1$ to the edges $v_i v_i^p$ for all $p \in [1, 4]$, color $2n$ to the edges $v_i^1 v_i^2$, $v_i^2 v_i^3$, $v_i^3 v_i^4$, and $v_i^1 v_i^4$, and color $2n + 1$ to the edges $v_i^1 v_i^3$ and $v_i^2 v_i^4$. By this edge-coloring, it can be verified that there exists a rainbow Steiner tree connecting every three vertices of $T_n \odot K_4$. □

Before we proceed further, we first define graphs H_1, H_2, H_3, H_4, and H_5 as given in Figure 2.

Lemma 3.2. *Let n and m be two integers with $n \geq 2$ and $m \geq 5$. Let T_n be a tree of order n and H be a connected graph of order m where $H_l \subseteq H$ for $l \in [1, 5]$. Then*

$$srx_3(T_n \odot H) \leq \begin{cases} nm - 2n, & l = 2; \\ nm - 2n - 1, & \textit{otherwise}. \end{cases}$$

Proof. For $l = 2$, we show that $srx_3(T_n \odot H) \leq nm - 2n$ by defining a strong 3-rainbow coloring $c : E(T_n \odot H) \to [1, nm - 2n]$ as follows.

1. For each $i \in [1, n]$, define $c(v_i v_i^1) = c(v_i v_i^2) = c(v_i v_i^3) = c(v_i^4 v_i^5) = 1 + (m - 3)(i - 1)$, $c(v_i v_i^4) = c(v_i v_i^5) = c(v_i^1 v_i^3) = c(v_i^2 v_i^3) = 2 + (m - 3)(i - 1)$, and $c(v_i^1 v_i^2) = n\,(m - 3) + 1$.

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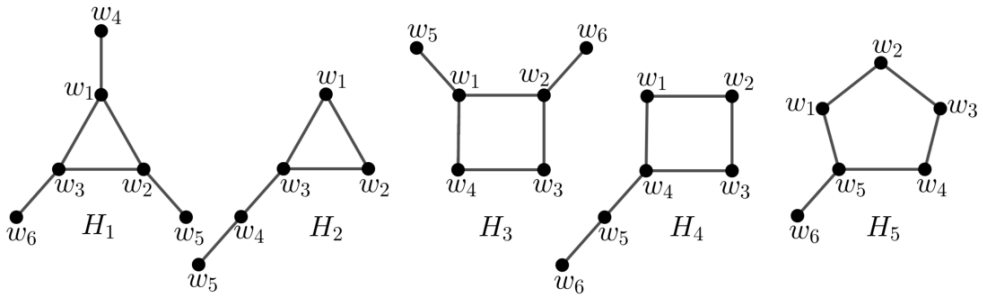


Figure 2. Graphs H_1, H_2, H_3, H_4, and H_5

2. If $m = 5$, then go directly to step 3. Otherwise, define $c(v_iv_i^p) = p - 3 + (m-3)(i-1)$ for each $i \in [1, n]$ and $p \in [6, m]$.

3. Color the remaining edges of H^i by some colors in $[1 + (m-3)i, (m-3)(i+1)]$ for each $i \in [1, n-1]$ and the remaining edges in H^n by some colors in $[1, m-3]$ such that two adjacent edges which are not contained in the same triangle have distinct colors.

4. Define $c(e_i) = i + n\,(m-3) + 1$ for $e_i \in E(T_n)$ with $i \in [1, n-1]$.

For other values of l, we show that $srx_3(T_n \odot H) \leq nm-2n-1$ by defining a strong 3-rainbow coloring $c : E(T_n \odot H) \to [1, nm-2n-1]$ as follows.

1. (a) For $l = 1$ and each $i \in [1, n]$, define $c(v_iv_i^1) = c(v_iv_i^4) = c(v_i^3v_i^6) = 1+(m-3)(i-1)$, $c(v_iv_i^2) = c(v_iv_i^5) = c(v_i^1v_i^4) = 2+(m-3)(i-1)$, and $c(v_iv_i^3) = c(v_iv_i^6) = c(v_i^2v_i^5) = 3 + (m-3)(i-1)$.

