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On the outer-independent double Italian domination number

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Abstract

An outer-independent Italian dominating function (OIIDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex $v \in V(G)$ with $f(v) = 0$ has at least two neighbors assigned 1 under f or one neighbor w with $f(w) = 2$, and the set $\{u \in V(G) | f(u) = 0\}$ is independent. An outer-independent double Italian dominating function (OIDIDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ such that if $f(v) \in \{0, 1\}$ for a vertex $v \in V(G)$, then $\sum_{u\in N[v]} f(u) \geq 3$ and the set $\{u \in V(G)|f(u) = 0\}$ is independent. The weight of an OIIDF (respectively, OIDIDF) f is the value $w(f) = \sum_{v\in V(G)} f(v)$. The minimum weight of an OIIDF (respectively, OIDIDF) on a graph G is called the outer-independent Italian (respectively, outer-independent double Italian) domination number of G. We characterize all trees T with outer-independent double Italian domination number twice the outer-independent Italian domination number. We also present lower bounds on the outer-independent double Italian domination number of a connected graph G in terms of the order, minimum and maximum degrees.

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1. Introduction

We consider simple connected graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n(G) = |V|$. The *open neighborhood* of a vertex v is the set $N(v) = \{u \in V(G) \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *degree* of vertex $v \in V$ is $\deg(v) = d(v) = |N(v)|$. The *maximum degree* and *minimum degree* among the vertices of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex*. A *strong support vertex* is a support vertex adjacent to more than one leaf. For a subset D of vertices of G, we denote by $G[D]$ the subgraph of G induced by D. A subset D of $V(G)$ is a *dominating set* in G if $\bigcup_{v \in D} N[v] = V(G)$. A set I of vertices of G is called *independent* if no pair of vertices of I are adjacent. If H is a subgraph of a graph G and f is a function defined on $V(G)$, then we denote by $f\mid_H$ the restriction of f on $V(H)$. For other notations and terminology not given here we refer to [14].

Cockayne et al. [12] introduced the concept of *Roman domination* in graphs, and since then many generalizations and related variations have been considered by researchers, see [7, 8, 9, 10, 11, 16, 19]. A variation of Roman domination, namely, double Roman domination is introduced by Beeler et al. [4]. A generalization of double Roman domination, namely Italian domination, is introduced by Chellali et al. in [6] and further studied in [15] and [17]. An *Italian dominating function* (IDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex $v \in V(G)$ with $f(v) = 0$ has at least two neighbors assigned 1 under f or one neighbor w with $f(w) = 2$. The *weight* of an IDF is the value $w(f) = \sum_{u \in V(G)} f(u)$. The minimum weight of an IDF on a graph G is called the *Italian domination number* and denoted by $\gamma_I(G)$. For an IDF f on G, let $V_i^f = \{v \in V(G) : f(v) = i\}$, for $i = 0, 1, 2$. We can equivalently write $f = (V_0^f, V_1^f, V_2^f)$ (or simply $f = (V_0, V_1, V_2)$).

Fan et al. [13] initiated the study of outer independent Italian domination in graphs. An *outer independent Italian dominating function* (OIIDF) on a graph G is an IDF on G that V_0^f is an independent set. Mojdeh and Volkmann [18] considered an extension of Italian domination as follows. For a graph G, a *double Italian dominating function* (DIDF) is a function $f : V \longrightarrow \{0, 1, 2, 3\}$ having the property for which every vertex $u \in V$, if $f(u) \in \{0, 1\}$, then $f(N[u]) \geq 3$. The weight of a DIDF f is the sum $w(f) = f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a DIDF in a graph G is the *double Italian domination number*, denoted by $\gamma_{\{R3\}}(G)$, or $\gamma_{dI}(G)$ for convenience. Abdollahzadeh Ahangar et al. [1] studied outer independent double Roman domination. An *outer independent double Roman dominating function* (OIDRDF) on G is an DRDF of G for which V_0^f is an independent set.

Azvin et al. [3] (see also Volkmann [20]) considered those double Italian dominating functions f such that $\{v \in V(G) \mid f(v) = 0\}$ is an independent set. An *outer independent double Italian dominating function* (OIDIDF) of graph G is a DIDF $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ for which V_0^f is independent. The *outer independent double Italian domination number* denoted by $\gamma_{oidI}(G)$ is the minimum weight of an OIDIDF on G. An OIDIDF on graph G with weight of $\gamma_{oidI}(G)$ is denoted by $\gamma_{oidI}(G)$-function. Benatallah [5] studied outer independent double Italian domination number and presented the following bounds on the outer independent double Italian domination number.

