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Multipartite Ramsey numbers for the union of stars

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Abstract

Let s and k be positive integers with $k \geq 2$ and $G_1, G_2, \ldots, G_k$ be simple graphs. The *set multipartite Ramsey number*, denoted by $M_s(G_1, G_2, \ldots, G_k)$, is the smallest positive integer c such that any k-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of G_i in color i for some $i \in \{1, 2, \ldots, k\}$. The *size multipartite Ramsey number*, denoted by $m_c(G_1, G_2, \ldots, G_k)$, is the smallest positive integer s such that any k-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of G_i in color i for some $i \in \{1, 2, \ldots, k\}$. In this paper, we establish some lower and upper bounds, and some exact values of multipartite Ramsey numbers for the union of stars.

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1. Introduction

For simple graphs $G_1, \ldots, G_k$, the *Ramsey number* $r(G_1, \ldots, G_k)$ is defined as the smallest positive integer n such that any k-coloring of the edges of a complete graph K_n on n vertices

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contains a monochromatic copy of G_i in color i for some i, $1 \leq i \leq k$. We refer to [4] for an overview of Ramsey theory. In 2004, Burger and Van Vuuren [1, 2] introduced the notion of set multipartite Ramsey number and size multipartite Ramsey number as variations of the classical Ramsey number. The extension to many colors is established in [3], and the extension in a more general setting is presented in [9, 12]. For $c \geq 2, s \geq 1$, denote by $K_{c\times s}$ the complete multipartite graph with c partite sets, each of which contains s vertices.

Definition 1. *[9, 12] Let s, k be positive integers with $k \geq 2$ and $G_1, G_2, \ldots, G_k$ be simple graphs. The* set multipartite Ramsey number*, denoted by $M_s(G_1, G_2, \ldots, G_k)$, is the smallest positive integer c such that any k-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of G_i in color i for some $i \in \{1, 2, \ldots, k\}$.*

The size multipartite Ramsey number*, denoted by $m_c(G_1, G_2, \ldots, G_k)$, is the smallest positive integer s such that any k-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of G_i in color i for some $i \in \{1, 2, \ldots, k\}$.*

In the case of $G_1 = G_2 = \ldots = G_k = G$, the two aforementioned Ramsey numbers are abbreviated to $M_s(G; k)$ and $m_c(G; k)$, respectively.

In 2018, Perondi and Carmelo [9] completely determined the set and size multipartite Ramsey number of stars. Their results are presented in Theorems 1.1 and 1.2.

Theorem 1.1. *[9] Let $s \geq 1$ and $k, n_1, \ldots, n_k \geq 2$, and let $N = \sum_{i=1}^{k} n_i$. Then*

$$M_s(K_{1,n_1}, \ldots, K_{1,n_k}) = \begin{cases} \frac{N-k}{s} + 1 & \text{if } (N-k)/s \text{ is even, } s \text{ is odd, and} \\ & \quad n_i \text{ is even for some } i; \\ \left\lfloor \frac{N-k}{s} \right\rfloor + 2 & \text{otherwise.} \end{cases}$$

Theorem 1.2. *[9] Let $c, k, n_1, \ldots, n_k \geq 2$, and let $N = \sum_{i=1}^{k} n_i$. Then*

$$m_c(K_{1,n_1}, \ldots, K_{1,n_k}) = \begin{cases} \frac{N-k}{c-1} & \text{if } (N-k)/(c-1) \text{ is odd, } c \text{ is odd, and} \\ & \quad n_i \text{ is even for some } i; \\ \left\lfloor \frac{N-k+1}{c-1} \right\rfloor + 2 & \text{otherwise.} \end{cases}$$

The size multipartite Ramsey numbers of small stars versus the union of stars have been studied. In [8], Lusiani et al. determined the size multipartite Ramsey numbers $m_j(G, K_{1,2})$, for $j = 2, 3$, where G is a union of two, three, or four distinct stars. In [6, 7], Lusiani et al. determined the size multipartite Ramsey numbers $m_j(mK_{1,n}, H)$ where $H = K_{1,2}$ or $H = K_{1,3}$. In this paper, we shall consider multipartite Ramsey numbers for the union of stars in a more general setting.

2. Main Results

We start with providing some upper bounds of the multipartite Ramsey number for the union of stars in the following two theorems.

