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On some covering graphs of a graph

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Abstract

For a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, let S be the covering set of G having the maximum degree over all the minimum covering sets of G. Let $N_S[v] = \{u \in S : uv \in E(G)\} \cup \{v\}$ be the closed neighbourhood of the vertex v with respect to S. We define a square matrix $A_S(G) = (a_{ij})$, by $a_{ij} = 1$, if $|N_S[v_i] \cap N_S[v_j]| \geq 1, i \neq j$ and 0, otherwise. The graph G^S associated with the matrix $A_S(G)$ is called the maximum degree minimum covering graph (MDMC-graph) of the graph G. In this paper, we give conditions for the graph G^S to be bipartite and Hamiltonian. Also we obtain a bound for the number of edges of the graph G^S in terms of the structure of G. Further we obtain an upper bound for covering number (independence number) of G^S in terms of the covering number (independence number) of G.

Keywords: Covering graph, maximum degree, covering set, maximum degree minimum covering graph, covering number, independence number
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1. Introduction

Let G be finite, undirected, simple graph with n vertices and m edges having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. When the graph G is to be specified, the number of edges is denoted by $m(G)$. A subset S of the vertex set $V(G)$ is said to be covering set of G if every edge of G is incident to at least one vertex in S. A covering set with minimum cardinality among all covering sets

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of G is called the minimum covering set of G and its cardinality is called the (vertex)covering number of G, denoted by α_0. Let $C(G) = \{S \subset V(G)$: S is a minimum covering set of $G\}$. If $U = \{u_1, u_2 \ldots, u_r\}$ is a subset of $V(G)$ and $d_U(u_i),\ i = 1, 2, \ldots, r$ denote the degree of the vertex u_i in G, which is in U, then we call $d_U(u_1) \leq d_U(u_2) \leq \cdots \leq d_U(u_r)$ as the degree sequence of U. If $U = \{u_1, u_2 \ldots, u_r\}$ and $W = \{w_1, w_2 \ldots, w_r\}$ be any two subsets of $V(G)$ having degree sequences $d_U(u_1) \leq d_U(u_2) \leq \cdots \leq d_U(u_r)$ and $d_W(w_1) \leq d_W(w_2) \leq \cdots \leq d_W(w_r)$, respectively, then we say the degrees of U dominates the degrees of W if $d_W(w_i) \leq d_U(u_i)$ for all $i = 1, 2, \ldots, r$. The minimum covering set $S = \{v_1, v_2, \ldots, v_k\}$ of G is said to be a maximum degree minimum covering set (shortly MDMC-set) of the graph G if the degrees of the vertices in S dominates the degrees of the vertices in any other minimum cover of G. Let $C_{MD}(G) = \{S \subset V(G)$: S is a maximum degree minimum covering set of $G\}$. Further, let $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i adjacent to v_j and 0 otherwise, be the adjacency matrix of the graph G and let $N_S[v] = \{u \in S \subset C(G) : uv \in E(G)\} \cup \{v\}$ be the closed neighbourhood of the vertex $v \in V(G)$ with respect to S. We define a square matrix $A_S(G) = (a_{ij})$ of order n, by

$$a_{ij} = \begin{cases} 1, & \text{if } |N_S[v_i] \cap N_S[v_j]| \geq 1, i \neq j \text{ ,} \\ 0, & \text{otherwise.} \end{cases}$$

Now corresponding to every $(0, 1)$-square matrix of order n with zero diagonal entries there is a simple graph on n vertices, therefore corresponding to the n-square matrix $A_S(G)$ defined above we have a simple graph of order n, we denote such a graph by G^S and call it the minimum covering graph (MC-graph) of G. As the minimum covering set of a graph G need not be unique, it can be seen that if S_1 and S_2 are any two minimum covering sets of G, with different degree sequences, then the minimum covering graphs (MC-graphs) G^{S_1} and G^{S_2} are non isomorphic. However, if S_1 and S_2 have the same degree sequences, then the MC-graphs G^{S_1} and G^{S_2} are isomorphic.

For example, consider the graph G as shown in Figure 1, the set of minimum covering sets of G is $C(G) = \{S_1 = \{1, 3, 4\}, S_2 = \{2, 3, 4\}, S_3 = \{5, 1, 3\}, S_4 = \{6, 2, 4\}\}$. Among these covering sets the pairs $S_1,\ S_2$ and $S_3,\ S_4$ are degree equivalent and $S_1,\ S_2$ are maximum degree minimum covering sets (MDMC-sets) of G. That is, $S_1,\ S_2 \in C_{MD}(G)$. Let G^{S_i}, $i = 1, 2, 3, 4$ be the minimum covering graphs of G with respect to S_i. Clearly G^{S_1} and G^{S_2} are isomorphic; G^{S_3} and G^{S_4} are isomorphic, while as G^{S_1} is not isomorphic to G^{S_3}; and G^{S_2} is not isomorphic to G^{S_4} (see Figure 1 below).

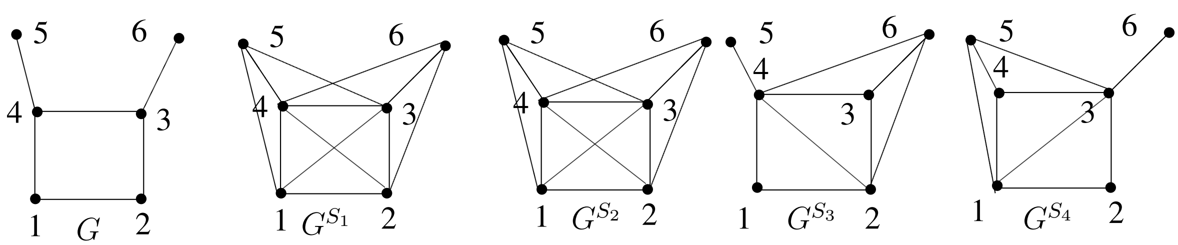

Figure 1. Graph G and its minimum covering sets.

From this example, it follows that for minimum covering sets having different degree sequences, we obtain different MC-graphs. Therefore, to get a unique (up to isomorphism) MC-graph of the graph G, we consider the MDMC-set of the graph G. The unique graph G^S in this case is called the maximum degree minimum covering graph (MDMC-graph) of G. It is clear from the definition of G^S that if two vertices u and v are adjacent in G, they are adjacent in G^S and if u and v are non adjacent in G they are adjacent in G^S if they share at least one common neighbour with S. So, it follows that G^S is connected if and only if G is connected. Also, since G and G^S are the graphs on the same vertex set, it follows that G is a spanning subgraph of G^S.

