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The locating chromatic number of (k, n)-split cycle graph and its barbell operation

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Abstract

The locating chromatic number remains an active topic in graph theory. It combines the concepts of partition dimension and proper vertex coloring. A necessary condition for determining the locating chromatic number is that each vertex must have a unique color code under a minimal coloring. This paper investigates the locating chromatic number of the (k, n)-split cycle graph and its barbell operation.

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1. Introduction

Let $G = (V, E)$ be a finite and connected graph. A $l-$*coloring* of G is a function $c : V(G) \longrightarrow 1, 2, \cdots, l$, where $c(u) \neq c(v)$ for any two adjacent vertices $u \neq v$ in G. Let $\Pi = \{C_1, C_2, \cdots, C_l\}$

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be a partition of $V(G)$, where C_i is the set of all vertices colored by the color i for $1 \leq i \leq l$. The *color code* $c_\Pi(v)$ of a vertex v in G is defined as the $l-$ordinate $(d(v,C_1), d(v,C_2), \cdots, d(v,C_l))$, where $d(v,C_i) = \min\{d(v,x); x \in C_i\}$ for $1 \leq i \leq l$. The $l-$coloring c of G such that all vertices have different color codes is called a *locating coloring* of G. The *locating chromatic number* of G, denoted by $\chi_L(G)$, is the minimum l such that G has a locating coloring.

The locating chromatic number was studied by Chartrand et al. [4] for paths, cycles, complete multipartite graphs, double stars. Next, Chartrand et al. [5] gave a characterization of all graphs of order n with locating chromatic number $(n-1)$. Asmiati et al. [1] obtained the locating chromatic number of amalgamation of stars, firecracker graphs [2], whereas Irawan et al.[8] for origami graphs, and Syofyan et al.[12] for homogeneous lobster. In 2018, Asmiati et al. [3] determined the locating chromatic number of barbell graphs contains complete graph and generalized Petersen graphs. For other operations, Baskoro and Purwasih [6] determined the locating chromatic number of corona product, Behtoei and Anbarloei [7] for join of graphs, and Sudarsana et al.[11] for shadow of a connected graph. Next, Ridwan et al. [10] discussed some general connections among partition dimension and locating chromatic number of graphs.

The following definition of $(k,n)-$split cycle graph is taken from [9]. A $(k,n)-$split cycle graph has the vertex set $V = \{v_i, v_i^j; i \in [1,n], j \in [1,k]\}$ and the edge set $E = \{v_iv_{i+1}; i \in [1,n-1]\} \cup \{v_nv_1\} \cup \{v_iv_{i+1}^j; i \in [1,n-1], j \in [1,k]\} \cup \{v_nv_1^j; j \in [1,k]\} \cup \{v_{i+1}v_i^j; i \in [1,n-1], j \in [1,k]\} \cup \{v_1v_n^j; j \in [1,k]\}$.

Prawinasti et al. [9] determined the locating chromatic number of $(1,n)-$split cycle graph for $n \geq 3$. In this paper, we do further results about the locating chromatic number of$(k,n)-$split cycle graph, for $k \geq 2$ and its barbell operation. The barbell of a $(k,n)-$split cycle graph is formed by taking two copies of the $(k,n)-$split cycle graph and connecting them by a bridge, denoted by $B(k,n)-$split cycle graph. A $B(k,n)-$split cycle graph has the vertex set $V = \{v_i, w_i, v_i^j, w_i^j; i \in [1,n], j \in [1,k]\}$ and the edge set $E = \{v_iv_{i+1}, w_iw_{i+1}; i \in [1,n-1]\}$ $\cup\{v_nv_1, w_nw_1\} \cup \{v_iv_{i+1}^j, w_iw_{i+1}^j; i \in [1,n-1], j \geq 1, i=j\} \cup \{v_nv_1^j, w_nw_1^j; j \geq 1\} \cup$ $\{v_{i+1}v_i^j, w_{i+1}w_i^j; i \in [1,n-1], j \geq 1, i=j\} \cup \{v_1v_n^j, w_1w_n^j; j \geq 1\} \cup \{e\}$, where $e = (v^k_{\frac{n+1}{2}}, w^k_{\frac{n+1}{2}})$ is a bridge for odd n and $e = (v^k_{\frac{n}{2}}, w^k_{\frac{n}{2}})$ for even n.