   (b) For $l = 3$ and each $i \in [1, n]$, define $c(v_iv_i^1) = c(v_iv_i^5) = c(v_i^3v_i^4) = 1+(m-3)(i-1)$, $c(v_iv_i^2) = c(v_iv_i^6) = c(v_i^1v_i^5) = 2+(m-3)(i-1)$, and $c(v_iv_i^3) = c(v_iv_i^4) = c(v_i^2v_i^6) = 3 + (m-3)(i-1)$.

   (c) For $l = 4$ and each $i \in [1, n]$, define $c(v_iv_i^1) = c(v_iv_i^2) = c(v_i^3v_i^4) = 1+(m-3)(i-1)$, $c(v_iv_i^3) = c(v_iv_i^4) = c(v_i^1v_i^2) = c(v_i^5v_i^6) = 2+(m-3)(i-1)$, and $c(v_iv_i^5) = c(v_iv_i^6) = 3 + (m-3)(i-1)$.

   (d) For $l = 5$ and each $i \in [1, n]$, define $c(v_iv_i^1) = c(v_iv_i^2) = c(v_i^5v_i^6) = 1+(m-3)(i-1)$, $c(v_iv_i^3) = c(v_iv_i^4) = c(v_i^1v_i^2) = 2+(m-3)(i-1)$, and $c(v_iv_i^5) = c(v_iv_i^6) = c(v_i^3v_i^4) = 3 + (m-3)(i-1)$.

2. If $m = 6$, then go directly to step 3. Otherwise, define $c(v_iv_i^p) = p - 3 + (m-3)(i-1)$ for each $i \in [1, n]$ and $p \in [7, m]$.

3. Color the remaining edges of H^i by some colors in $[1 + (m-3)i, (m-3)(i+1)]$ for each $i \in [1, n-1]$ and the remaining edges of H^n by some colors in $[1, m-3]$ such that two adjacent edges which are not contained in the same triangle have distinct colors.

4. Define $c(e_i) = i + n\,(m-3)$ for $e_i \in E(T_n)$ with $i \in [1, n-1]$.

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To verify that the edge-coloring c is a strong 3-rainbow coloring, let S be any set of three vertices of $T_n \odot H$. We consider all possible locations of the vertices in S: (i) all vertices lie in the same copy H^i, (ii) two vertices lie in the same copy H^i and the third lies outside H^i, (iii) the vertices lie in different copies of H or in T_n, and (iv) all vertices lie in T_n. In each case, a rainbow Steiner tree connecting S can be constructed. Thus, the claim follows. □

Lemma 3.3. *Let n, m, and h be three integers with $n \geq 2$ and $m, h \geq 6$. Let T_n be a tree of order n and H be a connected graph of order m and girth h. Then $srx_3(T_n \odot H) \leq nm - 2n - 1$.*

Proof. We show that $srx_3(T_n \odot H) \leq nm - 2n - 1$ by defining a strong 3-rainbow coloring $c : E(T_n \odot H) \to [1, nm - 2n - 1]$ as follows.

1. For each $i \in [1, n]$, define $c(v_i v_i^1) = c(v_i v_i^2) = c(v_i^3 v_i^4) = 1 + (m-3)(i-1)$, $c(v_i v_i^3) = c(v_i v_i^4) = c(v_i^1 v_i^2) = c(v_i^5 v_i^6) = 2+(m-3)(i-1)$, and $c(v_i v_i^5) = c(v_i v_i^6) = 3+(m-3)(i-1)$.
2. If $m = 6$, then go directly to step 3. Otherwise, define $c(v_i v_i^p) = p - 3 + (m-3)(i-1)$ for each $i \in [1, n]$ and $p \in [7, m]$.
3. Color the remaining edges of H^i by some colors in $[1 + (m-3)i, (m-3)(i+1)]$ for each $i \in [1, n-1]$ and the remaining edges of H^n by some colors in $[1, m-3]$ such that two adjacent edges have distinct colors.
4. Define $c(e_i) = i + n\,(m-3)$ for $e_i \in E(T_n)$ with $i \in [1, n-1]$.

To verify that the edge-coloring c is a strong 3-rainbow coloring of $T_n \odot H$, let S be any set of three vertices of $T_n \odot H$. We consider all possible locations of the vertices in S: (i) all vertices of S lie in the same copy H^i, (ii) two vertices of S lie in H^i and the third lies outside H^i, (iii) the vertices of S lie in different copies of H or in T_n, and (iv) all vertices of S lie in T_n. In each case, a rainbow Steiner tree connecting S can be constructed. Thus, the claim follows. □

Now we are ready to characterize graphs H with $srx_3(T_n \odot H) = nm - n - 1$ that are not trees.