The motivation behind our interest in the study of minimum covering graphs of a graph G is to explore some interesting properties of G which changes (or does not change) when edges between non-adjacent vertices are added in G under some definite rule.

$$\text{Let } B_S = (b_{ij}), \text{ where } b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ adjacent to } v_j, \\ 1, & \text{if } v_i = v_j \in S, \\ 0, & \text{otherwise,} \end{cases}$$

be a matrix of order $|S| \times n$, whose rows are indexed by the vertices in any MDMC-set S of the graph G and whose columns are indexed by the vertices of G. Define an n-square matrix R as the product of B_S^t and B_S, that is, $R = B_S^t B_S$, where B_S^t is the transpose of B_S. It is easy to see that the ij^{th}-entry of the matrix $R = (r_{ij})$ is

$$r_{ij} = \begin{cases} |N_S[v_i] \cap N_S[v_j]|, & \text{if } i \neq j, \\ |N_S[v_i]|, & \text{if } i = j. \end{cases}$$

The matrix R is a sort of covering matrix of G, so we call it as the covering matrix of G. Replacing each non-zero entry in the matrix R by 1 and diagonal entries by 0, we obtain the matrix $A_S(G)$ defined above. From this it follows that except for diagonal elements the matrix A_S is the $(0, 1)$ analogue of the matrix R (see Spectral Graph Theory and the Inverse Eigenvalue Problem of a Graph [2, 3]). This gives another motivation for the study/discussion of the graphs associated with the matrix $A_S(G)$.

Since the graph G^S associated with the $A_S(G)$ is the spanning supergraph of the graph G, then clearly $|V(G)| = |V(G^S)|$ and $m(G^S) \geq m(G)$. At the first sight, the following problems about MDMC-graph G^S of the graph G will be of interest.

1. Knowing the graph G and MDMC-set S, what can we say about the degrees of the vertices of G^S.
2. To obtain an upper bound for the number of edges $m(G^S)$ of G^S.
3. Is the graph G^S always Hamiltonian, Eulerian, bipartite.
4. If $\alpha_0^S,\ \alpha_1^S,\ \beta_0^S$ and β_1^S ($\alpha_0,\ \alpha_1,\ \beta_0$ and β_1) are the vertex covering number, the edge covering number, the vertex independence number and the edge independence number of G^S (respectively G), then to find the relation between these parameters.
5. To find the relation between the chromatic and domination numbers of the graphs G^S and G.
6. How the spectra of G^S and G under various graph matrices are related.
7. When is the graph G^S regular.
8. If $G_1 \cong G_2$, then obviously $G_1^S \cong G_2^S$. In case $G_1^S \cong G_2^S$, what about G_1 and G_2 are they isomorphic.
9. What can be the relation between the vertex connectivity (edge connectivity) of G^S and G.
10. How the line graph of G^S and the line graph of G are related.

11. For any two graphs G and H, what are the possible relations between the graphs G^S and H^S under various graph operations with G and H.

There are many other graph theoretical and spectral questions that one can ask about the graph G^S. Here we answer some of these questions.

The subgraph of G whose vertex set U and whose edge set is the set of those edges of G that have both ends in U is denoted by $\langle U \rangle$ and is called the subgraph of G induced by U. A subset U of $V(G)$ is called an independent set of G if no two vertices of U are adjacent in G. An independent set with maximum cardinality among all the independent sets of G is called the maximum independent set and its cardinality is called the (vertex)independence number of G, denoted by β_0.

In the rest of this paper, the set $S \subset V(G)$ will denote the MDMC-set of the graph G, unless otherwise stated. If two vertices u and v are adjacent, we denote it by $u \sim v$ and the edge between them by $e = uv$. We denote the complete graph on n vertices by K_n, the empty graph on n vertices by $\overline{K_n}$, the path on n-vertices by P_n, the cycle on n vertices by C_n, the complete bipartite graph with partite sets of cardinalities p and q, $p + q = n$ by $K_{p,q}$ and the graph obtained by joining each vertex of K_p with every vertex of $\overline{K_q}$ by $K_p \vee \overline{K_q}$, such a graph is called the complete split graph. For other undefined notations and terminology from graph theory, the readers are referred to [1, 6].

The paper is organized as follows. In Section 2, some basic properties of G^S are considered. In Section 3, we study the degree sequence and obtain an upper bound for the number of edges in G^S in terms of the structure of G and characterise the extremal graphs. In Section 4, we obtain the conditions for the MDMC-graph G^S to be bipartite and Hamiltonian. Lastly, in Section 5, we obtain an upper bound for the covering number (independence number) of G^S in terms of the covering number (respectively independence number) of G and discuss the equality case.

2. Basic properties of MDMC-graphs

In this section, we discuss some basic properties of the MDMC-graph of a graph G. Let G^S be the MDMC-graph of G with respect to MDMC-set $S = \{v_1, v_2, \ldots, v_k\}$. Using the fact that G^S is obtained by adding edges between the non adjacent vertices of G which share a common neighbour in S, we have the following relations which can easily verified:

For any MDMC-set S, the MDMC-graphs of the complete graph and empty graph are respectively the complete graph and empty graph that is, $K_n^S = K_n$ and $\overline{K_n}^S = \overline{K_n}$. For the complete bipartite graph $K_{p,q}$ with $p \leq q$, the MDMC-set S is the partite set with cardinality p and the MDMC-graph $K_{p,q}^S$ is the complete split graph $K_q \vee \overline{K_p}$. In particular $K_{1,n-1}^S = K_n$. For the path $P_n = \{u_1, u_2, \ldots, u_n\}$, if n is odd, the MDMC-set is $S = \{u_2, u_4, \ldots, u_{n-1}\}$ and the MDMC-graph P_n^S is the graph $P_n \cup \{u_1u_3, u_3u_5, \cdots, u_{n-2}u_n\}$. Clearly P_n^S consists of $\lfloor \frac{n}{2} \rfloor$ copies of K_3. On the other hand if n is even, the MDMC-set is $S = \{u_2, u_4, \ldots, u_{n-2}, u_{n-1}\}$ and the MDMC-graph P_n^S is the graph $P_n \cup \{u_1u_3, u_3u_5, \ldots, u_{n-3}u_{n-1}, u_{n-2}u_n\}$. It is easy to see that P_n^S consists of $\frac{n}{2}$ copies of K_3. For the cycle $C_n = \{u_1, u_2, \ldots, u_n, u_1\}$, if n is even, the MDMC-set is $S = \{u_2, u_4, \ldots, u_{n-2}\}$ and the MDMC-graph C_n^S is the graph $C_n \cup \{u_1u_3, u_3u_5, \ldots, u_{n-1}u_1\}$.

On the other hand if $n \geq 5$ is odd, the MDMC-set of C_n^S is $S = \{u_1, u_3, \ldots, u_{n-2}, u_n\}$ and the MDMC-graph C_n^S is the graph $C_n \cup \{u_2u_4, u_4u_6, \ldots, u_{n-3}u_{n-1}, u_{n-1}u_1, u_nu_2\}$. We have seen that the MDMC-graph of a complete graph is the complete graph itself, however if W_n is the wheel graph on n vertices, then $W_n^S = K_n$. Therefore, we have the following observation.