The following basic theorems are needed to determine the lower bound of the locating chromatic number of a graph. The set of neighbors of a vertex q in G is denoted by $N(q)$.

Theorem 1.1. *(see [4]). Let c be a locating coloring in a connected graph G. If u and v are two distinct vertices of G such that $d(u,w) = d(v,w)$ for all $w \in V(G) - \{u,v\}$, then $c(u) \neq c(v)$. In particular, if u and v are nonadjacent vertices such that $N(u) = N(v)$, then $c(u) \neq c(v)$.*

Locating chromatic number of $(1,n)-$split cycle graph for $n \geq 3$ is given in the following theorem.

Theorem 1.2. *(see [9]). Let G be a $(1,n)-$split cycle graph with $n \geq 3$. Then the locating chromatic number of G is 4 if n is odd and 5 if n is even. .*

2. Main results

The following theorems give the locating chromatic number of $(k,n)-$split cycle graph and $B(k,n)-$split cycle graph, respectively.

Theorem 2.1. *For $n \geq 3$ and $k \geq 2$, the locating chromatic number of $(k,n)-$split cycle graph is $(k+2)$ if n is odd and $(2k+3)$ if n is even.*

Proof. First, we determine the lower bound for the locating-chromatic number of $(k,n)-$split cycle graph, $n \geq 3$ and $k \geq 2$. Observe that $d(u, v_i^h) = d(u, v_i^l)$ for all $u \in V(G) - \{v_i^h, v_i^l\}$, where $h, l \geq 1$ and $h \neq l$. Then, by Theorem 1.1, we have $c(v_i^h) \neq c(v_i^l)$. As a result, we need at least $(k+2)$ colors for $(k - spl(C_n))$, $n \geq 3$ and $k \geq 2$. Similarly, we have $(k,n)-$split cycle graph $\geq (2k+3)$ for even n.

Next, We now construct an upper bound of locating chromatic number for $(k,n)-$split cycle graph, $n \geq 3$ and $k \geq 2$. Consider the following two cases.

Case 1. n is odd. Let c be a coloring using $(k+2)$ colors as follows :

$$c(v_i) = \begin{cases} 1, & \text{for } i = 1; \\ 2, & \text{for odd } i \geq 3; \\ 3, & \text{for even } i\ , i \geq 2. \end{cases}$$

$$c(v_i^2) = \begin{cases} 1, & \text{for } i = 1; \\ 2, & \text{for odd } i; \\ 3, & \text{for even } i. \end{cases}$$

$c(v_i^1) = 4$ for $i \in [1,n]$

$c(v_i^j) = j + 2$ for $j \in [3,k], i \in [1,n]$

The color codes of $(k,n)-$split cycle graph for odd n are :

$$c_\pi(v_i) = \begin{cases} i-1, & 1^{st} \text{ ordinate for } i \leq \frac{n+1}{2}; \\ n-i+1, & 1^{st} \text{ ordinate for } i > \frac{n+1}{2}; \\ 0, & 2^{nd} \text{ ordinate for even } i; \\ & 3^{rd} \text{ ordinate for odd } i\ , i \geq 3; \\ 1, & \text{other ordinates.} \end{cases}$$

$$c_\pi(v_i^j) = \begin{cases} i-1, & 1^{st} \text{ ordinate for } 2 \leq i \leq \frac{n+1}{2},\ j = 1, 3; \\ & 1^{st} \text{ ordinate for } i \leq \frac{n+1}{2},\ j = 2; \\ n-i+1, & 1^{st} \text{ ordinate for } i > \frac{n+1}{2},\ j \geq 1; \\ 0, & 4^{th} \text{ ordinate for } i \geq 1,\ j = 1; \\ & 2^{nd} \text{ ordinate for even } i\ , j = 2; \\ & 3^{rd} \text{ ordinate for odd } i\ , i \geq 3,\ j = 2; \\ & (j+2)^{th} \text{ ordinate for } j \geq 3, i \geq 1, n; \\ 1, & 2^{nd} \text{ ordinate for odd i},\ j \geq 1 \\ & 3^{rd} \text{ ordinate for } i = 1 \text{ and odd } i,\ j \geq 1 \\ 2, & \text{other ordinates.} \end{cases}$$

Since all vertices of $(k,n)-$split cycle graph for odd n, $n \geq 3$ and $k \geq 2$ have distinct color codes, then c is a locating coloring using $k+2$ colors. As a result, $(k,n)-$split cycle graph $\leq k+2$.