Theorem 3.2. *Let n, m, a, b, and c be five integers with $n \geq 2$, $m \geq 4$, $a \geq 1$, and $b, c \geq 0$. Let T_n be a tree of order n and H be a connected graph of order m that is not a tree. Then $srx_3(T_n \odot H) = nm - n - 1$ if and only if $H \cong K_4$ for $n = 2$ or $H \cong H_{m,l}(a, b, c)$ with $l \in [1, 2]$ or $H \cong C_5$.*

Proof. If $H \cong K_4$ for $n = 2$ or $H \cong H_{m,l}(a, b, c)$ with $l \in [1, 2]$, then it follows by Lemma 3.1 and Theorem 2.3 that $srx_3(T_n \odot H) = nm - n - 1$, respectively. This leaves us to show that $srx_3(T_n \odot C_5) = 4n - 1$. By using Theorem 2.1, $srx_3(T_n \odot C_5) \leq 4n - 1$. Thus, we only need to prove the lower bound. By using Observation 2.2, we need at least $3n$ distinct colors to color edges $v_i v_i^p$ for all $i \in [1, n]$ and $p \in [1, 5]$. Since any rainbow Steiner $\{v_i, v_j, v_k\}$-tree for distinct $i, j, k \in [1, n]$ should be contained in T_n, it follows by Observation 2.3 that $srx_3(T_n \odot C_5) \geq 4n-1$.

Conversely, suppose that H is a connected graph and not a tree with $srx_3(T_n \odot H) = nm-n-1$. Let $h \geq 3$ be the girth of H. If $h \geq 6$, then it follows by Lemma 3.3 that $srx_3(T_n \odot H) \leq nm - 2n - 1$, a contradiction. Therefore, we consider the following three cases.

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Case 1. $h = 3$
Suppose that $H \ncong K_4$ for $n = 2$ and $H \ncong H_{m,1}(a, b, c)$. Then $H \cong K_4$ for $n \geq 3$ or $H_k \subseteq H$ with $k \in [1, 2]$ (graph H_k is given in Figure 2). If $H \cong K_4$ for $n \geq 3$, then it follows by Lemma 3.1 that $srx_3(T_n \odot K_4) \leq 2n + 1 < 3n - 1$, a contradiction. If $H_k \subseteq H$ with $k \in [1, 2]$, then it follows by Lemma 3.2 that $srx_3(T_n \odot H) \leq nm - 2n$, a contradiction.

Case 2. $h = 4$
Suppose that $H \ncong H_{m,2}(a, b, c)$. Then $H_k \subseteq H$ with $k \in [3, 4]$ (graph H_k is given in Figure 2). However, $srx_3(T_n \odot H) \leq nm - 2n - 1$ by Lemma 3.2, a contradiction.

Case 3. $h = 5$
Suppose that $H \ncong C_5$. Then $H_5 \subseteq H$ (graph H_5 is given in Figure 2). However, $srx_3(T_n \odot H) \leq nm - 2n - 1$ by Lemma 3.2, a contradiction. □

4. Conclusion

We first provided a sharp upper bound for $srx_3(G \odot H)$ by constructing a strong 3-rainbow coloring of $G \odot H$ through an algorithmic approach. In particular, we showed that $nm - 1$ or $nm - n - 1$ is the sharp upper bound for $srx_3(T_n \odot H)$. We also characterized graphs H with $srx_3(T_n \odot H) = nm - 1$ and graphs H that are not trees with $srx_3(T_n \odot H) = nm - n - 1$.

An interesting open problem that remains is the characterization of trees T_m satisfying $srx_3(T_n \odot T_m) = nm - n - 1$. It is also natural to extend this study by determining the exact values of $srx_3(G \odot H)$ for other connected graphs G and H. Such results may lead to a broader characterization of graphs G and H for which $srx_3(G \odot H)$ attains its upper bound. Furthermore, it would be interesting to investigate possible generalizations of these results to the strong k-rainbow index $srx_k(G \odot H)$ for $k > 3$, as well as to other graph products.

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