Electronic Journal of Graph Theory and Applications **13** (2) (2025), 295–307

Electronic Journal of
Graph Theory and Applications

On the sigma chromatic number of the ideal-based zero divisor graphs of the ring of integers modulo *n*

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Abstract

The objective of this paper is to investigate a particular graph coloring, called sigma coloring, as applied to ideal-based zero-divisor graphs. Given a commutative ring R with (nonzero) identity and a proper ideal I of R, the graph $\Gamma_I(R)$ is defined as an undirected graph with vertex set $\{x \in R \setminus I : xy \in I \text{ for some } y \in R \setminus I\}$ and edge set $\{xy : xy \in I\}$. On the other hand, given a graph G, a sigma coloring $c : V(G) \to \mathbb{N}$ is a coloring that satisfies $\sigma(u) \neq \sigma(v)$ for any two adjacent vertices u, v in G, where $\sigma(x)$ denotes the sum of all colors $c(y)$ among all neighbors y of a vertex x. The sigma chromatic number of G is denoted by $\sigma(G)$ and is defined as the fewest number of colors needed for a sigma coloring of G. In this paper, we completely determine the sigma chromatic number of ideal-based zero-divisor graphs of rings of integers modulo n.

Keywords: sigma coloring, Ideal-based zero-divisor graphs
Mathematics Subject Classification: 05C15, 05C76, 05C25
DOI: 10.5614/ejgta.2025.13.2.5

1. Introduction

Zero-divisor graphs of finite rings continue to be a fascinating topic that interests both graph and ring theorists. A development in this area is the introduction of ideal-based zero-divisor graphs as an extension of the usual zero-divisor graphs. Studies on the structure and graph parameters of

Received: 16 June 2022, Revised: 28 June 2025, Accepted: 13 July 2025.

ideal-based zero-divisor graphs have been done, but there are still a broad range of open problems on this topic that researchers will find promising.

Recall that the *zero-divisor graph* of a commutative ring R, as defined in [1], is denoted by $\Gamma(R)$ and is defined as the graph whose vertices are the nonzero zero divisors of R and in which two vertices x and y are adjacent if and only if $xy = 0$ in R. The zero-divisor graphs of commutative rings have been studied with respect to different topics such as domination [2], and sigma colorings [9], and normalized Laplacian spectrum [12].

In [13], Redmond generalized the notion by defining *ideal-based zero-divisor graphs* as follows: Given a commutative ring R with (nonzero) identity and a proper ideal I of R, the graph $\Gamma_I(R)$ is defined as an undirected graph with vertex set $\{x \in R \setminus I : xy \in I \text{ for some } y \in R \setminus I\}$ and edge set $\{xy : xy \in I\}$. Ideal-based zero-divisor graphs have been investigated with respect to different notions such as parameter and girth [6], domination [11], complementedness [14], and crosscap [16]. However, there are seemingly few studies that focus on colorings of ideal-based zero divisor graphs.

On the other hand, given a graph G, a *sigma coloring* $c : V(G) \to \mathbb{N}$ is a coloring that satisfies $\sigma(u) \neq \sigma(v)$ for any two adjacent vertices u, v in G, where $\sigma(x)$ denotes the sum of all colors $c(y)$ among all neighbors y of a vertex x. The *sigma chromatic number* of G is denoted by $\sigma(G)$ and is defined as the fewest number of colors needed for a sigma coloring of G. The notion of sigma coloring was first studied by Chartrand et al. in [4]. Clearly, $\sigma(G) \leq \chi(G)$ for any graph G. Moreover, it has been shown by Dehghan et al. in [5] that for every integer $k \geq 3$, it is **NP**-complete to decide whether $\sigma(G) = k$ for a given graph G. The computational complexity of this sigma coloring problem motivates a number of works on this topic; see [7, 8, 15]), for example.

Example 1.1. *Let R be the ring $\mathbb{Z}_{24}$, I be the ideal $\langle 8 \rangle = \{0, 8, 16\}$, and $G = \Gamma_I(R)$. To illustrate, the elements 2 and 20 of R become vertices of G since $2 \cdot 20 = 16 \in I$. On the other hand, the element 3 of R is not a vertex of G since $3 \cdot x \notin I$ for all $x \in R \setminus I$. The graph G is shown in Figure 1 together with an example of a sigma coloring of G.*

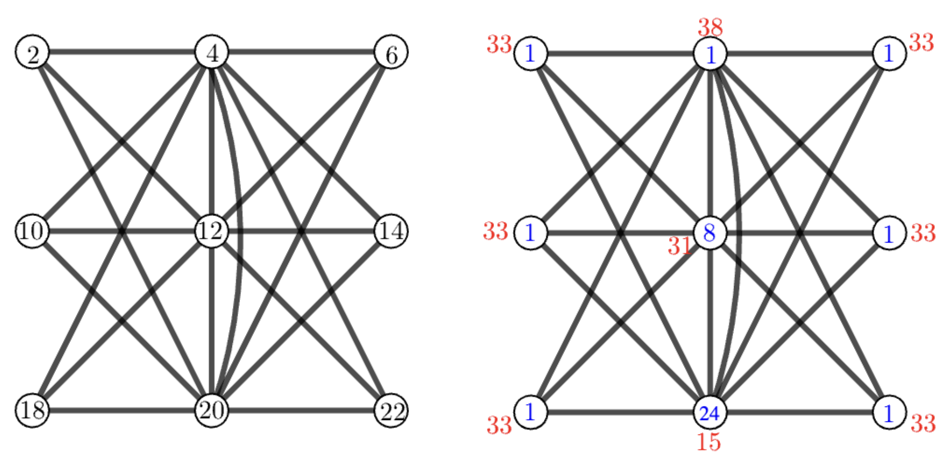

Figure 1. The graph $G = \Gamma_I(R)$ (left), where $R = \mathbb{Z}_{24}$ and $I = \langle 8 \rangle$, and a sigma coloring of G (right), where the color and color sum of each vertex are shown inside and beside the vertex, respectively.