Thus $\chi_L((k,n)-$split cycle graph$)= k+2$.

Case 2. n is even. Let c be a coloring using $(2k+3)$ colors for $k \geq 2$ as follows :

$$c(v_i) = \begin{cases} 1, & \text{for } i=1; \\ 2, & \text{for even } i,\ 2 \leq i \leq n-1; \\ 3, & \text{for odd } i,\ i \geq 3; \\ 4, & \text{for } i=n. \end{cases}$$

$$c(v_i^j) = \begin{cases} 3, & \text{for } i=1 \text{ and } j=1; \\ 4, & \text{for } i=n \text{ and } j=1; \\ 2j+3, & \text{for } 1 \leq j \leq k,\ 2 \leq i \leq n-1; \\ 2j+2, & \text{for } 2 \leq j \leq k,\ i=1,n. \end{cases}$$

The color codes of $(k,n)-$split cycle graph for even n are :

$$c_\pi(v_i) = \begin{cases} i-1, & 1^{st} \text{ ordinate for } i \leq \frac{n}{2}; \\ & (4+2j)^{th} \text{ ordinate for } 3 \leq i \leq \frac{n}{2},\ n \geq 6,\ j \geq 1; \\ n-i+1, & 1^{st} \text{ ordinate for } i > \frac{n}{2}; \\ i, & 4^{th} \text{ ordinate for } i \leq \frac{n}{2}; \\ n-i, & (4+2j)^{th} \text{ ordinate for } \frac{n}{2}+1 \leq i \leq n-2,\ n \geq 6,\ j \geq 1; \\ 0, & 2^{nd} \text{ ordinate for even i},\ 2 \leq i \leq n-2; \\ & 3^{rd} \text{ ordinate for odd i},\ i \geq 3; \\ 2, & 2^{nd} \text{ ordinate for } i=n; \\ & 3^{rd} \text{ ordinate for } i=1; \\ 1, & \text{other ordinates.} \end{cases}$$

$$c_\pi(v_i^j) = \begin{cases} i-1, & 1^{st} \text{ ordinate for } 2 \leq i \leq \frac{n}{2},\ j \geq 1; \\ & (4+2j)^{th} \text{ ordinate for } 4 \leq i \leq \frac{n}{2},\ n \geq 8,\ j \geq 1; \\ n-i+1, & 1^{st} \text{ ordinate for } i > \frac{n}{2},\ j \geq 1; \\ i, & 4^{th} \text{ ordinate for } i \leq \frac{n}{2},\ j \geq 1; \\ n-i, & 4^{th} \text{ ordinate for } i > \frac{n}{2},\ j=1; \\ & 4^{th} \text{ ordinate for } \frac{n}{2}+1 \leq i \leq n-1,\ j \geq 2; \\ & (4+2j)^{th} \text{ ordinate for } \frac{n}{2}+1 \leq i \leq n-3,\ j \geq 1,\ n \geq 8; \\ 0, & 3^{rd} \text{ ordinate for } i=1,\ j=1; \\ & (2j+3)^{th} \text{ ordinate for } j \geq 1,\ 2 \leq i \leq n-1; \\ & (2j+2)^{th} \text{ ordinate for } j \geq 2,\ i=1 \text{ and } n; \\ 1, & 2^{nd} \text{ ordinate for odd i},\ j \geq 1; \\ & 3^{rd} \text{ ordinate for even } i,\ j \geq 1; \\ 2, & \text{other ordinates.} \end{cases}$$

Since all vertices in $(k,n)-$split cycle graph for even n have distinct color codes, then c is a locating coloring using $2k+3$ colors for $k \geq 1$. As a result, $\chi_L((k,n)-$split cycle graph$) \leq 2k+3$. Thus $\chi_L((k,n)-$split cycle graph$) = 2k+3$ for even n. $\square$

Theorem 2.2. *The locating chromatic number of $B(k,n)-$split cycle graph, $n \geq 3$ and $k \geq 2$ is $(k+3)$ for odd $n \geq 3$ and $(2k+4)$ for otherwise.*