Theorem 2.1. *Let $k, s, l_i, n_{i,j}$ be natural numbers for $i \in \{1, 2, \ldots, k\}$, $j \in \{1, 2, \ldots, l_i\}$, and $k \geq 2$. Let $L = \sum_{i=1}^{k} l_i$ and $T = \sum_{i=1}^{k} \sum_{j=1}^{l_i} n_{i,j}$. If $s > L - k$, then*

$$M_s(\bigcup_{j=1}^{l_1} K_{1,n_{1,j}}, \bigcup_{j=1}^{l_2} K_{1,n_{2,j}}, \ldots, \bigcup_{j=1}^{l_k} K_{1,n_{k,j}}) \leq \left\lceil \frac{T-k+1}{s} \right\rceil + 1.$$

Proof. Let $c = \left\lceil \frac{T-k+1}{s} \right\rceil + 1$. Let $\psi : E(K_{c\times s}) \to \{1, 2, \ldots, k\}$ be any k-coloring of the edges of $K_{c\times s}$. For any $i \in \{1, 2, \ldots, k\}$ and any $v \in V(K_{c\times s})$, let $d_i(v) = |\{x|\psi(vx) = i\}|$. Since $d(v) = (c-1)s = \left\lceil \frac{T-k+1}{s} \right\rceil s \geq T-k+1$, then by the pigeon hole principle, for any $v \in V(K_{c\times s})$ there exists $i \in \{1, 2, \ldots, k\}$ such that $d_i(v) \geq \sum_{j=1}^{l_i} n_{i,j}$. Since $s > L - k$, then by the pigeon hole principle, there exists $i \in \{1, 2, \ldots, k\}$ such that there are l_i non adjacent pairwise vertices $x_1, x_2, \ldots, x_{l_i} \in V(K_{c\times s})$ such that $d_i(x_a) \geq \sum_{j=1}^{l_i} n_{i,j}$ for every $a \in \{1, 2, \ldots, l_i\}$. Hence, there is a monochromatic copy of $\bigcup_{j=1}^{l_i} K_{1,n_{i,j}}$ in color i. □

Theorem 2.2. *Let $k, c, l_i, n_{i,j}$ be natural numbers for $i \in \{1, 2, \ldots, k\}$, $j \in \{1, 2, \ldots, l_i\}$, and $c \geq 2$. Let $L = \sum_{i=1}^{k} l_i$ and $T = \sum_{i=1}^{k} \sum_{j=1}^{l_i} n_{i,j}$. If $\left\lceil \frac{T-k+1}{c-1} \right\rceil > L - k$, then*

$$m_c(\bigcup_{j=1}^{l_1} K_{1,n_{1,j}}, \bigcup_{j=1}^{l_2} K_{1,n_{2,j}}, \ldots, \bigcup_{j=1}^{l_k} K_{1,n_{k,j}}) \leq \left\lceil \frac{T-k+1}{c-1} \right\rceil.$$

Proof. Let $s = \left\lceil \frac{T-k+1}{c-1} \right\rceil$. Let $\psi : E(K_{c\times s}) \to \{1, 2, \ldots, k\}$ be any k-coloring of the edges of $K_{c\times s}$. For any $i \in \{1, 2, \ldots, k\}$ and any $v \in V(K_{c\times s})$, let $d_i(v) = |\{x|\psi(vx) = i\}|$. Since $d(v) = (c-1)s = (c-1)\left\lceil \frac{T-k+1}{c-1} \right\rceil \geq T - k + 1$, then by the pigeon hole principle, for any $v \in V(K_{c\times s})$ there exists $i \in \{1, 2, \ldots, k\}$ such that $d_i(v) \geq \sum_{j=1}^{l_i} n_{i,j}$. The rest of the proof is similar to that in Theorem 2.1. □

In 1979, Grossman [5] studied the (classical) Ramsey number for the union of stars and proved the following exact value of this Ramsey number.

Theorem 2.3. *[5] Let n, m be two integers with $n \geq m \geq 1$, and let $K_{1,n}, K_{1,m}$ be two stars. Then*

$$r(K_{1,n} \cup K_{1,m}, K_{1,n} \cup K_{1,m}) = \max\{n + 2m, 2n + 1, n + m + 3\}.$$

Recently in 2019, Perondi and Carmelo [10, 11] showed that the classical Ramsey number could be used to obtain some lower bound of the multipartite Ramsey numbers as stated in the following theorems.