Lemma 2.1. If G contains a dominant vertex, that is, a vertex of degree $n-1$, then $G^S = K_n$.
Proof. Suppose that G contains a vertex v of degree $n-1$. Then the set S being an MDMC-set must contain the vertex v. Since every other vertex of G is adjacent to v, it follows that each vertex of G shares at least one vertex with S. Therefore by the definition of G^S, the result follows. □

From the definition, it is clear that if $G_1 \cong G_2$, then $G_1^{S_1} \cong G_2^{S_2}$, where S_1 and S_2 are respectively the MDMC-sets in G_1 and G_2. However if $G_1^{S_1} \cong G_2^{S_2}$, then G_1 need not be isomorphic to G_2, as is clear from Lemma 2.1.

3. Degrees and conditions for MDMC-graph to be bipartite

Let $S = \{v_1, v_2 \ldots, v_k\}$ be an MDMC-set of G. For $i = 1, 2, \ldots, n$, let $d(v_i)$ and $d'(v_i)$ be respectively, the degrees of the vertices of the graphs G and G^S. For any two vertices v_i and v_j, let $\pi_{v_i}(v_j) = \{v_k \in V(G) : v_k$ is adjacent to v_j; and v_k is not adjacent to $v_i\}$, that is, $\pi_{v_i}(v_j)$ is the set of neighbours of v_j which are not the neighbours of v_i and let $\theta(v_i) = \sum_{v_iv_jv_sv_tv_i} 1$ be the number of 4-cycles in G containing the vertex v_i, with $v_j, v_t \in S$ and v_i not adjacent to v_s. Using the fact G^S is obtained from G by adding edges between non-adjacent vertices which have a common neighbour in S, we have the following observations.

$$d'(v_i) = \begin{cases} \sum\limits_{\substack{v_j \in S \\ v_j \sim v_i}} d(v_j) - \theta(v_i), & \text{if } v_i \in V(G) - S, \\ d(v_i), & \text{if } v_i \in S, \end{cases} \tag{1}$$

if S is an independent set in G and

$$d'(v_i) = d(v_i) + \sum_{\substack{v_j \in S \\ v_j \sim v_i}} |\pi_{v_i}(v_j)| - \theta(v_i), \text{ for all } v_i \in V(G), \tag{2}$$

if S is not an independent set in G.

Using this observation, we have the following result.

Theorem 3.1. If $d(v_1), d(v_2), \ldots, d(v_n)$ are the degrees of the vertices of graph G having MDMC-set $S = \{v_1, v_2, \ldots, v_k\}$, then $d'(v_)), d'(v_2), \ldots, d'(v_n)$ are the degrees of the vertices of the graph G^S, where for $i = 1, 2, \ldots, n$, $d'(v_i)$ are given by equation (1), if S is independent and by equation (2), if S is not independent.

Example 3.2. Consider the graph G in Figure 1, the degrees of the vertices of G are $3, 3, 2, 2, 1, 1$, with MDMC-set $S = \{v_1, v_3, v_4\}$, where v_i corresponds to vertex i. Since the set S is not independent in G, the degree of the vertices v_1, v_2, v_3, v_4, v_5 and v_6 in G^S are given by $d'(v_1) = d(v_1) + \sum\limits_{\substack{v_j \in S \\ v_j \sim v_1}} |\pi_{v_1}(v_j)| - \theta(v_1) = 2 + |\pi_{v_1}(v_4)| - \theta(v_1) = 2 + 2 - 0 = 4$, as v_4 is the only vertex in S adjacent to v_1 and there is no 4-cycle $v_1 v_j v_r v_s v_1$, with $v_j, v_s \in S$ and v_1 not adjacent to v_r. Also $d'(v_2) = d(v_2) + \sum\limits_{\substack{v_j \in S \\ v_j \sim v_2}} |\pi_{v_2}(v_j)| - \theta(v_2) = 2 + |\pi_{v_2}(v_1)| + |\pi_{v_2}(v_3)| - \theta(v_2) = 2 + 1 + 2 - 1 = 4$, as $v_1, v_3 \in S$ are adjacent to v_2 and there is only one 4-cycle of the form $v_2 v_j v_r v_s v_2$, with $v_j, v_s \in S$ and v_2 not adjacent to v_r. Proceeding similarly, it can be seen that the degrees of the vertices v_3, v_4, v_5 and v_6 are respectively as $5, 5, 3$ and 3. Thus the degrees of the vertices of the graph G^S are $5, 5, 4, 4, 3, 3$, which is clear from the graph G^{S_1} in Figure 1.

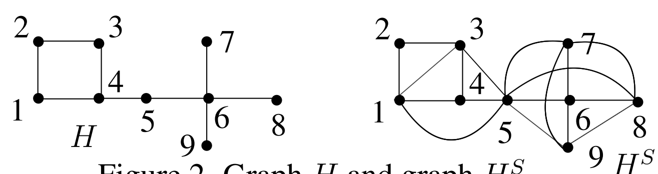

Figure 2. Graph H and graph H^S.

Example 3.3. Consider the graph H in Figure 2, the degrees of the vertices of the graph H are $4, 3, 2, 2, 2, 2, 2, 1, 1, 1$, with MDMC-set $S = \{v_2, v_4, v_6\}$, where v_i corresponds to vertex i. Since the set S is independent in H, the degree of the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ and v_9 in H^S are given by $d'(v_2) = d(v_2) = 2$, $d'(v_4) = d(v_4) = 3$, $d'(v_6) = d(v_6) = 4$, $d'(v_1) = \sum\limits_{\substack{v_j \in S \\ v_j \sim v_1}} d(v_j) - \theta(v_1) = d(v_2) + d(v_4) - \theta(v_1) = 2 + 3 - 1 = 4$, as $v_2, v_4 \in S$ are adjacent to v_1 and there is only one 4-cycle of the form $v_1 v_j v_r v_s v_1$, with $v_j, v_s \in S$ and v_1 not adjacent to v_r. Proceeding similarly, it can be seen that the degrees of the vertices v_3, v_5, v_7, v_8 and v_9 are $4, 7, 4, 4$ and 4. Thus the degrees of the vertices of the graph H^S are $7, 4, 4, 4, 4, 4, 4, 3, 2$, which is clear from the Figure 2.

We now obtain an upper bound for the number of edges $m(G^S)$ of the graph G^S and characterise the extremal graphs which attain this bound.

Theorem 3.4. For $k < n$, let $S = \{v_1, v_2, \ldots, v_k\}$, be the MDMC-set of the graph G and let G^S be the MDMC-graph of G.

(i) If S is an independent set in G, then $2m(G^S) \leq k(n-k)(n-k+1) - \sum\limits_{v_i \in V(G) - S} \theta(v_i)$, with equality if and only if $G \cong K_{k,n-k}$, and

(ii) if S is not an independent set in G, then $2m(G^S) \leq 2m + k(\Delta - 1)(n-1) - \sum\limits_{v_i \in V(G)} \theta(v_i)$, with equality if and only if G is a graph with each vertex $v_i \in S$ of same degree $\Delta = \max\{d_i, i = 1, 2, \ldots, n\}$ and $\langle S \rangle = K_k$, such that $\pi_{v_i}(v_j) = \phi$, for all $v_i \in V(G)$ and $v_j \in S$.