Proof. First, we determine the lower bound of locating-chromatic number for $B(k,n)-$split cycle graph with odd n. Since $B(k,n)-$split cycle graph contains $(k,n)-$split cycle graph, then by Theorem 2.1, we need at least $k+2$ colors. Suppose that c is a $(k+2)$-locating coloring of $B(k,n)-$split cycle graph. $B(k,n)-$split cycle graph contains two $(k,n)-$split cycle graphs and $c(v_i^s) \neq c(v_i^t)$, where $s \neq t$, $s,t \geq 0$. Since we use $(k+2)$ colors, then we have $c(v_i^s) = c(w_i^s)$ such that $c_\pi(v_i^s) = c_\pi(w_i^s)$, a contradiction. As a result, $\chi_L(B(k,n)-$split cycle graph$) \geq k+3$ for odd n. The case is similar for even n.

Next, we determine the upper bound of the locating chromatic number for $B(k,n)-$split cycle graph, $n \geq 3$ and $k \geq 2$. Consider the following two cases.

Case 1. n is odd. Let c be a coloring using $(k+2)$ colors as follows:

$$c(v_i) = \begin{cases} 1, & \text{for } i = 1; \\ 2, & \text{for even } i; \\ 3, & \text{for odd } i,\ i \geq 3. \end{cases}$$

$$c(w_i) = \begin{cases} 1, & \text{for } i = 1; \\ 2, & \text{for even } i; \\ 3, & \text{for odd } i,\ i \geq 3. \end{cases}$$

$c(v_i^1) = 4$, for $i \geq 1$

$c(w_i^1) = 5$, for $i \geq 1$

$$c(v_i^2) = \begin{cases} 1, & \text{for } i = 1; \\ 2, & \text{for even } i,\ i \neq \frac{n+1}{2}; \\ 3, & \text{for odd } i,\ i \neq 1 \text{ and } i \neq \frac{n+1}{2}; \\ 4, & \text{for } i = \frac{n+1}{2}. \end{cases}$$

$$c(w_i^2) = \begin{cases} 1, & \text{for } i = 1; \\ 2, & \text{for even } i,\ i \neq \frac{n+1}{2}; \\ 3, & \text{for odd } i,\ i \neq 1 \text{ and } i \neq \frac{n+1}{2}; \\ 5, & \text{for } i = \frac{n+1}{2}. \end{cases}$$

Case $k \geq 3$. $c(v_i^2) = \begin{cases} 1, & \text{for } i = 1; \\ 2, & \text{for even } i; \\ 3, & \text{for odd } i,\ i \geq 3. \end{cases}$

$$c(v_i^2) = \begin{cases} 1, & \text{for } i = 1; \\ 2, & \text{for even } i; \\ 3, & \text{for odd } i,\ i \geq 3. \end{cases}$$

$c(v_i^j) = j + 2$, for $i \geq 1$ and $3 \leq j \leq k$.
$c(w_i^j) = j + 3$, for $i \geq 1$ and $3 \leq j \leq k$.
The color codes of $B(k, n)-$split cycle graph for odd n are:

$$c_\pi(v_i) = \begin{cases} i-1, & 1^{st} \text{ ordinate for } i \leq \frac{n+1}{2},\ n \geq 3; \\ n-i+1, & 1^{st} \text{ ordinate for } i > \frac{n+1}{2},\ n \geq 3; \\ \frac{n+1}{2}+1-i, & 5^{th} \text{ ordinate for } i < \frac{n+1}{2},\ n \geq 3,\ j=1; \\ & (j+3)^{th} \text{ ordinate for } i < \frac{n+1}{2},\ n \geq 3,\ j \geq 2; \\ i+1-\frac{n+1}{2}, & 5^{th} \text{ ordinate for } i > \frac{n+1}{2},\ n \geq 3,\ j=1; \\ & (j+3)^{th} \text{ ordinate for } i > \frac{n+1}{2},\ n \geq 3,\ j \geq 2; \\ 0, & 2^{nd} \text{ ordinate for even } i; \\ & 3^{rd} \text{ ordinate for odd } i\ , i \geq 3; \\ 3, & 5^{th} \text{ cordinate for } i = \frac{n+1}{2},\ n \geq 3,\ j=1; \\ & (j+3)^{th} \text{ ordinate for } i = \frac{n+1}{2},\ n \geq 3,\ j \geq 2; \\ 1, & \text{other ordinates.} \end{cases}$$