As previously mentioned, in [9], zero-divisor graphs of rings of integers modulo n were studied in relation to sigma colorings. An interesting observation from [9] is that the structure of such zero-divisor graphs allows for a natural way of constructing their sigma colorings using the minimum number of colors possible. Given that the notion of ideal-based zero divisor graphs is a generalization of zero-divisor graphs, it is then natural to ask if the structure of ideal-based zero divisor graphs can also allow for a similarly natural way of constructing their sigma colorings.

Thus, in this paper, we aim to continue the work done in [9] by investigating the sigma chromatic number of ideal-based zero-divisor graphs of rings of integers modulo n. Note that if n has prime factorization $p_1^{n_1}p_2^{n_2}\cdots p_m^{n_m}$, where $m, n_1, n_2, ..., n_m$ are positive integers and $p_1, p_2, ..., p_m$ are distinct primes, then the ring $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_m^{n_m}}$. Thus, we focus our attention on such rings.

In Section 2, we review important results from [9] on the structure of $\Gamma(\mathbb{Z}_n)$. In Section 3, we discuss the important relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$ and we establish results particular to the structure of $\Gamma_I(\mathbb{Z}_n)$. Finally, in Section 4, we present the main results on the sigma chromatic number of ideal-based zero-divisor graphs $\Gamma_I(\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_m^{n_m}})$.

All graphs considered in this work are simple, undirected, connected, and finite. Unless otherwise stated, notations will follow [3]. If two vertices u and v are adjacent in a graph G, then we write $u \sim_G v$. The *closed neighborhood* $N_G[u]$ of a vertex u in a graph G is given by $N_G[u] = N_G(u) \cup \{u\}$. Given two sets A and B, we denote by $A+B$ the set $\{a+b : a \in A, b \in B\}$; in the case that A is a singleton $\{a\}$, then we may also write $a + B$ for $\{a\} + B$.

2. On the structure of $\Gamma(\mathbb{Z}_n)$

As established in [13], ideal-based zero-divisor graphs are closely related to zero-divisor graphs of factor rings. Since for a proper ideal I of $\mathbb{Z}_M$, the factor ring $\mathbb{Z}_M/I$ is isomorphic to some ring $\mathbb{Z}_n$, understanding the structure of $\Gamma(\mathbb{Z}_n)$ is of primary importance in this work. In this section, we present important resuls from [9] on the structure of $\Gamma(\mathbb{Z}_n)$. The reader may refer to [9] for the proofs of these results.

For this entire section, we let m be a positive integer, and let $R = \prod_{k=1}^{m} \mathbb{Z}_{p_k^{n_k}}$, where $m, n_1, n_2, ...,$ n_m are positive integers and $p_1, p_2, ..., p_m$ are distinct primes.

The Sets $R_{i_1,i_2,...,i_m}$

We define the following sets: For each $k \in \{1, 2, ..., m\}$, let $i_k \in \{0, 1, ..., n_k\}$. We denote by $R_{i_1,i_2,...,i_m}$ the set of all $(x_1, x_2, ..., x_m) \in R$ such that for each $k = 1, 2, ..., m$, we have

1. $x_k = 0$ if $i_k = n_k$, and
2. $p_k^{i_k} | x_k$ and $p_k^{i_k+1} \nmid x_k$ if $i_k = 0, 1, ..., n_k - 1$.

Note that $R_{n_1,n_2,...,n_m} = \{(0, 0, ..., 0)\}$ while the set $R_{0,0,...,0}$ is the set of elements that are not zero divisors of R. Hence,

$$V(\Gamma(R)) = \bigcup_{\substack{(i_1,i_2,...,i_m)\neq(0,0,...,0),\\ (i_1,i_2,...,i_m)\neq(n_1,n_2,...,n_m)}} R_{i_1,i_2,...,i_m}. \tag{1}$$

For convenience, throughout this paper, we will refer to the sets $R_{i_1,i_2,...,i_m}$, where $(i_1, i_2, ..., i_m) \notin \{(0, 0, ..., 0), (n_1, n_2, ..., n_m)\}$, as *blocks* of $V(\Gamma(R))$.

Let $H = \Gamma(R)$. We present below some important results on the properties of H in relation to the blocks $R_{i_1,i_2,...,i_m}$.

Proposition 2.1 ([9]). *Let $(i_1, i_2, ..., i_m) \notin \{(0, 0, ..., 0), (n_1, n_2, ..., n_m)\}$. The following statements hold.*

1. *If $u, v \in R_{i_1,i_2,...,i_m}$, then $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.*
2. *The vertices in $R_{i_1,i_2,...,i_m}$ are adjacent to all the vertices in $R_{j_1,j_2,...,j_m}$ if and only if $i_k + j_k \geq n_k$ for all $k = 1, 2, ..., m$.*
3. *The vertices in $R_{i_1,...,i_m}$ form a clique in $\Gamma(R)$ if and only if $2i_k \geq n_k$ for all $k = 1, 2, ..., m$.*

Proposition 2.2 ([9]). *For $(i_1, i_2, ..., i_m) \notin \{(0, 0, ..., 0), (n_1, n_2, ..., n_m)\}$,*

$$|R_{i_1,i_2,...,i_m}| = \prod_{k=1}^{m} \phi(p_k^{n_k-i_k}) = \prod_{k=1}^{m} \lceil p_k^{n_k-i_k-1}(p_k - 1)\rceil. \tag{2}$$

Corollary 2.1 ([9]). *Let $u \in R_{i_1,i_2,...,i_m}$ and $v \in R_{j_1,j_2,...,j_m}$. If $R_{i_1,i_2,...,i_m} \neq R_{j_1,j_2,...,j_m}$, then $\deg u \neq \deg v$.*

3. On the structure of $\Gamma_I(\mathbb{Z}_n)$

We begin with three important results from [13] that provide key properties of ideal-based zero-divisor graphs.