Proof. (i). For $i = 1, 2, \ldots, n$, let $d(v_i)$ and $d'(v_i)$ be respectively the degrees of the vertices of the graphs G and G^S. Since $\sum_{v_i \in V(G)} d_i = 2m$ and S is an independent set in G, from equation (1) it follows that

$$\begin{aligned}
2m(G^S) &= \sum_{v_i \in V(G^S)} d'(v_i) = \sum_{v_i \in S} d'(v_i) + \sum_{v_i \in V(G^S)-S} d'(v_i) \\
&= \sum_{v_i \in S} d(v_i) + \sum_{v_i \in V(G)-S} \left(\sum_{\substack{v_j \in S \\ v_i \sim v_j}} d(v_j) - \theta(v_i) \right) \\
&\leq k(n-k) + k(n-k)(n-k) - \sum_{v_i \in V(G)-S} \theta(v_i) \\
&= k(n-k)(n-k+1) - \sum_{v_i \in V(G)-S} \theta(v_i).
\end{aligned}$$

Equality will occur if and only if

$$\sum_{v_i \in S} d(v_i) = k(n-k) \qquad \textit{and} \qquad \sum_{\substack{v_j \in S \\ v_i \sim v_j}} d(v_j) = k(n-k)(n-k).$$

Since S is an independent set with $|S| = k$, the first of these equalities will hold if each vertex in S is of degree $n-k$. Also the set $V(G) - S$ is an independent set in G as the set S is independent covering set. So for the second of these equalities to hold it follows from the first equality and the fact that the set $V(G) - S$ is an independent set in G having cardinality $n-k$, each vertex in $V(G) - S$ is of degree k. Thus, it follows that the sets S and $V(G) - S$ are independent, such that each vertex in S is of degree $n-k$ and each vertex in $V(G) - S$ is of degree k. This is only possible if and only if $G \cong K_{k,n-k}$. Conversely, if $G \cong K_{k,n-k}$, then it is easy to see that equality occurs.

(ii). Now, if S is not an independent set in G, it follows from equation (2) that

$$\begin{aligned}
2m(G^S) &= \sum_{v_i \in V(G^S)} d'(v_i) = \sum_{v_i \in S} d'(v_i) + \sum_{v_i \in V(G^S)-S} d'(v_i) \\
&= \sum_{v_i \in S} \left(d(v_i) + \sum_{\substack{v_j \in S \\ v_i \sim v_j}} |\pi_{v_i}(v_j)| - \theta(v_i) \right) + \sum_{v_i \in V(G)-S} \left(d(v_i) + \sum_{\substack{v_j \in S \\ v_i \sim v_j}} |\pi_{v_i}(v_j)| - \theta(v_i) \right) \\
&= 2m + \sum_{v_i \in S} \sum_{\substack{v_j \in S \\ v_i \sim v_j}} |\pi_{v_i}(v_j)| + \sum_{v_i \in V(G)-S} \sum_{\substack{v_j \in S \\ v_i \sim v_j}} |\pi_{v_i}(v_j)| - \sum_{v_i \in V(G)} \theta(v_i) \\
&\leq 2m + \sum_{v_i \in S} \sum_{\substack{v_j \in S \\ v_i \sim v_j}} (d(v_j) - 1) + \sum_{v_i \in V(G)-S} \sum_{\substack{v_j \in S \\ v_i \sim v_j}} (d(v_j) - 1) - \sum_{v_i \in V(G)} \theta(v_i)
\end{aligned}$$

$$\leq 2m + k(k-1)(\Delta-1) + k(n-k)(\Delta-1) - \sum_{v_i \in V(G)} \theta(v_i)$$
$$= 2m + k(n-1)(\Delta-1) - \sum_{v_i \in V(G)} \theta(v_i).$$

Equality occurs if and only if $|\pi_{v_i}(v_j)| = d(v_j) - 1 = \Delta - 1$, $\sum\limits_{v_i \in S} \sum\limits_{\substack{v_j \in S \\ v_i \sim v_j}} (d(v_j) - 1) = k(k-1)(\Delta-1)$ and $\sum\limits_{v_i \in V(G)-S} \sum\limits_{\substack{v_j \in S \\ v_i \sim v_j}} (d(v_j) - 1) = k(n-k)(\Delta-1)$. The first of these equalities implies that $v_j \in S$ has no common neighbour with any $v_i \in V(G)$ and $d(v_j) = \Delta$. The second equality implies that $\langle S \rangle$ is a complete graph on k-vertices and the third equality implies that every vertex $v_i \in V(G) - S$ is adjacent to each vertex in S. Combining all these we obtain the graph as mentioned in the hypothesis. □

The following is an immediate consequence of part (i) of the Theorem 3.4.

Corollary 3.5. If n is even and $S = \{v_1, v_2, \ldots, v_{\frac{n}{2}}\}$ is an independent MDMC-set in G, then $2m(G^S) \leq \frac{1}{8}n^2(n+2) - \sum\limits_{v_i \in V(G)-S} \theta(v_i)$, with equality if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Let $N(v_i) = \{v_j \in V(G) : v_j \sim v_i\}$ be the neighbourhood of v_i in G and let G be a tree with r-pendent vertices. We have the following observation about the number of edges in G^S.

Theorem 3.6. Let G be a tree with r-pendant vertices and let $S = \{v_1, v_2, \ldots, v_k\}$ be a MDMC-set in G.

(i) If S is an independent set, then $2m(G^S) \leq \Delta(2n - k - r)$, with equality if and only if every vertex in S is of degree $\Delta = \max\{d_i, i = 1, 2, \ldots, n, \}$ and

(ii) if S is not an independent set, then $2m(G^S) \leq 2m + 2(\Delta-1)(k-1) + \Delta(2n - 2k - r)$, with equality if and only if every vertex in S is of degree Δ and $\langle S \rangle = P_k$, a path of length $k-1$.

Proof. (i). If S is an independent MDMC-set in G which is a tree with r-pendant vertices, then $d'(v_1) = d(v_i)$, for all $v_i \in S$ and no pendant vertex of G is in S. If $v_i \in V(G) - S$ is not a pendant vertex, then $\sum\limits_{\substack{v_j \in S \\ v_i \sim v_j}} d(v_j) - \theta(v_i) \leq d(v_j) + d(v_s)$, where $v_j, v_s \in N(v_i) \cap S$ and if $v_i \in V(G) - S$ is a pendant vertex, then $\sum\limits_{\substack{v_j \in S \\ v_i \sim v_j}} d(v_j) - \theta(v_i) \leq d(v_j)$, for $v_j \in N(v_i) \cap S$. Therefore,

$$2m(G^S) = \sum_{v_i \in S} d(v_i) + \sum_{v_i \in V(G)-S} \sum_{\substack{v_j \in S \\ v_i \sim v_j}} d(v_j)$$
$$\leq k\Delta + r\Delta + 2(n-k-r)\Delta$$
$$= \Delta(2n - k - r).$$

