Electronic Journal of Graph Theory and Applications **10** (2) (2022), 385–392

Electronic Journal of Graph Theory and Applications

Maximum average degree of list-edge-critical graphs and Vizing's conjecture

Joshua Harrelson[a], Hannah Reavis[b]

[a]*Faculty of Mathematics and Statistics, Middle Georgia State University, United States*
[b]*Department of Mathematics and Statistics, Middle Georgia State University, United States*

joshua.harrelson@mga.edu, hannah.reavis@mga.edu

Abstract

Vizing conjectured that $\chi'_\ell(G) \leq \Delta + 1$ for all graphs. For a graph G and nonnegative integer k, we say G is a k-list-edge-critical graph if $\chi'_\ell(G) > k$, but $\chi'_\ell(G - e) \leq k$ for all $e \in E(G)$. We use known results for list-edge-critical graphs to verify Vizing's conjecture for G with $mad(G) < \frac{\Delta+3}{2}$ and $\Delta \leq 9$.

Keywords: list-edge-coloring, maximum average degree, discharging
Mathematics Subject Classification: 05C07, 05C10, 05C15
DOI: 10.5614/ejgta.2022.10.2.4

1. Introduction

We consider only simple graphs in this paper. It will be convenient for us to define for a graph G, the vertex set $V_x = \{v \in V(G) \mid d(v) = x\}$ and the set $V_{[x,y]} = \{v \in V(G) \mid x \leq d(v) \leq y\}$. An *edge-coloring* of G is a function which maps one color to every edge of G such that adjacent edges receive distinct colors. A k-edge-coloring of G is an edge-coloring of G which maps a total of k colors to $E(G)$. The chromatic index $\chi'(G)$ is the minimum k such that G is k-edge-colorable. Vizing's Theorem [10] gives us $\chi'(G) \leq \Delta + 1$ for all graphs G where Δ is the maximum degree of G.

We are interested in a variation of edge-coloring called list-edge-coloring. A list-edge-coloring is an edge-coloring with the extra constraint that each edge can only be colored from a preassigned

Received: 6 August 2021, 13 April 2022, Accepted: 5 May 2022.

list of colors. Specifically, we say an *edge-list-assignment* of G is a function which maps a set of colors to every edge in G. If L is an edge-list-assignment of G, then we refer to the set of colors mapped to $e \in E(G)$ as the list, $L(e)$. We say that G is *L-colorable* if G can be properly edge-colored with every edge e receiving a color from $L(e)$. We say that G is *k-list-edge-colorable* if G is L-colorable for all L such that $|L(e)| \geq k$ for all $e \in E(G)$. We note this concept is referred to as k-edge-choosable in other papers. The list-chromatic index, $\chi'_\ell(G)$, is the minimum k such that G is k-list-edge-colorable. So, we want to achieve a list-edge-coloring for all list-assignments L with minimal list-size k.

It is easy to see that $\chi'_\ell(G) \geq \chi'(G) \geq \Delta$ for all graphs. The List-Edge Coloring Conjecture proposes that $\chi'_\ell(G) = \chi'(G)$, but this has only been verified for a few special families of graphs, such as Galvin's result for the family of bipartite graphs [6]. In this paper, we will focus on a relaxation of the LECC proposed by Vizing.

Conjecture 1 (Vizing [9])**.** *If G is a graph, then $\chi'_\ell(G) \leq \Delta + 1$.*

This conjecture has been verified for all graphs with $\Delta \leq 4$. The $\Delta = 3$ case was proved by Vizing [9] in 1976 and independently by Erdős, Rubin, and Taylor [5] in 1979. The $\Delta = 4$ case of Conjecture 1 was proved in 1998 by Juvan, Mohar, Škrekovski [8].

The *average degree* of a graph G is $ad(G) = \frac{\sum d(v)}{v(G)}$. The *maximum average degree* of a graph G is $mad(G) = max\{ad(H) : H \subseteq G\}$. That is, $mad(G)$ is the maximum of the set of average degrees of all subgraphs G.