Theorem 3.1 ([13]). *Let I be an ideal of a ring R, and let $x, y \in R - I$. Then:*

1. *If $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$, then x is adjacent to y in $\Gamma_I(R)$.*
2. *If x is adjacent to y in $\Gamma_I(R)$ and $x + I \neq y + I$, then $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$.*
3. *If x is adjacent to y in $\Gamma_I(R)$ and $x + I = y + I$, then $x^2, y^2 \in I$.*

Corollary 3.1 ([13]). *If x and y are (distinct) adjacent vertices in $\Gamma_I(R)$, then all (distinct) elements of $x + I$ and $y + I$ are adjacent in $\Gamma_I(R)$. If $x^2 \in I$, then all the distinct elements of $x + I$ are adjacent in $\Gamma_I(R)$.*

The following proposition from [13] provides an important relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$.

Proposition 3.1 ([13]). *Let I be an ideal of a ring R. Let $\Lambda \subseteq R$ be a set of coset representatives of R/I that are in $V(\Gamma(R/I))$. For each $i \in I$, define a graph G_i with vertex set $\{a + i : a \in \Lambda\}$, and where $a + i \sim_{G_i} b + i$ if and only if $a + I \sim_{\Gamma(R/I)} b + I$ (i.e., $ab \in I$).*

Then $\Gamma_I(R) = G$, where G is defined as follows:

$$V(G) = \bigcup_{i\in I} V(G_i) = \bigcup_{a\in\Lambda}(a + I), \tag{3}$$

and where $E(G)$ is defined as follows:

1. *$E(G)$ contains all the edges in G_i for each $i \in I$,*
2. *for distinct $a, b \in \Lambda$ and for any $i, j \in I$, $a + i \sim_G b + j$ if and only if $a + I \sim_{\Gamma(R/I)} b + I$,*
3. *for $a \in \Lambda$ and distinct $i, j \in I$, $a + i \sim_G a + j$ if and only if $a^2 \in I$.*

We also recall the following definition from [13]: Using the notation in Proposition 3.1, we call the subset $a + I$ a *column* of $\Gamma_I(R)$. Moreover, $a + I$ is called a *connected column* if $a^2 \in I$.

For the rest of this section, we let $R = \prod_{k=1}^{m} \mathbb{Z}_{p_k^{n_k}}$, where $m, n_1, n_2, ..., n_m$ are positive integers and $p_1, p_2, ..., p_m$ are distinct primes. Moreover, let I be a proper ideal given by

$$I = \langle p_1^{e_1} \rangle \times \langle p_2^{e_2} \rangle \times \cdots \times \langle p_m^{e_m} \rangle, \tag{4}$$

where $e_i \in \{0, 1, ..., n_i\}$ for $i \in \{1, 2, ..., m\}$. Moreover, we set $G = \Gamma_I(R)$ and $H = \Gamma(R/I)$. Finally, we assume that I is not prime so that G is not empty (see Proposition 2.2 in [13]).

The Sets $S_{i_1,i_2,...,i_m}$

Consider R/I:

$$R/I = \big\{(x_1, x_2, ..., x_m) + I : x_k \in \{0, 1, .., p_k^{e_k} - 1\} \subseteq \mathbb{Z}_{p_k^{n_k}} \text{ for } k = 1, 2, ..., m\big\}. \tag{5}$$

Then $R/I \cong \mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}} \times \cdots \times \mathbb{Z}_{p_m^{e_m}}$, and we can choose the set Λ of coset representatives to come from $\prod_{k=1}^{m}\{0, 1, ..., p_k^{e_k} - 1\}$. In the previous section, we defined the set $R_{i_1,i_2,...,i_m}$, which consists of elements of R. In R/I, we can construct a similar set, which will consist of cosets. We define instead a collection of coset representatives as follows. If $i_k \in \{0, 1, \ldots, e_k\}$ for each $k \in \{1, 2, \ldots, m\}$, and $(i_1, i_2, ..., i_m) \notin \{(0, 0, \ldots, 0), (e_1, e_2, \ldots, e_m)\}$, let $R'_{i_1,i_2,...,i_m}$ be the set of all $(x_1, x_2, ..., x_m) \in \Lambda$ such that for each $k = 1, 2, ..., m$, we have

1. $x_k = 0$ if $i_k = e_k$, and
2. $p_k^{i_k} | x_k$ and $p_k^{i_k+1} \nmid x_k$ if $i_k = 0, 1, ..., e_k - 1$.

Then the blocks of $\Gamma(R/I)$ are the sets $\{x + I : x \in R'_{i_1,i_2,...,i_m}\}$, where $i_k \in \{0, 1, ..., e_k\}$ for all $k \in \{1, 2, ..., m\}$ and $(0, 0, ..., 0) \neq (i_1, i_2, ..., i_m) \neq (e_1, e_2, ..., e_m)$.

We now consider $G = \Gamma_I(R)$. We define the set $S_{i_1,i_2,...,i_m} \subseteq R$ as

$$\begin{aligned} S_{i_1,i_2,...,i_m} &= R'_{i_1,i_2,...,i_m} + I \\ &= \bigcup_{(x_1,x_2,...,x_m) \in R'_{i_1,i_2,...,i_m}} [(x_1, x_2, ..., x_m) + I] \\ &= \bigcup_{(x_1,x_2,...,x_m) \in R'_{i_1,i_2,...,i_m}} \{(x_1, x_2, ..., x_m) + i : i \in I\}. \end{aligned} \tag{6}$$

By Proposition 3.1,

$$V(G) = \bigcup_{\substack{(i_1,i_2,...,i_m) \neq (0,0,...,0) \\ (i_1,i_2,...,i_m) \neq (e_1,e_2,...,e_m)}} S_{i_1,i_2,...,i_m}. \tag{7}$$

Thus, the collection $\{S_{i_1,i_2,...,i_m} : i_k \in \{0,1,...,e_k\}$ for all $k \in \{1,2,...,m\}$, and $(0,0,...,0) \neq (i_1,i_2,...,i_m) \neq (e_1,e_2,...,e_m)\}$ forms a partition of $V(\Gamma_I(R))$, and we refer to the sets $S_{i_1,i_2,...,i_s}$ as *blocks* of $V(\Gamma_I(R))$.