$$c_\pi(v_i^j)=\begin{cases} i-1, & 1^{st} \text{ ordinate for } 2\le i\le \frac{n+1}{2},\ n\ge 3,\ j\ge 3;\\ & 1^{st} \text{ ordinate for } i\le \frac{n+1}{2},\ n\ge 3,\ j=2;\\ n-i+1, & 1^{st} \text{ ordinate for } i> \frac{n+1}{2},\ n\ge 3,\ j\ge 1;\\ (\frac{n+1}{2})+1-i, & 5^{th} \text{ ordinate for } i<(\frac{n+1}{2})-1,\ n\ge 5,\ k\le 2;\\ & (j+5)^{th} \text{ ordinate for } i<(\frac{n+1}{2})-1,\ n\ge 5,\ k\ge 3,\ j\ge 1;\\ i+1-(\frac{n+1}{2}), & 5^{th} \text{ ordinate for } i>(\frac{n+1}{2})+1,\ n\ge 5,\ j\le 2;\\ & (j+5)^{th} \text{ ordinate for } i>(\frac{n+1}{2})+1,\ n\ge 5,\ k\ge 3,\ j\ge 1;\\ 0, & 4^{th} \text{ ordinate for } i=1,...,n,\ n\ge 3,\ j=1;\\ & 2^{nd} \text{ ordinate for even } i,\ i\ne \frac{n+1}{2},\ n\ge 3,\ k=2,\ j=2;\\ & 2^{nd} \text{ ordinate for even } i,\ n\ge 3,\ k\ge 3,\ j=2;\\ & 3^{rd} \text{ ordinate for odd } i,\ i\ge 3,\ i\ne \frac{n+1}{2},\ n\ge 3,\ k\ge 2,\ j=2;\\ & 3^{rd} \text{ ordinate for odd } i,\ i\ge 3,\ n\ge 3,\ k\ge 3,\ j=2;\\ & 4^{th} \text{ ordinate for } i=\frac{n+1}{2},\ n\ge 3,\ k\le 2,\ j=2;\\ & (j+2)^{th} \text{ ordinate for } i=1,...,n,\ k\ge 3,\ j\ge 3;\\ 1, & 5^{th} \text{ ordinate for } i=\frac{n+1}{2},\ n\ge 3,\ k\le 2,\ k=j;\\ & 2^{nd} \text{ ordinate for odd } i,\ n\ge 3,\ j\ge 1;\\ & 3^{rd} \text{ ordinate for even } i \text{ and } i=1,\ n\ge 3,\ j\ge 1;\\ & (j+3)^{th} \text{ ordinate for } i=\frac{n+1}{2},\ n\ge 3,\ k\ge 3,\ k=j;\\ 3, & 5^{th} \text{ ordinate for } i=1 \text{ and } i=n,\ n=3,\ k\le 2,\ k=j;\\ & (j+3)^{th} \text{ ordinate for } i=1 \text{ and } i=n,\ n=3,\ k\ge 3,\ k=j;\\ & (j+4)^{th} \text{ ordinate for } i\ge 1,\ n=3,\ k\ge 2,\ j\le k-1;\\ & (j+4)^{th} \text{ ordinate for } i=\frac{n+1}{2},\ n\ge 5,\ k\ge 2,\ j\le k-1;\\ 4, & (j+4)^{th} \text{ ordinate for } i=(\frac{n+1}{2})-1 \text{ and } i=(\frac{n+1}{2})+1,\ n\ge 5,\ k\ge 2,\ j\ge 1;\\ 2, & \text{other ordinates.} \end{cases}$$

$$c_\pi(w_i)=\begin{cases} i-1, & 1^{st} \text{ ordinate for } i\le \frac{n+1}{2},\ n\ge 3;\\ n-i+1, & 1^{st} \text{ ordinate for } i> \frac{n+1}{2},\ n\ge 3;\\ (\frac{n+1}{2})+1-i, & 4^{th} \text{ ordinate for } i<\frac{n+1}{2},\ n\ge 3,\ k\le 2;\\ i+1-(\frac{n+1}{2}), & 4^{th} \text{ ordinate for } i>\frac{n+1}{2},\ n\ge 3,\ k\le 2;\\ (\frac{n+1}{2})+3-i, & 4^{th} \text{ ordinate for } i<\frac{n+1}{2},\ n\ge 3,\ k\ge 2;\\ i+3-(\frac{n+1}{2}), & 4^{th} \text{ ordinate for } i>\frac{n+1}{2},\ n\ge 3,\ k\ge 3;\\ 0, & 2^{nd} \text{ ordinate for even } i,\ n\ge 3;\\ & 3^{rd} \text{ ordinate for odd } i,\ i\ge 3,\ n\ge 3;\\ 3, & 4^{th} \text{ coordinate for } i=\frac{n+1}{2},\ n\ge 3,\ k\le 2;\\ 5, & 4^{th} \text{ ordinate for } i=\frac{n+1}{2},\ n\ge 3,\ k\ge 3;\\ 1, & \text{other ordinates.} \end{cases}$$