It is easy to see that equality occurs if and only if $d(v_j) = \Delta$, for all $v_j \in S$.
(ii). If S is not an independent MDMC-set in G and $v_i \in S$, then for the vertices v_j and v_s, we have

$$\sum_{\substack{v_j \in S \\ v_i \sim v_j}} |\pi_{v_i}(v_j)| - \theta(v_i) \leq \begin{cases} d(v_j) + d(v_s) - 2, & \text{if } v_i \text{ has two neighbours in } S\text{ ,} \\ d(v_j) - 1, & \text{if } v_i \text{ has one neighbour in } S. \end{cases}$$

If $F = N(v_i) \cap S$, then for $v_i \in V(G) - S$, there is a vertex $v_s \in F$ so that

$$\sum_{\substack{v_j \in S \\ v_i \sim v_j}} |\pi_{v_i}(v_j)| - \theta(v_i) \leq \begin{cases} d(v_j) + d(v_s), & \text{if } v_j, v_s \in F,\ v_i \text{ not a pendant vertex,} \\ d(v_j), & \text{if } v_j \in F,\ v_i \text{ a pendant vertex.} \end{cases}$$

Therefore, we have

$$\begin{aligned} 2m(G^)S &= \sum_{v_i \in V(G)} d(v_i) + \sum_{v_i \in V(G)} \sum_{\substack{v_j \in S \\ v_i \sim v_j}} |\pi_{v_i}(v_j)| - \sum_{v_i \in V(G)} \theta(v_i) \\ &\leq 2m + (k-2)(2\Delta - 2) + 2(\Delta - 1) + 2\Delta(n - k - r) + r\Delta \\ &= 2m + 2(\Delta - 1)(k - 1) + \Delta(2n - 2k - r). \end{aligned}$$

Equality will occur if and only if each vertex of S is of degree Δ and $\langle S \rangle$ is connected. Since G is a tree, therefore $\langle S \rangle$ must be a path on k vertices and every vertex not in S should be a pendant vertex. □

A graph G is said to be bipartite if its vertex set $V(G)$ can be partitioned in two disjoint subsets V_1 and V_2, such that every edge in G has one end in V_1 and another in V_2. It is well known that a graph G is bipartite if and only if it contains no odd cycles (cycles with odd number of vertices) [5]. The following result characterizes the bipartite MDMC-graphs.

Theorem 3.7. Let G be a connected graph and S be an MDMC-set in G. Then G^S is bipartite if and only if $G \cong K_2$.
Proof. Let G^S be the MDMC-graph of G. Since G is connected, it follows that the graph G^S is connected. If G^S is bipartite, then it contains no odd cycles. We claim that G contains no vertex v_i, such that $d(v_i) \geq 2$. If possible suppose there is a vertex (say) $v_j \in V(G)$, such that $d(v_j) \geq 2$. By definition, the graph G^S is obtained from the graph G by adding edges between the non-adjacent vertices in G which share a neighbour in S, so we have the following cases to consider.

Since $d(v_j) \geq 2$, there are at least two vertices $v_r, v_s \in V(G)$ which are adjacent to v_j. Clearly v_r is not adjacent to v_s, because if they are adjacent, then $v_j v_r v_s v_j$ will be a 3-cycle in G and hence in G^S, which is bipartite. If $v_j \in S$, then v_r and v_s share a common neighbour v_j in S and so they are adjacent in G^S. Therefore, giving a 3-cycle in G^S, which is bipartite, a contradiction.

On the other hand, if $v_j \notin S$, then both v_r and v_s must be in S. Since v_r is not adjacent to v_s, there must exist vertices $v_l, v_t \in V(G)$, such that v_l is adjacent to v_r; and v_t is adjacent to v_s, for otherwise S can not be an MDMC-set in G. Therefore, it follows that v_j and v_l share a common neighbour in S and so will be adjacent in G^S, giving rise to a 3-cycle, again a contradiction. Thus, if the connected graph G^S is bipartite, then the graph G is connected with no vertex of degree greater than or equal to two. It is easy to see that the only possible graph with this property is K_2. Converse follows from the fact that $K_2^S = K_2$. □

4. Characterization of Hamiltonian MDMC-graphs

A graph G is said to be Eulerian if and only if each of its vertex is of even degree [1, 6, 7]. If the graph G is Eulerian and S is a MDMC-set in G, then the graph G^S need not be Eulerian. For example, consider the 4-cycle C_4 which is Eulerian, but $C_4^S = K_4 - e$, where e is an edge, is not Eulerian. It is clear from the degrees of the vertices of the graph G^S that if the MDMC-set S is an independent set in G, then G^S is Eulerian if and only if every vertex in S is of even degree and there are even number of 4-cycles of the form $v_iv_jv_rv_sv_i$, with $v_j, v_s \in S$ and v_i is not adjacent to v_r. However, if S is not independent in G, the characterization of G^S to be Eulerian seems a difficult problem and so we have the following.

Problem 4.1. If S is not an independent set in G, characterize the graphs G such that G^S is Eulerian?

A graph G is said to be Hamiltonian if it contains a spanning cycle (a cycle which passes through all the vertices) [1, 7]. Since the graph G is a spanning subgraph of the graph G^S, it follows that if G is Hamiltonian then G^S is also Hamiltonian. However, if G^S is Hamiltonian, then G need not be so. For example the graph $G^S = K_n$ is Hamiltonian, but the graph $G = K_{1,n-1}$ is non-Hamiltonian with MDMC-set S consisting of a single vertex. The Hamiltoniancity of the graph G^S depends in general on the MDMC-set S of the graph G, which can be seen in the following result.

Theorem 4.2. Let G be a connected graph and let $S = \{v_1, v_2, \ldots, v_k\},\ k < n$ be an MDMC-set in G:

(I) If S is an independent set, then G^S is Hamiltonian if every vertex of the graph $\langle S \rangle$ lies on a cycle and there is no non-pendent cut edge, otherwise it is non-Hamiltonian.

(II) If S is not an independent set, then the graph G^S is Hamiltonian, if either $\langle S \rangle$ is a connected subgraph of G or $\langle S \rangle$ consists of a connected component together with some isolated vertices which lie on cycles and there is no non-pendent cut edge.

Proof. (I). Let $S = \{v_1, v_2, \ldots, v_k\}$ be an independent set in G and let each vertex of the induced subgraph $\langle S \rangle$ lie on some cycle in G. Suppose that G does not contain a non-pendent cut edge. Since S is an independent set, the graph G is either 1-connected or 2-connected [1, 7].

Case (i). If G is a 1-connected graph having no pendant vertex then there will be a vertex $v_i \in S$ which is the cut vertex and which lies on at least two cycles. Let $u_{i_j},\ w_{i_j}$ be the neighbours of the vertex v_i on the cycles $H_j,\ j \geq 2$. Clearly these vertices will be mutually adjacent in the graph G^S and thus forms a cycle around v_i, which traces all these vertices. This cycle together with the cycles in G containing v_i gives a Hamiltonian cycle in G^S around v_i. Since every vertex of S is either a cut vertex, which lies on more than one cycle or is a non cut vertex which lies on one or more cycles in G, it follows that the above process can be continued for each of the vertices $v_i \in S$, which is a cut vertex. In this way we obtain Hamiltonian cycles around each of the cut vertices $v_i \in S$. These cycles together with the cycles holding other vertices of S in G gives a Hamiltonian cycle in G^S. On the other hand if G has pendant vertices, then again there will be a vertex $v_i \in S$ which is the cut vertex and which lies on at least two cycles; or at least two cycles and some pendant edges; or a cycle and some pendant edges.