Motivated by Vizing and the List Edge Coloring Conjecture, Woodall conjectured [11] if G has $mad(G) < \Delta - 1$, then $\chi'_\ell(G) = \Delta$. Together with Borodin and Kostochka, Woodall [2] was able to verify his conjecture when $mad(G) < \sqrt{2\Delta}$.

We say that a graph G is *k-list-edge-critical* if $\chi'_\ell(G) > k$, and $\chi'_\ell(G-e) \leq k$ for all $e \in E(G)$. By taking advantage of known results for *list-edge-critical* graphs, we relax Woodall's conjecture by bounding $\Delta(G) \leq 9$ to verify Conjecture 1 when $mad(G) < \frac{\Delta(G)+3}{2}$.

2. Main Result

In 1990, Borodin verified Conjecture 1 for planar graphs with $\Delta \geq 9$ (see [3]). This was improved to planar graphs with $\Delta \geq 8$ by Bonamy in 2015 (see [1]). In 2010, before Bonamy's result, Cohen and Havet wrote a new proof of Borodin's theorem which reduced the argument to about a single page (see [4]). Their new proof used the minimality of list-edge-critical graphs and a clever discharging argument. We state one of their lemmas below.

Lemma 2.1 (Cohen & Havet [4])**.** *If G is $(\Delta+1)$-list-edge-critical, then $deg(u)+deg(v) \geq \Delta+3$.*

Lemma 2.1, together with Borodin, Kostochka, Woodall's generalization [2] of Galvin's Theorem, were used to prove the following lemma. This lemma is listed as Lemma 9 in [7] and was used to achieve edge-precoloring results.

Lemma 2.2 (Harrelson, McDonald, Puleo [7])**.** *Let $a_0, a, b_0 \in \mathcal{N}$ such that $a_0 > 2$, $b_0 > a$, and $a + b_0 = \Delta + 3$. If G is $(\Delta + 1)$-list-edge-critical, then*

$$2\sum_{i=a_0}^{a}|V_i| < \sum_{j=b_0}^{\Delta}(a+j-\Delta-2)|V_j|.$$

We apply Lemma 2.2 directly to graphs of bounded maximum average degree to prove our main result.

Theorem 2.1. *If G has $\Delta(G)=\Delta\leq 9$ and $mad(G)<\frac{\Delta+3}{2}$, then $\chi'_\ell(G)\leq\Delta+1$.*

Proof. Let $m=\frac{\Delta+3}{2}$ and assign integers, which we will call an initial charge, to every vertex and an artificial, global pot P. We denote and define these initial charges as follows: $\alpha(P)=0$ and $\alpha(v)=d(v)$ for all $v\in V(G)$. Let $\alpha(G)$ denote the sum of all initial charges. We know $ad(G)=\frac{\sum d(v)}{v(G)}$, rather $\alpha(G)=ad(G)\cdot v(G)<m\cdot v(G)$. We will apply a discharging step and denote $\alpha'(v)$ as the final charge for $v\in V(G)$ after discharing. We will also use $\alpha'(P)$ and $\alpha'(G)$ to denote the final charges of P and G, respectively, after the discharging step. To get a contradiction, we will prove $\alpha'(G)\geq m\cdot v(G)$ by showing $\alpha'(P)>0$ and $\alpha'(v)\geq m$ for all $v\in V(G)$.

We note that this theorem is known for $\Delta\leq 4$ so we may assume $5\leq\Delta\leq 9$. For each of these values of Δ, we provide Tables 1 through 5. Each table provide a list of triples (a_0,a,b_0) and their resulting inequalities from Lemma 2.2. Each table also presents the discharging step and verifies $\alpha'(v)\geq m$ for all $v\in V(G)$. We let x_i be the sum of coefficients of V_i from the first table. For all values of Δ, we discharge in the following way; If $deg(v)=i\geq m$, then v will give x_i to P. If $deg(v)=i<m$, then v will take x_i from P.

For all values of Δ, we verify $\alpha'(P)>0$ by using only strict inequalities and noting the lesser side of every inequality only contains vertices with degree less than m and the greater side every inequality only contains vertices with degree greater than m. This means more charge is put into P than is taken from P due to how we defined x_i in our discharging step.