$$c_\pi(w_i^j) = \begin{cases} i-1, & 1^{st} \text{ ordinate for } 2 \leq i \leq \frac{n+1}{2},\ j=1 \text{ and } k \geq 3; \\ & 1^{st} \text{ ordinate for } i \leq \frac{n+1}{2},\ n \geq 3,\ j=2; \\ n-i+1, & 1^{st} \text{ ordinate for } i > \frac{n+1}{2},\ n \geq 3,\ k \geq 1; \\ (\frac{n+1}{2})+1-i, & 4^{th} \text{ coordinate for } i < (\frac{n+1}{2})-1,\ n \geq 5,\ k \leq 2; \\ i+1-(\frac{n+1}{2}), & 4^{th} \text{ ordinate for } i > (\frac{n+1}{2})+1,\ n \geq 5,\ k \leq 2; \\ (\frac{n+1}{2})+3-i, & 4^{th} \text{ ordinate for } i < (\frac{n+1}{2})-1,\ n \geq 5,\ k \geq 3; \\ i+3-(\frac{n+1}{2}), & 4^{th} \text{ ordinate for } i > (\frac{n+1}{2})+1,\ n \geq 5,\ k \geq 3; \\ 0, & 5^{th} \text{ ordinate for } i \geq 1,\ n \geq 3,\ k=1,\ j=1; \\ & 2^{nd} \text{ ordinate for even } i,\ i \neq \frac{n+1}{2},\ n \geq 3,\ k=2,\ j=2; \\ & 2^{nd} \text{ ordinate for even } i,\ n \geq 3,\ k \geq 3,\ j=2; \\ & 3^{rd} \text{ ordinate for odd } i,\ i \geq 3,\ i \neq \frac{n+1}{2},\ n \geq 3,\ k=2,\ j=2; \\ & 3^{rd} \text{ ordinate for odd } i,\ i \geq 3,\ n \geq 3,\ k \geq 3,\ j=2; \\ & 5^{th} \text{ ordinate for } i = \frac{n+1}{2}, \imath \geq 3,\ n \geq 3,\ k \geq 3,\ j=2; \\ & (j+3)^{th} \text{ ordinate for } i \geq 1,\ n \geq 2,\ k \geq 3,\ j \geq 3; \\ 1, & 4^{th} \text{ ordinate for } i = \frac{n+1}{2},\ n \geq 3,\ k \leq 2,\ k=j; \\ & 2^{nd} \text{ ordinate for odd } i,\ n \geq 3,\ k \geq 1; \\ & 3^{rd} \text{ ordinate for even } i \text{ and } i=1,\ n \geq 3,\ k \geq 1; \\ 3, & 4^{th} \text{ ordinate for } i=1 \text{ and } i=n,\ n=3,\ k \leq 2,\ k=j; \\ & 4^{th} \text{ ordinate for } i \geq 1,\ n=3,\ k=2,\ j=k-1; \\ & 4^{th} \text{ ordinate for } i = \frac{n+1}{2},\ n \geq 5,\ k=2,\ j=k-1; \\ & 4^{th} \text{ ordinate for } i = \frac{n+1}{2},\ n=3,\ k \geq 3,\ k=j; \\ 4, & 4^{th} \text{ ordinate for } i = (\frac{n+1}{2})-1 \text{ and } i = (\frac{n+1}{2})+1,\ n \geq 5,\ k \leq 2,\ j \geq 1; \\ 5, & 4^{th} \text{ ordinate for } i \geq 1,\ n=3,\ k \geq 3,\ j \leq k-1; \\ & 4^{th} \text{ ordinate for } i=1 \text{ and } i=n,\ n=3,\ k \geq 3,\ k=j; \\ & 4^{th} \text{ ordinate for } i = \frac{n+1}{2},\ n \geq 5, k \geq 4,\ j \geq 1; \\ 6, & 4^{th} \text{ ordinate for } i = (\frac{n+1}{2})-1 \text{ and } i = (\frac{n+1}{2})+1,\ n \geq 5,\ k \leq 4,\ j \geq 1; \\ 2, & \text{other ordinates.} \end{cases}$$