Subcase (i). If v_i lies on at least two cycles and there are pendant edges on the other vertices in S, then G^S is Hamiltonian follows from the above case and the fact that every pendant vertex will be adjacent to at least two vertices on the cycle in G^S and the pendant vertices on the same vertex will be mutually adjacent in G^S.

Subcase (ii). If v_i lies on at least two cycles and some pendant edges, then there can be pendant edges on the other vertices in S. Let $u_{i_j},\ w_{i_j}$ be the neighbours of the vertex v_i on the cycles H_j, $j \geq 2$ and t_{i_j}, $j \geq 1$ be the neighbours of v_i which corresponds to pendant edges. Since v_i is the common neighbour of the vertices u_{i_j} ,w_{i_j}, $j \geq 2$ and t_{i_j}, $j \geq 1$, they will be mutually adjacent in G^S and thus forms a cycle around v_i which traces all these vertices. Also since every pendant vertex will be adjacent to at least two vertices on the cycle in G^S and the pendant vertices on the same vertex will be mutually adjacent in G^S they will form a cycle. Since G is connected these cycles together gives a Hamiltonian cycle in G^S.

Subcase (iii). The case when v_i lies on a cycle and some pendant edges follows similar to the cases considered above.

Case (ii). If G is 2-connected with no pendant vertices, since S is independent with each vertex on a cycle, the graph G is itself Hamiltonian and so will be the graph G^S. On the other hand if G is a 2-connected graph having pendant vertices, then the graph G will contain a cycle tracing all the vertices of G other than the pendant vertices. Also any pendant vertex at $v_i \in S$ in G will be adjacent to at least two vertices on the cycle in G^S and the pendant vertices adjacent at the same vertex will be mutually adjacent in G^S, so they will induce a complete graph with the neighbours of v_i in G^S. These complete graphs at each such vertex $v_i \in S$ together with the cycle containing the vertices of S gives the Hamiltonian cycle in G^S.

Now, suppose that S is an independent set in G having at least one vertex say v_t which does not lie on a cycle in G. Let $u_i \in V(G) - S,\ i \geq 2$ be the neighbours of v_t in G. Clearly none of u_i will be on a cycle in G, because if some u_i lie on a cycle in G then it must be in S, which is not the case. Since G is connected, at least one of u_i, say u_1, will be adjacent to some $v_j \in S$. In the graph G^S all the $u_i' s$ are mutually adjacent and thus $u_i' s$ together with v_t induces a complete graph. Let this complete graph be H_1 . Also the vertex u_1 will be adjacent to all the neighbours of the vertex v_j and thus forms another complete graph H_2. The complete graphs H_1 and H_2 so obtained have the property that they have one common vertex namely u_1 and there is no edge having one end in H_1 and another in H_2. Thus, in G^S the induced subgraph H on the vertex set $V(H_1) \cup V(H_2)$ will disturb the Hamiltonicity of G^S (because a graph obtained by fusing a vertex of a Hamiltonian graph with a vertex of another Hamiltonian graph is not Hamiltonian) [1, 3, 7] .

(**II**). Let S be not independent set in G such that the induced subgraph $\langle S \rangle$ is connected. Without loss of generality, assume that $\langle S \rangle = P_k = v_1 v_2 \dots v_k$. We have the following cases to consider.

Case (i). Let us suppose that the graph G has no cycle. Let $v_1,\ v_2$ be any two vertices of S and let u_i, $i = 1, 2, \ldots, d_1$ and w_j, $j = 1, 2, \ldots, d_2$ be respectively the neighbours of the vertices v_1 and v_2, where $u_1 = v_2$ and $w_1 = v_1$. Since G is acyclic, the vertices $u_i,\ i = 1, 2, \ldots, d_1$ are mutually non-adjacent in G with a common neighbour $v_1 \in S$, so they are mutually adjacent in G^S. Indeed these vertices together with v_1 will induce a complete graph of order $d_1 + 1$, say H_1. Similarly, the neighbours w_j, $j = 1, 2, \ldots, d_2$ of v_2 will be mutually adjacent in G^S and so together with v_2 induces a complete graph of order $d_2 + 1$, say H_2. Let $H = \langle V(H_1) \cup V(H_2)\rangle$. We claim that H is Hamiltonian. Being complete graphs, both H_1 and H_2 are Hamiltonian. Let $v_1u_2 \ldots u_{d_1}u_1v_1$ be a Hamiltonian cycle in H_1 and $v_2w_2 \ldots w_{d_2}w_1v_2$ be a Hamiltonian cycle in H_2. Since $u_1 = v_2$ and $w_1 = v_1$, we get $v_1u_2 \ldots u_{d_1}u_1 = v_2w_2 \ldots w_{d_2}w_1 = v_1$ as a Hamiltonian cycle in H. Thus if $k = 2$, the graph $G^S = H$ is Hamiltonian. Assume that the result holds if $S = \{v_1, v_2, \ldots, v_{k-1}\}$. We show it also holds for $S = \{v_1, v_2, \ldots, v_k\}$. For $i = 1, 2, \ldots, k$, let H_i be the complete graph induced by the neighbours of v_i together with v_i. Let $U = \langle V(H_1) \cup V(H_2) \cdots \cup V(H_{k-1})\rangle$. By induction hypothesis, the graph U is Hamiltonian. Let $X = \langle V(U) \cup V(H_k)\rangle$. By the case $k = 2$, it follows that the graph X is Hamiltonian. Since $X = G^S$, it follows that the graph G^S is Hamiltonian.

Case (ii). On the other hand if G contains cycles, then the vertices in S can have common neighbours. Let u_i $(1 \le i \le t)$ and w_j $(1 \le j \le r)$, $t + r = d_1$ be the neighbours of $v_1 \in S$; and q_i $(1 \le i \le p)$ and w_j $(1 \le j \le r)$, $p + r = d_2$ be the neighbours of $v_2 \in S$, where $u_t = v_2$ and $q_p = v_1$. As two non-adjacent vertices having a common neighbour in S are made adjacent in G^S, it follows that the graph Y_1 induced by the neighbours of v_1 together with v_1 will be a complete graph of order $d_1 + 1$ and therefore Hamiltonian. Let $v_1u_1u_2 \ldots u_tw_1w_2 \ldots w_rv_1$ be a Hamiltonian cycle in Y_1. Similarly let $v_2q_1q_2 \ldots q_pw_1w_2 \ldots w_rv_2$ be a Hamiltonian cycle in the graph Y_2 induced by the neighbours of v_2 together with v_2. Then $v_1u_1u_2 \ldots u_t = v_2w_1w_2 \ldots w_rq_1q_2 \ldots q_p = v_1$ is a Hamiltonian cycle in $Y = \langle V(Y_1) \cup V(Y_2)\rangle$. Proceeding inductively as above, we see that the result follows in this case as well.