Theorem 2.4. *[10] Let k and s be positive integers, where $k \geq 2$. For simple graphs $G_1, \ldots, G_k$,*

$$\left\lfloor \frac{r(G_1, \ldots, G_k) - 1}{s} \right\rfloor + 1 \leq M_s(G_1, \ldots, G_k).$$

Theorem 2.5. *[11] Suppose that $m_c(G_1, \ldots, G_k)$ exists. The following connection holds*

$$\left\lfloor \frac{r(G_1, \ldots, G_k) - 1}{c} \right\rfloor + 1 \leq m_c(G_1, \ldots, G_k).$$

Substituting the Grossman's result for the union of stars (Theorem 2.3) into the general bounds in Theorems 2.4 and 2.5, we obtain Corollaries 2.1 and 2.2.

Corollary 2.1. *Let $n, m, k, c,$ and s be positive integers where $n \geq m \geq 1$ and $k \geq 2$, then*

$$M_s(K_{1,n} \cup K_{1,m}; 2) \geq \left\lfloor \frac{\max\{n+2m-1, 2n, n+m+2\}}{s} \right\rfloor + 1$$

and

$$m_c(K_{1,n} \cup K_{1,m}; 2) \geq \left\lfloor \frac{\max\{n+2m-1, 2n, n+m+2\}}{c} \right\rfloor + 1.$$

Corollary 2.2. *Let $n, m, t,$ and $(n-1)m/2$ be positive integers.*

- *If $t \leq (m-1)/2$, $m \geq 7$, and $n \geq 3$, then $M_m(K_{1,(n-1)m/2+1} \cup K_{1,t}; 2) \geq n$.*
- *If $t \leq (n-2)/2$, $n \geq 8$, and $m \geq 3$, then $m_n(K_{1,(n-1)m/2+1} \cup K_{1,t}; 2) \geq m + 1 + \left\lfloor \frac{-m+2}{n} \right\rfloor$.*

We provide some lower bounds of the set and size multipartite Ramsey numbers for the union of stars in the next theorem.

Theorem 2.6. *Let $m, n, k, l_i, t_{i,j}$ be natural numbers for $i \in \{1, 2, \ldots, k\}$, $j \in \{1, 2, \ldots, l_i\}$, and $m, n, k \geq 2$. Let $G_i = \bigcup_{j=1}^{l_i} K_{1,t_{i,j}}$ where $t_{i,1} = (n-1)m/k + 1$ for $i = 1, 2, \ldots, k$. If (k divides m) or (n is even and k divides $n-1$), then*

$$M_m(G_1, G_2, \ldots, G_k) \geq n + 1$$

and

$$m_n(G_1, G_2, \ldots, G_k) \geq m + 1.$$

Proof. Consider the graph $K_{n \times m}$ where its vertices are partitioned into n classes $L_1, L_2, \ldots, L_n$, where $L_a = \{(a,1), (a,2), \ldots, (a,m)\}$ for $a \in \{1, 2, \ldots, n\}$. For $(i,j) \in V(K_{n \times m})$, define the function $g : V(K_{n \times m}) \to \{1, 2, \ldots, m\}$ as $g((i,j)) = j$.

Case 1. k divides m. Define an edge coloring $\psi : E(K_{n \times m}) \to \mathbb{Z}_k = \{\overline{0}, \overline{1}, \ldots, \overline{k-1}\}$ on graph $K_{n \times m}$ as follows:

$$\psi((a,i)(b,j)) = \overline{i+j}.$$

Since $\psi((a,i)(b,j)) = \psi((b,j)(a,i))$, then ψ is well defined. For $x \in V(K_{n \times m})$, $W \subseteq V(K_{n \times m})$, and $i \in \{0, 1, \ldots, k-1\}$, let $N_i(x, W) = \{y \in W | \psi(xy) = \overline{i}\}$. Let $a' \in \{1, 2, \ldots, n\}$ such that $x \notin L_{a'}$, then

$$\begin{aligned} |N_i(x, L_{a'})| &= |\{y \in L_{a'} |\, \psi(xy) = \overline{i}\}| \\ &= |\{j \in \{1, 2, \ldots, m\} |\, \overline{g(x) + j} = \overline{i}\}| \\ &= |\{j \in \{1, 2, \ldots, m\} |\, \overline{g(x)} = \overline{i-j}\}| \\ &= m/k. \end{aligned}$$

Therefore, we have $|N_i(x, V(K_{n \times m}))| = (n-1)m/k$.