Since all vertices in $B(k,n)-$split cycle graph for odd n have distinct color codes, then c is a locating coloring using colors for $k \geq 1$. As a result, $\chi_L(B(k,n)-$split cycle graph$) \leq 2k+3$. Thus $\chi_L(B(k,n)-$split cycle graph) = $2k+3$ for odd n.

Case 2. n is even. Let c be a coloring using $2k+4$ colors as follows :

$$c(v_i) = \begin{cases} 1, & \text{for } i=1; \\ 2, & \text{for even } i,\ 2 \leq i \leq n-2; \\ 3, & \text{for odd } i,\ i \geq 3; \\ 4, & \text{for } i=n. \end{cases}$$

$$c(w_i) = \begin{cases} 1, & \text{for } i = 1; \\ 2, & \text{for even } \ i, \ 2 \le i \le n-2; \\ 3, & \text{for odd } i, \ i \ge 3; \\ 4, & \text{for } i = n. \end{cases}$$

$$c(v_i^j) = \begin{cases} 3, & \text{for } i = 1, \ j = 1; \\ 4, & \text{for } i = n, \ j = 1; \\ 2j+2, & \text{for } i = 1 \text{ and } n, \ j \ge 2; \\ 2j+3, & \text{for } 2 \le i \le n-1, \ j \ge 1. \end{cases}$$

$$c(w_i^j) = \begin{cases} 3, & \text{for } i = 1, \ j = 1; \\ 4, & \text{for } i = n, \ j = 1; \\ 2j+3, & \text{for } i = 1 \text{ and } n, \ j \ge 2; \\ 2j+4, & \text{for } 2 \le i \le n-1, \ j \ge 1. \end{cases}$$

Since color $2k+4$ is only assigned to one $(k,n)-$split cycle graph of $B(k,n)-$split cycle graph, i.e. at the vertices v_i^s for some i, then the color codes of all of the vertices will be different. Since all vertices in $B(k,n)-$split cycle graph for even n have distinct color codes, then c is a locating coloring for $k \ge 1$. As a result, $\chi_L(B(k,n)-$split cycle graph$) \le 2k+4$. Thus $\chi_L(B(k,n)-$split cycle graph$) = 2k+4$ for even n. □

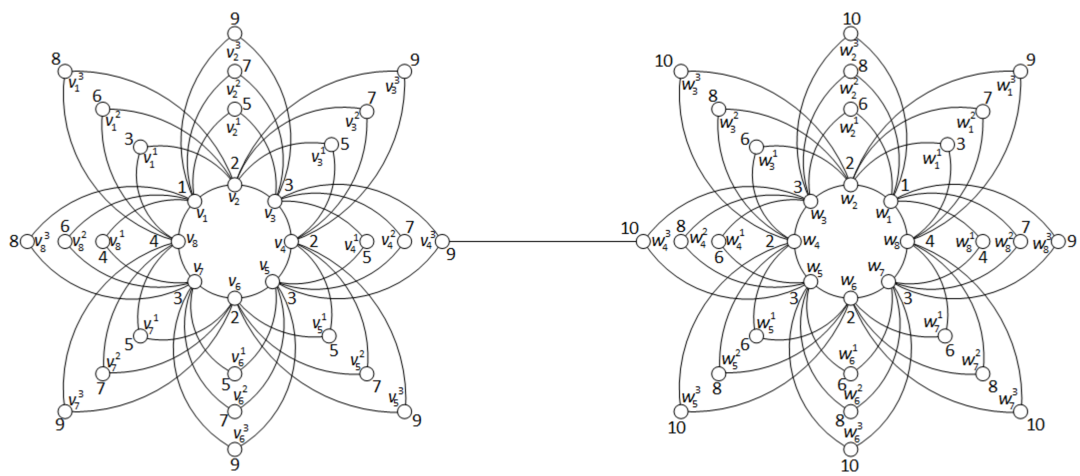

Figure 1. A minimum locating coloring of $B(3,8)-$split cycle graph

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