Before we proceed, we provide a summary of the different notations that have been introduced so far.

Notation	Definition
R	$\prod_{k=1}^{m} \mathbb{Z}_{p_k^{n_k}}$, a ring $m, n_1, n_2, ..., n_m \in \mathbb{Z}^+$; $p_1, p_2, ..., p_m$ are distinct primes
I	$\langle p_1^{e_1}\rangle \times \langle p_2^{e_2}\rangle \times \cdots \times \langle p_m^{e_m}\rangle$, a proper ideal of R, $e_i \in \{0,1,...,n_i\} \forall i$; I is assumed to be a prime ideal
G	$\Gamma_I(R)$
H	$\Gamma(R/I)$
Λ	$\prod_{k=1}^{m}\{0,1,...,p_k^{e_k}-1\}$, set of coset representatives for R/I
G_i	Given $i \in I$: $V(G_i) = \{x+i : x \in \Lambda\}$, $E(G_i) = \{\{x+i, y+i\} : xy \in I\}$
$R_{i_1,...,i_m}$	$\{(x_1,...,x_m) \in R : x_k = 0$ if $i_k = n_k; p_k^{i_k} \mid x_k$ and $p_k^{i_k+1} \nmid x_k$ if $i_k = 0,...,n_k-1\}$
$R'_{i_1,...,i_m}$	$\{(x_1,...,x_m) \in \Lambda : x_k = 0$ if $i_k = e_k; p_k^{i_k} \mid x_k$ and $p_k^{i_k+1} \nmid x_k$ if $i_k = 0,...,e_k-1\}$
$S_{i_1,...,i_m}$	$R'_{i_1,...,i_m} + I$

Table 1. Summary of different notations.

Example 3.1. *Let $R = \mathbb{Z}_{2^3} \times \mathbb{Z}_3$, which is isomorphic to the ring considered in Example 1.1. Correspondingly, let $I = 0 \times \mathbb{Z}_3 = \{(0,0),(0,1),(0,2)\}$, which is an ideal of R. Then the set of coset representatives for R/I can be chosen to be $\{0,1,...,7\} \times \{0\}$ and $\Lambda = \{(2,0),(4,0),(6,0)\}$. Note that $R_{1,0} = \{(2,1),(6,1),(2,2),(6,2)\}$ but $R'_{1,0} = \{(2,0),(6,0)\}$. Similarly, $R_{2,0} = \{(4,1),(4,2)\}$ but $R'_{2,0} = \{(4,0)\}$. Since $e_1 = 3$ and $e_2 = 0$, G can be constructed from the sets $R'_{1,0}$ and $R'_{2,0}$; this is in light of Proposition 3.1 and Equation* (7). *We present the graph of G in Figure 2, where the sets R'_{i_1,i_2}, S_{i_1,i_2} and subgraphs G_i are also illustrated.*

Observation 3.1. *For $(i_1,...,i_m)$ with $i_k \in \{0,1,...,e_k\}$ for all $k \in \{1,...,m\}$,*

$$|S_{i_1,i_2,...,i_m}| = |I| \times |R'_{i_1,i_2,...,i_m}|. \tag{8}$$

The following lemma is an analog of Proposition 2.1 for ideal-based zero divisor graphs.

Lemma 3.1. *Suppose $(i_1,i_2,...,i_m),(j_1,j_2,...,j_m) \notin \{(0,0,...,0),(e_1,e_2,...,e_m)\}$ and that $i_k, j_k \in \{0,1,...,e_k\}$ for all $k \in \{1,...,m\}$. Then the following statements hold.*

1. *If $u, v \in S_{i_1,i_2,...,i_m}$, then $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$.*
2. *Suppose $(i_1,i_2,...,i_m) \neq (j_1,j_2,...,j_m)$. Two vertices $u \in S_{i_1,i_2,...,i_m}$ and $v \in S_{j_1,j_2,...,j_m}$ if and only if $i_k + j_k \geq e_k$ for all $k = 1,2,...,m$.*
3. *Two distinct vertices in $S_{i_1,...,i_m}$ are adjacent in G if and only if $2i_k \geq e_k$ for all $k = 1,2,...,m$.*

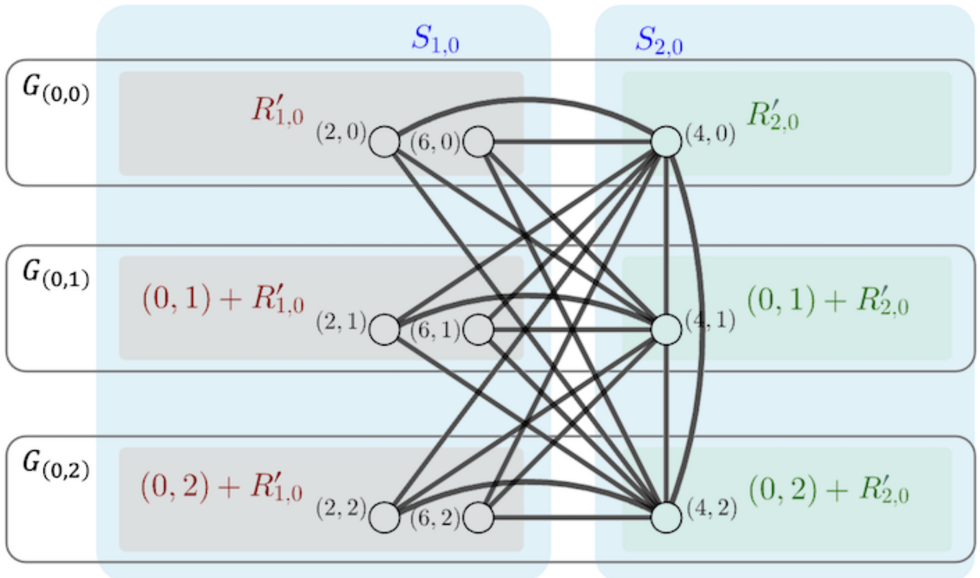

Figure 2. The graph $G = \Gamma_I(R)$, where $R = \mathbb{Z}_{2^3} \times \mathbb{Z}_3$ and $I = 0 \times \mathbb{Z}_3$.