Lastly, suppose that the graph induced by the vertices in S consists of a connected component and some isolated vertices, which lie on cycles and there is no non-pendent cut edge in G. Let $\langle S\rangle = \langle S_1\rangle \cup \{v_{t+1}, v_{t+2}, \ldots, v_k\}$, where $\langle S_1\rangle$ is the connected component of $\langle S\rangle$ induced by $v_1, v_2, \ldots, v_t$; and $v_{t+1}, v_{t+2}, \ldots, v_k$ are the isolated vertices, which lie on the cycles in G. The result now follows by using case (i) of part I and case (i) and (ii) of part II and the fact that G is connected. □

From the above theorem, it is clear that the Hamiltoniancity of the supergraph G^S depends upon the induced graph $\langle S\rangle$.

5. Independence and Covering number of MDMC-graphs

An independent set of vertices in G with maximum cardinality is called maximum independent set (or vertex independent set) and its cardinality is called independence number of G and is denoted by $\beta_0 = \beta_0(G)$ [1, 6, 7]. The cardinality of a minimum (vertex) covering set in G is called covering number of G and is denoted by $\alpha_0 = \alpha_0(G)$. It is easy to see that the set S is a minimum covering set in G if and only if $V(G) - S$ is a maximum independent set in G [1, 6, 7]. So if $|V(G)| = n$, then

$$\alpha_0 + \beta_0 = n. \tag{3}$$

We first obtain a connection between the vertex covering number α_0^S of G^S and the vertex covering number α_0 of G.

Theorem 5.1. Let S be an MDMC-set of a connected graph G ($G \neq K_n$) having independence number β_0 and covering number α_0 and let α_0^S be the covering number of the graph G^S. Then $\alpha_0^S = n - \alpha_0 = \beta_0$, if either S is independent; or $\langle S \rangle = P_k$ and G is acyclic.

Proof. For $k = \alpha_0$, let $S = \{v_1, v_2, \ldots, v_k\}$ be an independent MDMC-set of G and let $S' = V(G) - S = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$ be the complement of S in G. Since the set S is a maximum degree minimum covering set, it is a minimum covering set, therefore it follows from equation (3) the set S' is a maximum independent set of G, and $\alpha_0 + \beta_0 = n$, where $\beta_0 = n - k$. The set S being independent implies each vertex $v_i \in S$, $i = 1, 2, \ldots, k$ has its neighbours among the vertices $S' = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$, so the set S' is also a covering set of G. As the graph G^S is obtained from G by joining pairs of non-adjacent vertices which have a common neighbour in S, it follows that any two vertices $v_j, v_t \in S'$, $1 + k \leq j, t \leq n$, which have a common neighbour in S are adjacent in G^S, while as the vertices within S will remain non-adjacent in G^S and so the set S will be independent in G^S. Clearly the set S' is a covering set in G^S, because G^S is simply G together with some additional edges between the vertices in S'.

We claim that the set S' is a minimum covering set of G^S. If not, let X' be a covering set of G^S with $|X'| < |S'|$ and let $X = V(G^S) - X'$ be its complement in G^S. By equation (3) the set X is an independent set of G^S with $|S| < |X|$. Clearly the set X can not contain all the vertices $v_i \in S$, $i = 1, 2, \cdots, k$, because if it is so, then $X = S \cup \{u_i : u_i \in S', i \geq 1\}$. S-ince X is independent in G^S it is so in G and therefore some $u_i \in S'$ will not be adjacent with any of the vertices in S, which is not possible as S is an MDMC-set in G. So X must be of the from $X = \{u_1, u_2, \ldots, u_t, w_1, w_2, \ldots, w_r\}$, where $u_i \in S$, $w_j \in S'$ and $t + r > k$. If q_{i_j}, $(j = 1, 2, \ldots, d_i)$ are the neighbours of $v_i \in S$ for all $i = 1, 2, \ldots, k$, then in the graph G^S the vertices q_{i_j}, $(j = 1, 2, \ldots, d_i)$ will induce a complete graph together with v_i. For $i = 1, 2, \ldots, k$, let H_i be the complete graphs induced by the neighbours of v_i with v_i. Since independence number of a complete graph is one and independence number of a graph obtained by either fusing a vertex or an edge of two complete graphs is two, it follows that the independence number of the graph obtained by either fusing a vertex or an edge of H_i and H_j $(i, j = 1, 2, \ldots, k, i < j)$, in a chain is exactly k. Now G is connected implies that G^S is connected, and we have $G^S = \langle V(H_1) \cup V(H_2) \cup \cdots \cup V(H_n) \rangle$, in fact if $S = \{v_1, v_2, \ldots, v_k\}$, then G^S is obtained by either fusing an edge or a vertex (depending whether the neighbours of v_i lie on a cycle or not) of the complete graphs H_i corresponding to the vertices v_i, $i = 1, 2, \ldots, k$. So it follows that the independence number of the graph G^S is k, a contradiction, to the fact that X is an independent set of G^S with cardinality $|X| > |S| = k$. This verifies our claim. Thus it follows that the set S' is a minimum covering set of G. Since $|S'| = n - \alpha_0$, it follows from equation (3), $\alpha_0^S = \beta_0$.

On the other hand suppose that G is acyclic and for $(k = \alpha_0)$, $S = \{v_1, v_2, \ldots, v_k\}$, is an

MDMC-set of G, such that $\langle S\rangle = P_k$. Let $S' = V(G) - S = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$, be the complement of S in G. Since G is acyclic and $\langle S\rangle = P_k$, it follows that each of the vertices $v_j \in S'$ is a pendant vertex in G. Let f_{i_j}, $j = 1, 2, \ldots, d_i$, be the neighbours of the vertices v_i in G and let H_i, $i = 1, 2, \cdots, k$, be the complete graphs induced by the neighbours of v_i together with v_i, such that if v_t and v_s are consecutive in P_k, then the induced subgraph $H = \langle V(H_t) \cup V(H_s)\rangle$ has independence number two. Proceeding inductively, and using $G^S = \langle V(H_1) \cup V(H_2) \cup \cdots \cup V(H_k)\rangle$, we conclude that the independence number of the graph G^S is k. Now using equation (3) the result follows. □

Since for a bipartite graph α_0(vertex covering number)=β_1(edge independence number) and α_1(edge covering number)=β_0(vertex independence number)[7], we have the following observation.

Corollary 5.2. If G is a bipartite graph having vertex (edge) covering number α_0(respectively α_1) and vertex (edge) independence number β_0 (respectively β_1), then $\beta_1^S = \beta_0$, where β_1^S is the edge independence number (that is, matching number) of the graph G^S and S is an independent MDMC-set.