If $\Delta=9$, then we consider the ordered triples in the form of (a_0,a,b_0) and the system of inequalities resulting from Lemma 2.2 as displayed in Table 1. We note that the final charge of P is positive since adding all inequalities together yields:

$$x_3V_3+x_4V_4+x_5V_5<x_7V_7+x_8V_8+x_9V_9.$$

The final charges from Table 1 gives

$$a'(G)=\alpha'(P)+\sum_{v\in V(G)}\alpha'(v)>m\cdot v(G)$$

This is a contradiction for $\Delta=9$. We proceed through the remaining values of Δ using the same argument. We present a table for each value of Δ. Each table displays inequalities resulting from Lemma 2.2 and each table displays the discharging step to verify $\alpha'(v)>m$ and $\alpha'(P)>0$. Note that, for $\Delta=8$, we multiply the first inequality by 1/2.

This completes the proof of Theorem 2.1. □

Table 1. Inequalities and final charges for $\Delta = 9$.

Lemma 2.2 inequalities for $\Delta = 9$

(a_0, a, b_0)	Inequality
(3,5,7)	$V_3 + V_4 + V_5 < \frac{1}{2}V_7 + V_8 + \frac{3}{2}V_9$
(3,4,8)	$V_3 + V_4 < \frac{1}{2}V_8 + V_9$
(3,3,9)	$V_3 < \frac{1}{2}V_9$

Discharging for $\Delta = 9, m = 6$

$\alpha(v) = i$	x_i	$\alpha'(v)$
3	3	6
4	2	6
5	1	6
6	0	6
7	$\frac{1}{2}$	$\frac{13}{2}$
8	$\frac{3}{2}$	$\frac{13}{2}$
9	$\frac{6}{2}$	6

Table 2. Inequalities and final charges for $\Delta = 8$.

Lemma 2.2 inequalities for $\Delta = 8$

(a_0, a, b_0)	Inequality
(3,5,6)	$\frac{1}{2}[V_3 + V_4 + V_5] < \frac{1}{2}[\frac{1}{2}V_6 + \frac{2}{2}V_7 + \frac{3}{2}V_8]$
(3,4,7)	$V_3 + V_4 < \frac{1}{2}V_7 + \frac{2}{2}V_8$
(3,3,8)	$V_3 < \frac{1}{2}V_8$

Discharging for $\Delta = 8, m = \frac{11}{2}$

$\alpha(v) = i$	x_i	$\alpha'(v)$
3	$\frac{5}{2}$	$\frac{11}{2}$
4	$\frac{3}{2}$	$\frac{11}{2}$
5	$\frac{1}{2}$	$\frac{11}{2}$
6	$\frac{1}{4}$	$\frac{23}{4}$
7	1	6
8	$\frac{9}{4}$	$\frac{23}{4}$

Table 3. Inequalities and final charges for $\Delta = 7$

Lemma 2.2 inequalities for $\Delta = 7$

(a_0, a, b_0)	Inequality
(3,4,6)	$V_3 + V_4 < \frac{1}{2}V_6 + \frac{2}{2}V_7$
(3,3,7)	$V_3 < \frac{1}{2}V_7$

Discharging for $\Delta = 7, m = 5$

$\alpha(v) = i$	x_i	$\alpha'(v)$
3	2	5
4	1	5
5	0	5
6	$\frac{1}{2}$	$\frac{11}{2}$
7	$\frac{3}{2}$	$\frac{11}{2}$

Table 4. Inequalities and final charges for $\Delta = 6$.

Lemma 2.2 inequalities for $\Delta = 6$

(a_0, a, b_0)	Inequality
(3,4,5)	$V_3 + V_4 < \frac{1}{2}V_5 + \frac{2}{2}V_6$
(3,3,6)	$V_3 < \frac{1}{2}V_6$

Discharging for $\Delta = 6, m = \frac{9}{2}$

$\alpha(v) = i$	x_i	$\alpha'(v)$
3	2	5
4	1	5
5	$\frac{1}{2}$	$\frac{9}{2}$
6	$\frac{3}{2}$	$\frac{9}{2}$

Table 5. Inequalities and final charges for $\Delta = 5$.