Proof.

1. Suppose $u \in x + I$ and $v \in y + I$ for some $x, y \in R'_{i_1,i_2,\ldots,i_m}$. It is sufficient to show that $N_G(u) \setminus \{v\} \subseteq N_G(v) \setminus \{u\}$. Let $w \in N_G(u) \setminus \{v\}$, which implies that $(w + I)(u + I) = (w + I)(x + I) = I$.

   Case 1. Suppose $u \sim_G v$ and $w + I = y + I$.
   Then $(x+I)(y+I) = (u+I)(v+I) = I$ and we must have $xy \in I$. Since $x, y \in R'_{i_1,i_2,\ldots,i_m}$, we must have $2i_k \geq e_k$ for all $k \in \{1, 2, \ldots, m\}$ by Proposition 2.1. It follows that $x^2, y^2 \in I$ and that $(x + I)(x + I) = (y + I)(y + I) = I$. Since $w + I = y + I$, then $(w + I)(v + I) = (y + I)(y + I) = I$, which implies that $w \in N_G(v)$.

   Case 2. Suppose $u \not\sim_G v$ or $w + I \neq y + I$.
   First, if $u \not\sim_G v$, then $u(w - v) \notin I$ since $uv \notin I$ while $uw \in I$. Then $w - v \notin I$ and we must have $w + I \neq y + I$. Then $w + I \in N_H(x + I) \setminus \{y + I\} = N_H(y + I) \setminus \{x + I\}$ by Proposition 2.1(i). Then $(w + I)(v + I) = (w + I)(v + I) = I$ and $w \in N_G(v)$.

2. First, we must have $u \in x+I$ for some $x \in R'_{i_1,i_2,\ldots,i_m}$ and $v \in y+I$ for some $y \in R'_{j_1,j_2,\ldots,j_m}$. Clearly, $x+I \neq y+I$. Then Theorem 3.1(i)-(ii) and Proposition 2.1(ii) imply that $i_k+j_k \geq e_k$ for all $k = 1, 2, \ldots, m$ if and only if $u + I = x + I$ and $y + I = v + I$ are adjacent in $\Gamma(R/I)$ if and only if $u \sim_G v$.

3. We consider two cases.

   Case 1. Suppose $u \in x + I$ and $v \in y + I$ for some distinct $x, y \in R'_{i_1,i_2,\ldots,i_m}$. Again, we have $x + I \neq y + I$. Then Theorem 3.1(i)-(ii) and Proposition 2.1(iii) imply that $2i_k \geq e_k$ for all $k = 1, 2, \ldots, m$ if and only if $u + I = x + I$ and $y + I = v + I$ are adjacent in $\Gamma(R/I)$ if and only if $u \sim_G v$.

Case 2. Suppose $u, v \in x + I$ for some distinct $x, y \in R'_{i_1,i_2,...,i_m}$. Then $u \sim_G v$ if and only if $uv \in I$ if and only if $x^2 \in I$ if and only if $2i_k \geq e_k$ for all $k = 1, 2, ..., m$. $\square$

The following is an immediate consequence of Lemma 3.1.

Corollary 3.2. *Under the same conditions as in Lemma 3.1:*

1. *All the vertices in $S_{i_1,i_2,...,i_m}$ are adjacent to all the vertices in $S_{j_1,j_2,...,j_m}$ if and only if $i_k + j_k \geq e_k$ for all $k = 1, 2, ..., m$.*
2. *The vertices in $S_{i_1,i_2,...,i_m}$ form a clique in $\Gamma_I(R)$ if and only if $2i_k \geq e_k$ for all $k = 1, 2, ..., m$.*

Example 3.2. *Consider $R = \mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2}$ with $I = \langle 2^2 \rangle \times \langle 3^1 \rangle$. Figure 3 shows $\Gamma_I(R)$ depicted using the blocks S_{i_1,i_2} with their subsets $(i, j) + R'_{i_1,i_2}$, where $(i, j) \in I$. The subgraphs $G_{(i,j)}$, as defined in Proposition 3.1, are also shown. Note that, by Corollary 3.2, the block $S_{1,1}$ is a clique.*