From Theorem 5.1, it follows that, if G is a graph having vertex covering number same as the vertex independence number, then the supergraph G^S also has vertex covering number same as the vertex independence number. The importance of this fact can be seen as follows.

In a graph G that represents a road network between cities (with straight roads and no isolated vertices), we can interpret the problem of finding a minimum vertex cover as the problem of placing the minimum number of policemen to guard the entire road network. Suppose that the cardinality of an independent minimum vertex cover S (or a minimum vertex covering set S with $\langle S\rangle = P_k$ and G is a acyclic) for G is known. If we want to construct roads between the non-adjacent cities, with out effecting the cardinality of the minimum vertex cover, then in order to obtain such a road network we need to construct the graph G^S.

If S is an MDMC-set of the graph G, define Ω=$\{(v_i, v_j) : v_i, v_j \in S, v_i \sim v_j, i < j$ and v_i, v_j lie on a 3-cycle$\}$. If $k_0 = |\Omega|$, then we have the following observation.

Lemma 5.3. Let α_0 and β_0 be respectively the vertex covering number and the vertex independence number of a connected graph G and let S be an MDMC-set of G. If α_0^S is the vertex covering number of the graph G^S and $\langle S\rangle = P_k$, then $\alpha_0^S \leq n - \alpha_0 + k_0$.

Proof. For $k = \alpha_0$, let $S = \{v_1, v_2, \ldots, v_k\}$, be an MDMC-set of the graph G, such that $\langle S\rangle = P_k$. Let $\Omega = \{(v_i, v_j) : v_i, v_j \in S, v_i \sim v_j, i < j$ and v_i, v_j lie on a 3-cycle$\}$ and $k_0 = |\Omega|$. Let c_q, $q = 1, 2, \ldots, k_0$ be 3-cycles in G containing the vertices $v_i, v_j \in S, v_i \sim v_j$. Since each of these 3-cycles c_q consumes exactly two vertices from S, it follows that the number of vertices of S covered by these 3-cycles are at most $2k_0$, and so the number of vertices of S not lying on a 3-cycle are at least $k - 2k_0$. For $i < j, (1 \leq i, j < k)$ and $q = 1, 2, \ldots, k_0$, let $u_{i_s}^q$, $(s = 1, 2, \ldots, d_i)$ and $w_{j_s}^q$, $(s = 1, 2, \ldots, d_j)$ be respectively the neighbours of the vertices v_i and v_j, which lie on

c_q and let f_{l_s}, $(s = 1, 2, \ldots, d_l, l \geq 1)$ be the neighbours of the vertices $v_l \in S$, which does not lie on a 3-cycle, since the graph G^S is obtained by joining pairs of non-adjacent vertices in G which have a common neighbour in S. Let $H_{i,j}$ $(i < j, 1 \leq i, j < k)$ be the subgraph induced by the neighbours of v_i and v_j together with v_i and v_j and let X_l $(l \geq 1)$, be the subgraph induced by the neighbours of v_l together with v_l in G^S. It is easy to see that the independence number of the subgraph X_l $(l \geq 1)$, is one, while as the independence number of the subgraph $H_{i,j}$ $(i < j, 1 \leq i, j < k)$ is at least one. So if β_0^S is the independence number of the graph G^S, then $\beta_0^S \geq 1.k_0 + 1.(n - 2k_0) = k - k_0$. Now using $\alpha_0^S + \beta_0^S = n$, it follows that $\alpha_0^S \leq n - k + k_0$. $\square$

Since adding edges between the vertices in S can decrease the vertex independence number, but it can simultaneously increase the number k_0, therefore, we have the following observation.

Corollary 5.4. Let S be an MDMC-set of a connected graph G having vertex covering number α_0 and vertex independence number β_0. If α_0^S is the vertex covering number of the graph G^S and $\langle S \rangle$ is connected, then $\alpha_0^S \leq n - \alpha_0 + k_0$.

Let G_1 and G_2 be any two graphs having vertex covering numbers α_0^1 and α_0^2, respectively, then the vertex covering number $\alpha_0(G) = \alpha_0$ of the graph $G = G_1 \cup G_2$, the disjoint union of G_1 and G_2 is $\alpha_0 = \alpha_0^1 + \alpha_0^2$. In fact, if α_0^i is the vertex covering number of G_i, $i = 1, 2, \ldots, k$, then the vertex covering number of $G = \bigcup\limits_{i=1}^{k} G_i$ is

$$\alpha_0 = \sum_{i=1}^{k} \alpha_0^i. \tag{4}$$

Theorem 5.5. Let S be an MDMC-set of a connected graph G having vertex covering number α_0 and vertex independence number β_0. If α_0^S is the vertex covering number of the graph G^S, then $\alpha_0^S \leq n - \alpha_0 + k_0$.

Proof. For $k = \alpha_0$, let $S = \{v_1, v_2, \ldots, v_k\}$ be an MDMC-set of the graph G and let $\langle S \rangle = \bigcup\limits_{i=1}^{t} S_i \cup Y$, where S_i are the connected components and Y is the set of isolated vertices of the induced subgraph $\langle S \rangle$. Suppose that $|Y| = g$ and $|S_i| = k_i$, $i = 1, 2, \ldots, t$. Then $g + \sum_{i=1}^{t} k_i = k$. Let G_i, $i = 1, 2, \ldots, t$ be the connected components of the graph G corresponding to the covering subsets S_i and H be the part of the graph G corresponding to the covering subset Y. By Theorem 5.1, Lemma 5.3 and equation (4) it follows that

$$\begin{aligned}\alpha_0^S &= \alpha_0(\bigcup_{i=1}^{t} G_i) + \alpha_0(H) = \sum_{i=1}^{k} \alpha_0(G_i) + \alpha_0(H) \\ &\leq \sum_{i=1}^{k} (|G_i| - k_i + k_{i_0}) + (|H| - g) = n - k + k_0,\end{aligned}$$

where $k_0 = \sum_{i=1}^{k} k_{i_0}$ and $k_{i_0} = |\Omega_i|$. □

For bipartite graphs, we have the following.

Corollary 5.6. If G is a connected bipartite graph having vertex (edge) covering number α_0 (respectively α_1) and vertex (edge) independence number β_0 (respectively β_1), then $\beta_1^S \leq \beta_0 + k_0$, where β_1^S is the edge independence number (or matching number) of the graph G^S and S is a MDMC-set.

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References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North-Holland, 1976.

[2] D. M. Cvetcović, M. Doob, I. Gutman and A. Torgašev, *Recent Results in the Theory of Graph Spectra*, North-Holland, Amsterdam, 1998.

[3] F. Harary, *Graph Theory*, Narosa Publishing House India, 2001.

[4] L. Hogben, Editor, *Handbook of Linear Algebra*, Chapman and Hall/CRC Press, Boca Raton, 2007.

[5] D. Konig, Graphen and Matrizen, *Math. Fiz. Lapok* **38** (1931) 116-119.

[6] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, Orient Blackswan, Hyderabad, India 2012.

[7] D. B. West, *Introduction to Graph Theory*, Second Edition, Pearson Prentice Hall, 2002.