Lemma 2.2 inequalities for $\Delta = 5$

(a_0, a, b_0)	Inequality
(3,3,5)	$V_3 < \frac{1}{2}V_5$

Discharging for $\Delta = 5, m = 4$

$\alpha(v) = i$	x_i	$\alpha'(v)$
3	1	4
4	0	4
5	$\frac{1}{2}$	$\frac{9}{2}$

3. Conclusion

The application of Lemma 2.2 can be improved for some values of $\Delta(G)$ presented in Theorem 2.1 to yield slightly greater values of $mad(G)$. We can also apply Lemma 2.2 to any value of $\Delta(G)$, but this will lower the bound on $mad(G)$. Specifically, we can find optimum values of $mad(G)$ given $\Delta(G)$ for graphs of higher max-degree by "reverse-engineering" the inequalities of Lemma 2.2 as shown in the following example for $\Delta(G) = 10$.

Example 1. Finding an optimal $mad(G)$ for $\Delta(G) = 10$.

Proof. Let $mad(G) < m$ for some m, let $\alpha(P) = 0$, and let $\alpha(v) = d(v)$ for all $v \in V(G)$. We wish to determine the largest number m such that $\alpha'(P) > 0$ and $\alpha'(v) \geq m$ for all $v \in V(G)$. We begin by presenting a table of triples and their resulting inequalities from Lemma 2.2; however, we multiply each inequality by an arbitrary constant.

Table 6. *Lemma 2.2 inequalities for* $\Delta = 10$

(a_0, a, b_0)	Inequality
$(3, 6, 10)$	$c_1(V_3 + V_4 + V_5 + V_6 < \frac{1}{2}V_7 + \frac{2}{2}V_8 + \frac{3}{2}V_9 + \frac{4}{2}V_{10})$
(3,5,10)	$c_2(V_3 + V_4 + V_5 < \frac{1}{2}V_8 + \frac{2}{2}V_9 + \frac{3}{2}V_{10})$
(3,4,10)	$c_3(V_3 + V_4 < \frac{1}{2}V_9 + \frac{2}{2}V_{10})$
(3,3,10)	$c_4(V_3 < \frac{1}{2}V_{10})$

As in Theorem 2.1, we let x_i be the sum of coefficients of V_i from this table. We will let "high-degree" vertices give charge to P while "low-degree" vertices take charge from P in the rules that follow. If $deg(v) = i \geq \lceil \frac{1}{2}\Delta + 2 \rceil$, then v gives x_i to P. If $deg(v) = i \leq \lfloor \frac{1}{2}\Delta + 1 \rfloor$, then v takes x_i from P. This yields the list of final charges displayed in Table 6. We set each final charge greater than or equal to m.

Table 7. Final charges for Example 1.

V_i	Final Charge $\geq m$	Name
V_3	$3 + c_1 + c_2 + c_3 + c_4 \geq m$	A
V_4	$4 + c_1 + c_2 + c_3 \geq m$	B
V_5	$5 + c_1 + c_2 \geq m$	C
V_6	$6 + c_1 \geq m$	D
.	.	.
.	.	.
.	.	.
V_{10}	$10 - \frac{4}{2}c_1 - \frac{3}{2}c_2 - \frac{2}{2}c_3 - \frac{1}{2}c_4 \geq m$	E

Increasing the constants c_1,c_2,c_3,c_4 increases the final charge of our "low-degree" vertices, but decreases the final charge of our "high-degree" vertices. We need all final charges to be greater than or equal to m so we must chose m carefully. While all vertices in $V_{[7,10]}$ give charge away, the vertices in V_{10} give the most, meaning inequality E has the strictest bound on m. With this in mind, we can find an optimal bound for m by adding inequalities in the following way:
$2E + A + B + C + D \implies 38 + 0x_1 + 0x_2 + 0x_3 \geq 6m \implies \frac{19}{3} \geq m$

We can now use this bound and the inequalities of the "low-degree" vertices from Table 6 to solve for c_1,c_2,c_3,c_4.