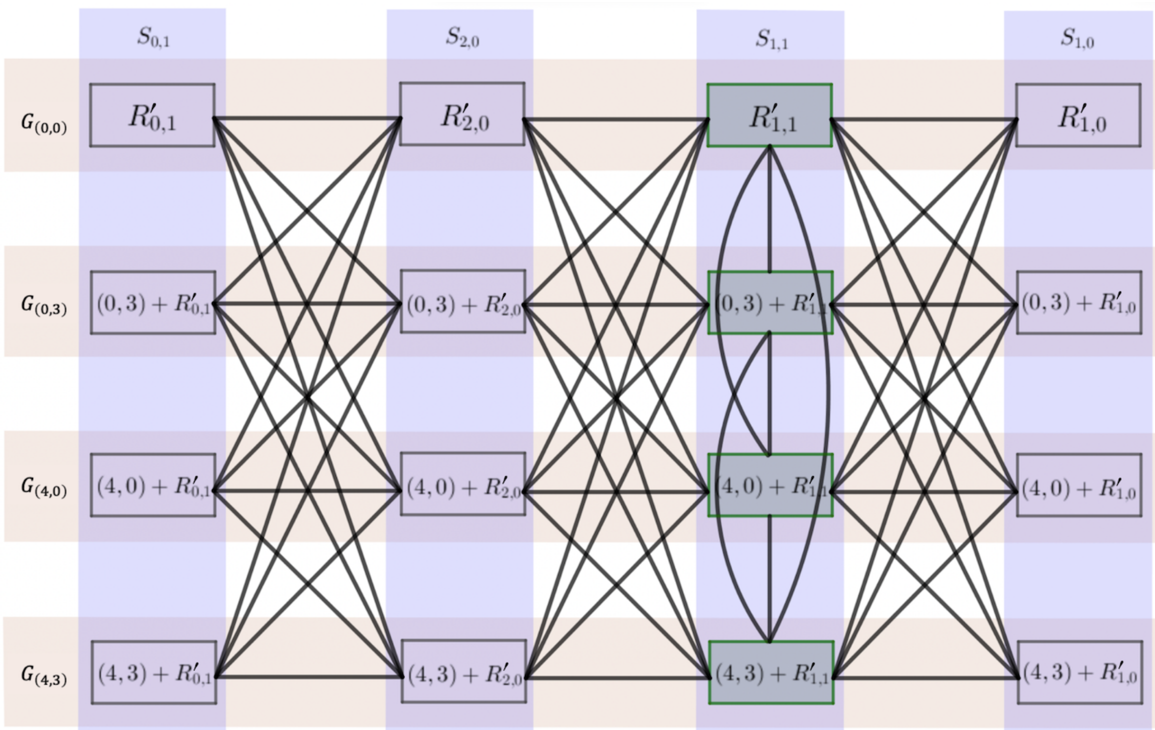

Figure 3. The ideal-based zero-divisor graph of $R = \mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2}$ with $I = \langle 2^2 \rangle \times \langle 3^1 \rangle$ depicted using blocks of $V(\Gamma_I(R))$ and the subgraphs $G_{(i,j)}$. The vertices in the green sets are pairwise adjacent; thus, the block $S_{1,1}$ is a clique.

The following lemma is a version of Lemma 4.1 in [10], which with Proposition 2.2, provides the degree of vertices in G.

Lemma 3.2. *Let $u \in x + I \subseteq S_{i_1,i_2,...,i_m} \subseteq V(G)$ for some $x \in R'_{i_1,i_2,...,i_m}$. Then*

$$\deg_G(u) = \begin{cases} |I| \times [1 + \deg_H(x + I)] - 1, & \text{if } 2i_k \geq e_k \ \forall k \in \{1, ..., m\}, \\ |I| \times \deg_H(x + I), & \text{otherwise.} \end{cases} \tag{9}$$

Proof. Recall the graphs G_i, where $i \in I$, described in Proposition 3.1. With our choice of coset representatives, note that $x \in V(G_0)$. Then $\deg_H(x+I) = \deg_{G_i}(x+i) = |N_{G_i}(x+i)|$ for any $i \in I$.

Suppose $u \in G_j$, where $j \in I$; then $u = x + j$. We first consider the case where $2i_k \geq e_k$ for all $k \in \{1, 2, ..., m\}$. Then $x^2 \in I$ and $x + I$ is a connected column; that is, $(x + i_1) \sim_G (x + i_2)$ for any distinct $i_1, i_2 \in I$. Then

$$N_G(u) = \left[\bigcup_{i \in I} N_{G_i}(x+i)\right] \cup \{x + i : i \neq j\} \tag{10}$$

and it follows that

$$\deg_G(u) = \left[\sum_{i \in I} |N_{G_i}(x+i)|\right] + (|I| - 1) = \left[\sum_{i \in I} \deg_H(x+I)\right] + (|I| - 1), \tag{11}$$

from which the desired conclusion follows.

On the other hand, suppose $2i_k < e_k$ for some $k \in \{1, 2, ..., m\}$. Then $x + I$ is not a connected column and $N_G(u) = \bigcup_{i \in I} N_{G_i}(x+i)$. The desired conclusion follows similarly. $\square$

Theorem 3.2. *Suppose $u \in S_{i_1,i_2,...,i_m}$ and $v \in S_{j_1,j_2,...,j_m}$, where $S_{i_1,i_2,...,i_m}$, $S_{j_1,j_2,...,j_m}$ are blocks of $V(G)$. Then $\deg u = \deg v$ if and only if $S_{i_1,i_2,...,i_m} = S_{j_1,j_2,...,j_m}$ (i.e. $i_k = j_k$ for all $k \in \{1, 2, ..., m\}$).*

Proof. Suppose $S_{i_1,i_2,...,i_m} = S_{j_1,j_2,...,j_m}$. By Lemma 3.1, we have $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$. If $u \sim_G v$, then $\deg_G(u) = |[N_G(u) \setminus \{v\}] \cup \{v\}| = |N_G(u) \setminus \{v\}| + 1 = |N_G(v) \setminus \{u\}| + 1 = |[N_G(v) \setminus \{u\}] \cup \{u\}| = \deg_G(v)$.

On the other hand, if $u \not\sim_G v$, then $N_G(u) = N_G(u) \setminus \{v\}$ and $N_G(v) = N_G(v) \setminus \{u\}$, from which the conclusion follows.