Case 2. n is even and k divides $n-1$. Consider the complete graph K_n where $V(K_n) = \{w_1, \ldots, w_n\}$. Since n is even, then the chromatic index of K_n is $\chi'(K_n) = n-1$. Consequently, there is an edge coloring $\tau : E(K_n) \rightarrow \mathbb{Z}_{n-1} = \{\overline{0}, \overline{1}, \ldots, \overline{n-2}\}$ satisfying that every two adjacent edges have different color. Define an edge coloring $\psi : E(K_{n\times m}) \rightarrow \mathbb{Z}_{n-1}$ on graph $K_{n\times m}$ as follows:

$$\psi((a,i)(b,j)) = \tau(w_a w_b).$$

Since $\psi((a,i)(b,j)) = \psi((b,j)(a,i))$, then ψ is well defined. Since k divides $n-1$, then we can define a function $\zeta : \mathbb{Z}_{n-1} \rightarrow \mathbb{Z}_k$ such that $\zeta(\bar{i}) = \bar{j}$ if and only if $i \equiv j \pmod{k}$. Hence, we have an edge coloring $\zeta \circ \psi : E(K_{n\times m}) \rightarrow \mathbb{Z}_k$ such that

$$|N_i(x, V(K_{n\times m}))| = |\{y \in V(K_{n\times m}) | \zeta \circ \psi(xy) = \bar{i}\}| = m(n-1)/k$$

for all $x \in V(K_{n\times m})$.

From all cases above, we conclude that we can find a k-coloring of the edges of $K_{n\times m}$ such that there is no monochromatic copy of G_i in color i for every $i \in \{1, 2, \ldots, k\}$. Therefore,

$$M_m(G_1, G_2, \ldots, G_k) \geq n+1$$

and

$$m_n(G_1, G_2, \ldots, G_k) \geq m+1.$$

□

In particular, the lower bounds in Theorem 2.6, for the case $k = 2$, are better than those in Corollary 2.2. Combining Theorems 2.1, 2.2, and 2.6, we obtain the following main result.

Theorem 2.7. *Let $m, n, k, l_i, t_{i,j}$ be natural numbers for $i \in \{1, 2, \ldots, k\}$, $j \in \{1, 2, \ldots, l_i\}$, $m, n \geq 3$, and $k \geq 2$. Let $G_i = \bigcup_{j=1}^{l_i} K_{1,t_{i,j}}$ where $t_{i,1} = (n-1)m/k + 1$ for $i = 1, 2, \ldots, k$. Let $L = \sum_{i=1}^{k} l_i$ and $T' = \sum_{i=2}^{k} \sum_{j=1}^{l_i} t_{i,j}$. If ($k$ divides m) or (n is even and k divides $n-1$), then we have the following:*

- *If $m > \max\{T', L-k\}$, then $M_m(G_1, G_2, \ldots, G_k) = n+1$.*
- *If $m \geq L-k$ and $n \geq T'+2$, then $m_n(G_1, G_2, \ldots, G_k) = m+1$.*

Proof. Theorem 2.6 provides the lower bounds. For the upper bounds, we use the fact that:

- If $m > \max\{T', L-k\}$, then by Theorem 2.1,

$$M_m(G_1, G_2, \ldots, G_k) \leq \left\lceil \frac{((n-1)m + k + T') - k + 1}{m} \right\rceil + 1 = n+1.$$

- If $m \geq L-k$ and $n \geq T'+2$, then by Theorem 2.2,

$$m_n(G_1, G_2, \ldots, G_k) \leq \left\lceil \frac{((n-1)m + k + T') - k + 1}{n-1} \right\rceil = m+1.$$

□

3. Conclusion and Open Problem

In this paper, we provide some upper bounds of the set and size multipartite Ramsey numbers (for k colors) for the union of stars, as stated in Theorem 2.1 and Theorem 2.2. We also provide a class of the union of stars achieving the upper bounds in Theorem 2.7. However, many parameters are still excluded in Theorem 2.7, which could lead to interesting further study.

Problem 1. *Let* $H_i = \bigcup_{j=1}^{l_i} K_{1,n_{i,j}}$ *for* $i = 1, 2, \ldots, k$ *with* $k \geq 2$*. Find the exact values of* $M_s(H_1, H_2, \ldots, H_k)$ *and* $m_c(H_1, H_2, \ldots, H_k)$ *for the remaining parameters excluded in Theorem 2.7.*

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