$$
\begin{array}{lll}
D & V_6 : 6 + c_1 \geq \dfrac{19}{3} & \implies c_1 = \frac{1}{3} \\
C & V_5 : 5 + c_1 + c_2 \geq \dfrac{19}{3} & \implies c_2 = 1 \\
B & V_4 : 4 + c_1 + c_2 + c_3 \geq \dfrac{19}{3} & \implies c_3 = 1 \\
A & V_3 : 3 + c_1 + c_2 + c_3 + c_4 \geq \dfrac{19}{3} & \implies c_4 = 1
\end{array}
$$

We have shown that $\alpha'(v) \geq \frac{19}{3}$ for our "low-degree" vertices in $V_{[3,6]}$. We only need to verify the values of c_1, c_2, c_3, c_4, and m give us appropriate inequalities for the "high-degree" vertices.

$$V_7 : 7 - \frac{1}{2}c_1 > \frac{19}{3}$$

$$V_8 : 8 - \frac{2}{2}c_1 - \frac{1}{2}c_2 > \frac{19}{3}$$

$$V_9 : 9 - \frac{3}{2}c_1 - \frac{2}{2}c_2 - \frac{1}{2}c_3 > \frac{19}{3}$$

$$V_{10} : 10 - \frac{4}{2}c_1 - \frac{3}{2}c_2 - \frac{2}{2}c_3 - \frac{1}{2}c_4 = \frac{19}{3}$$

So $m = \frac{19}{3}$ is a feasible bound for $mad(G)$ when $\Delta(G) = 10$. This means if a graph H has $\Delta(H) \leq 10$ and $mad(H) < \frac{19}{3}$, then $\chi'_\ell(H) \leq \Delta + 1$. □

Lemma 2.2 can be thought of as a generalization Cohen and Havet's argument in [4]. Both of these results use forbidden structures to force good counts of low and high degree vertices by relying on Galvin's Theorem [6]. In this sense, good counts are achieved from knowing the list-edge-colorability of bipartite graphs. We are interested in how the list-edge-colorability of other simple families of graphs could be used to develop counts to verify Vizing's Conjecture or even the LECC for a wider range of graphs than is currently known.

References

[1] M. Bonamy, Planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$-edge-choosable, *Seventh Euro. Conference in Comb., Graph Theory and App., CRM series, Edizioni della Normale* **16** (2013). https://doi.org/10.1137/130927449

[2] O.V. Borodin, A. V. Kostochka, and D. R. Woodall, List edge and list total colorings of multigraphs, *J. Combin. Theory Ser. B* **71** (1997), 184-204. https://doi.org/10.1006/jctb.1997.1780

[3] O.V. Borodin, A generalization of Kotzig's theorem on prescribed edge coloring of planar graphs, *Mat. Zametki* **48** (1990), 1186-1190. https://doi.org/10.1007/BF01240258

[4] N. Cohen and F. Havet, Planar graphs with maximum degree $\Delta \leq 9$ are $(\Delta + 1)$-edge-choosable-a short proof, *Discrete Math.* **310** (2010), 3049-3051. https://doi.org/10.1016/j.disc.2010.07.004

[5] P. Erdõs, A. Rubin, and H. Taylor, Choosability in graphs. *Congr. Numer.* **26** (1979), 125-157.

[6] F. Galvin, The list chromatic index of a bipartite multigraph, *Journal of Combinatorial Theory, Ser. B* **63** (1995), 153-158. https://doi.org/10.1006/jctb.1995.1011

[7] J. Harrelson, J. McDonald, and G. Puleo, List-edge-colouring planar graphs with precoloured edges, *European J. of Combin.* **75** (2019), 55-65. https://doi.org/10.1016/j.ejc.2018.07.003.

[8] M. Juvan, B. Mohar, and R. Škrekovski, List total colorings of graphs, *Combin. Probab. Comput.* **7** (1998), 181-188. https://doi.org/10.1017/S0963548397003210

[9] V.G. Vizing, Colouring the vertices of a graph with prescribed colours, *Diskret. Analiz* **29** (1976), 3-10 (In Russian).

[10] V.G. Vizing, On an estimate of the chromatic class of a $p-$graph, *Diskret. Analiz* **3** (1964), 25–30.

[11] D. Woodall, The average degree of a multigraph critical with respect to edge or total choosability, *Discrete Math.* **310**(6-7) (2010), 1167-1171. https://doi.org/10.1016/j.disc.2009.11.011