Now, suppose $S_{i_1,i_2,...,i_m} \neq S_{j_1,j_2,...,j_m}$. Moreover, suppose $u \in x + I$ for some $x \in R'_{i_1,i_2,...,i_m}$ and $v \in y + I$ for some $y \in R'_{j_1,j_2,...,j_m}$. Since $R'_{i_1,i_2,...,i_m} \neq R'_{j_1,j_2,...,j_m}$, Corollary 2.1 implies that $\deg_H(x+I) \neq \deg_H(y+I)$. We consider the following cases:

1. If $2i_k \geq e_k$ and $2j_k \geq e_k$ for all $k \in \{1, 2, ..., m\}$, then Lemma 3.1 implies the desired conclusion.
2. If $2i_{k_1} < e_{k_1}$ and $2j_{k_2} < e_{k_2}$ for some $k_1, k_2 \in \{1, 2, ..., m\}$, then Lemma 3.1 also implies the desired conclusion.
3. Without loss of generality, suppose $2i_k \geq e_k$ for all $k \in \{1, 2, ..., m\}$ and that there is a $j \in \{1, 2, ..., m\}$ for which $2i_j < e_j$. Then, by Lemma 3.1, $\deg_G(v)$ is divisible by $|I|$ while $\deg_G(u)$ is not. The conclusion follows. $\square$

4. The sigma chromatic number of $\Gamma_I(\mathbb{Z}_n)$

In [10], Mallika et al. established a relationship between the chromatic numbers of $\Gamma_I(R)$ and $\Gamma(R/I)$. More specifically, Mallika et al. proved that $2 \leq \chi(\Gamma(R/I)) \leq \chi(\Gamma_I(R)) \leq$

$|I| \times \chi(\Gamma(R/I))$. Moreover, they proved that $\chi(\Gamma(R/I)) = \chi(\Gamma_I(R))$ if $\Gamma_I(R)$ has no connected columns. In this section, we establish similar results for the sigma chromatic number of ideal-based zero-divisor graphs of rings of integers modulo n.

We begin by recalling the following important observation.

Observation 4.1 (Chartand et al. [4]). *If H is a complete subgraph of order k in a graph G such that $N[u] = N[v]$ for every two vertices u and v of H, then $\sigma(G) \geq k$.*

In this section, we will establish a formula for the sigma chromatic number of ideal-based zero-divisor graphs. As in the previous section, we let m be a positive integer and $R = \prod_{k=1}^{m} \mathbb{Z}_{p_k^{n_k}}$, where $m, n_1, n_2, ..., n_m$ are positive integers and $p_1, p_2, ..., p_m$ are distinct primes. Moreover, let I be a proper ideal given by

$$I = \langle p_1^{e_1} \rangle \times \langle p_2^{e_2} \rangle \times \cdots \times \langle p_m^{e_m} \rangle, \tag{12}$$

where $e_i \in \{0, 1, ..., n_i\}$ for $i \in \{1, 2, ..., m\}$. Since I is proper, we must have $(e_1, e_2, ..., e_m) \neq (n_1, n_2, ..., n_m)$ (i.e. I is not the zero ideal).

By Proposition 2.2 in [13], we also do not need to consider prime ideals as they would produce empty ideal-based zero-divisor graphs. Thus, we do not consider the case where $e_i = 1$ for exactly one $i \in \{1, 2, ..., m\}$ and $e_j = 0$ for all $j \neq i$.

Note that if $m = n_1 = 1$, then $R = \mathbb{Z}_{p_1}$ is a field and the only ideals are $\{0\}$ and $\langle 1 \rangle$, both of which we do not need to consider. On the other hand, if $m = 1$ and $n_1 \geq 2$, then the only relevant ideals to consider for the ring $\mathbb{Z}_{p_1^{n_1}}$ are those of the form $\langle p_1^k \rangle$, where $2 \leq k \leq n_1$; such cases fall under Theorem 4.2. Thus, the case $m = 1$ is not considered in the following result, which involves one class of ideals.

Theorem 4.1. *Let $m \geq 2$ and let $R = \prod_{k=1}^{m} \mathbb{Z}_{p_k^{n_k}}$, where $p_1, p_2, ..., p_m$ are distinct primes and $n_1, n_2, ..., n_m$ are positive integers. Let $I = \langle p_1^{e_1} \rangle \times \langle p_2^{e_2} \rangle \times \cdots \langle p_m^{e_m} \rangle$. If $e_{j_1} = e_{j_2} = \cdots = e_{j_s} = 1$ for some $j_1, j_2, ..., j_s$, with $2 \leq s \leq m$, and $e_i = 0$ for $i \notin \{j_1, j_2, ..., j_s\}$, then $\sigma(\Gamma_I(R)) = 1$.*

Proof. It is sufficient to consider the case where $(e_1, e_2, ..., e_s; e_{s+1}, ..., e_m) = (1, 1, ..., 1; 0, 0, ..., 0)$; that is, we only consider the ideal

$$I = \langle p_1 \rangle \times \langle p_2 \rangle \times \cdots \times \langle p_s \rangle \times \langle 1 \rangle \times \langle 1 \rangle \times \cdots \times \langle 1 \rangle. \tag{13}$$

Consider R/I; we pick the coset representatives so that

$$\begin{aligned} R/I = \{(x_1, x_2, ..., x_m) + I : x_k \in \{0, 1, 2, .., p_k - 1\} \subseteq \mathbb{Z}_{p_k^{n_k}}\ \forall k = 1, ..., s, \\ \text{and } x_k = 0 \text{ for } k = s+1, ..., m, \text{ if such } k \text{ exist.}\}. \end{aligned} \tag{14}$$

Then $R/I \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_s} \times \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_1$.

Let $G = \Gamma_I(R)$. As defined in the previous section, $V(G)$ can be partitioned into blocks $S_{i_1, i_2, ..., i_s, 0, 0, ..., 0}$, where $i_k \in \{0, 1\}$ for $k \in \{1, 2, .., s\}$ & $(0, 0, ..., 0) \neq (i_1, i_2, ..., i_s) \neq (1, 1, ..., 1)$.

If $2i_k \geq e_k$ for all $k \in \{1, 2, ..., m\}$, then $(i_1, i_2, ..., i_s) = (1, 1, ..., 1)$. This implies, by Corollary 3.2, that any two vertices belonging to the same block are not adjacent.

Now, we let c be a coloring of G for which $c(v) = 1$ for all $v \in V(G)$. Then $\sigma(v) = \deg_G(v)$ for all $v \in V(G)$. Let u and v be adjacent vertices in G. Then u and v belong to different blocks. By Theorem 3.2, we have $\sigma(u) \neq \sigma(v)$. Therefore, c is a sigma coloring of G that uses only one color. □

Theorem 4.2. *Let $m \geq 1$ and let $R = \prod_{k=1}^{m} \mathbb{Z}_{p_k^{n_k}}$, where $p_1, p_2, ..., p_m$ are distinct primes and $n_1, n_2, ..., n_m$ are positive integers. Let $I = \langle p_1^{e_1} \rangle \times \langle p_2^{e_2} \rangle \times \cdots \langle p_m^{e_m} \rangle$. If $n_j \geq e_j \geq 2$ for at least one $j \in \{1, 2, ..., m\}$, then*

$$\sigma(\Gamma_I(R)) = |I| \times \sigma(\Gamma(R/I)) = |I| \times \prod_{k=1}^{m} \left[p_k^{e_k - \lceil \frac{e_k}{2} \rceil - 1}(p_k - 1) \right]. \tag{15}$$

Proof. Let $G = \Gamma_I(R)$. Recall the partition of $V(G)$ given by $\{S_{i_1, i_2, ..., i_m} : i_k \in \{0, 1, ..., e_k\} \; \forall k \in \{1, 2, ..., m\}$, and $(0, 0, ..., 0) \neq (i_1, i_2, ..., i_m) \neq (e_1, e_2, ..., e_m)\}$, as described in Section 3.

Since $n_j \geq e_j \geq 2$ for at least one $j \in \{1, ..., m\}$, we have $(0, 0, ..., 0) \neq (\lceil \frac{e_1}{2} \rceil, \lceil \frac{e_2}{2} \rceil, ..., \lceil \frac{e_m}{2} \rceil)$ $\neq (e_1, e_2, ..., e_m)\}$. Let $S = S_{\lceil \frac{e_1}{2} \rceil, \lceil \frac{e_2}{2} \rceil, ..., \lceil \frac{e_m}{2} \rceil}$. By Observation 3.1 and Proposition 2.2,

$$|S| = |I| \times \left| R'_{\lceil \frac{e_1}{2} \rceil, \lceil \frac{e_2}{2} \rceil, ..., \lceil \frac{e_m}{2} \rceil} \right| = |I| \times \prod_{k=1}^{m} \left[p_k^{e_k - \lceil \frac{e_k}{2} \rceil - 1}(p_k - 1) \right]. \tag{16}$$

By Corollary 3.2, S is a clique in G. For $u, v \in S$, then $u \sim_G v$ and Lemma 3.1(i) implies that

$$N[u] = [N(u) \setminus \{v\}] \cup \{u, v\} = [N(v) \setminus \{u\}] \cup \{u, v\} = N[v]. \tag{17}$$

By Observation 4.1, $\sigma(G) \geq |S|$.

Let $d = \Delta(G) + 1$. We now define a coloring c of G as follows:

1. If $2i_j < e_j$ for some $j \in \{1, 2, ..., m\}$, set $c(S_{i_1, i_2, ..., i_m}) = \{1\}$.
2. If $2i_k \geq e_k$ for all $k \in \{1, 2, ..., m\}$, set

$$c(S_{i_1, i_2, ..., i_m}) = \{1, d, d^2, ..., d^{|S_{i_1, i_2, ..., i_m}| - 1}\}. \tag{18}$$

To illustrate, the sigma coloring depicted in Figure 1 has been constructed using the preceding definition. By Proposition 2.2, for all $(i_1, i_2, ..., i_m)$ satisfying $2i_k \geq e_k$ for all $k \in \{1, 2, ..., m\}$, we have $|R'_{\lceil \frac{e_1}{2} \rceil, \lceil \frac{e_2}{2} \rceil, ..., \lceil \frac{e_m}{2} \rceil}| \geq |R'_{i_1, i_2, ..., i_m}|$, which implies that $|S| \geq |S_{i_1, i_2, ..., i_m}|$. Thus, c uses exactly $|S|$ colors.

Suppose $u \sim_G v$. By the choice of colors (see Lemma 4.1 in [4]), $\deg_G u \neq \deg_G v$ implies that $\sigma(u) \neq \sigma(v)$. Now, suppose $\deg_G u = \deg_G v$. By Theorem 3.2, u and v belong to the same block, say $S_{i_1, i_2, ..., i_m}$. By Lemma 3.1(iii), we must have $2i_k \geq e_k$ for all $k \in \{1, 2, ..., m\}$. This implies that the vertices in $S_{i_1, i_2, ..., i_m}$ form a clique in G. Then

$$\sigma(u) = c(v) + \sum_{x \in N_G(u) \setminus \{v\}} c(x) \quad \text{and} \quad \sigma(v) = c(u) + \sum_{x \in N_G(v) \setminus \{u\}} c(x). \tag{19}$$

By Lemma 3.1(i), we have $\sum_{x\in N_G(u)\setminus\{v\}} c(x) = \sum_{x\in N_G(v)\setminus\{u\}} c(x)$. Moreover, by the construction of c, we have $c(v) \neq c(u)$; consequently, $\sigma(u) \neq \sigma(v)$. Therefore, c is a sigma coloring that uses $|S|$ colors. $\square$

Acknowledgement

The authors thank the Ateneo de Manila University for its support of the presentation of this paper at the International Conference on Graph Theory and Information Security (ICGTIS) V 2022. Moreover, the fourth author is grateful for the university's support via the Early Career Publication Support Grant. Finally, the authors thank the organizers of ICGTIS V 2022 for a wonderful and fruitful conference.

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