Theorem 1.1 ([5]). *For every graph G, $\gamma_{oiI}(G) \leq \gamma_{oidI}(G) \leq 2\gamma_{oiI}(G)$ and these bounds are sharp.*

In this paper we continue the study of the outer independent double Italian domination number of a graph. We characterize all trees T achieving equality in both bounds given in Theorem 1.1. We also prove two lower bounds on the outer independent double Italian domination number of a graph.

2. Trees T with $\gamma_{oidI}(T) = 2\gamma_{oiI}(T)$ or $\gamma_{oidI}(T) = \gamma_{oiI}(T) + 1$

We aim to characterize trees achieving the equality in each of the lower or upper bounds given in Theorem 1.1. We begin with the following result.

Observation 2.1. *If $\gamma_{oiI}(G) = \gamma_{oidI}(G)$ for a graph G, then $\delta(G) \geq 2$.*

Proof. Suppose that $\gamma_{oiI}(T) = \gamma_{oidI}(T)$. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{oidI}(T)$-function. If $|V_3| \neq 0$ then re-assigning 2 to any vertex of V_3 and 1 to any vertex of V_2 produces an OIIDF of T of weight less than $\gamma_{oiI}(T)$, a contradiction. Thus $|V_3| = 0$. Similarly, $|V_2| = 0$. Thus each vertex of V_0 is adjacent to at least three vertices of V_1 and each vertex of V_1 is adjacent to at least two vertices of V_1. Consequently, $\delta(T) \geq 2$. □

As the above observation shows, no tree achieves equality for the lower bound of Theorem 1.1. We will improve the lower bound of Theorem 1.1 for trees.

Proposition 2.1. *For any tree T of order $n \geq 2$, $\gamma_{oidI}(T) \geq \gamma_{oiI}(T) + 1$, with equality holds if and only if T is a star.*

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{oidI}(T)$-function. If $|V_3| \neq 0$ and $|V_2| \neq 0$ then re-assigning 2 to any vertex of V_3 and 1 to any vertex of V_2 produces an OIIDF of T of weight $\gamma_{oidI}(T) - 2$, implying that $\gamma_{oidI}(T) \geq \gamma_{oiI}(T) + 2$. Similarly, $\gamma_{oidI}(T) \geq \gamma_{oiI}(T) + 2$ if $|V_3| \geq 2$ or $|V_2| \geq 2$. Thus we may assume that $(|V_3|, |V_2|) \in \{(0,1), (1,0)\}$. If $(|V_3|, |V_2|) = (0,1)$, then re-assigning 1 to the vertex of V_2 produces an OIIDF of G of weight $\gamma_{oidI}(T) - 1$, implying that $\gamma_{oidI}(T) \geq \gamma_{oiI}(T) + 1$, and if $(|V_3|, |V_2|) = (1,0)$, then re-assigning 2 to the vertex of V_3 produces an OIIDF of G of weight $\gamma_{oidI}(T) - 1$, implying that $\gamma_{oidI}(T) \geq \gamma_{oiI}(T) + 1$.

Now assume that $\gamma_{oidI}(T) = \gamma_{oiI}(T) + 1$. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{oidI}(T)$-function. Following the above argument, we obtain that $(|V_3|, |V_2|) \in \{(0,1), (1,0)\}$. Suppose that $(|V_3|, |V_2|) = (0,1)$. Let $V_2 = \{y\}$. Since f is a $\gamma_{oidI}(T)$-function and T has at least two leaves, for any leaf $x \neq y$ we have $f(x) = 1$ and $x \in N(y)$. Since any vertex $x \neq y$ lies on a path from y to a leaf of T, we deduce that any vertex $x \neq y$ is a leaf. Consequently, T is a star centered at y. If $\deg(y) \geq 2$, then re-assigning 3 to y and 0 to any leaf produces an OIDIDF of T of weight less than $\gamma_{oidI}(T)$, a contradiction. Thus T is star of order 2. Next assume that $|V_3| = 1$ and $|V_2| = 0$. Let $V_3 = \{y\}$. Clearly, for any leaf x, $f(x) = 0$, and x is adjacent to y, since f is a $\gamma_{oidI}(T)$-function. Consequently, T is a star centered at y. The converse is obvious. □

We next aim to characterize trees achieving the equality in the upper bound of Theorem 1.1. The following lemma has an important role in the rest of this section.

Lemma 2.1. *If T is a tree for which $\gamma_{oidI}(T) = 2\gamma_{oiI}(T)$ and $f = (V_0, V_1, V_2)$ is a $\gamma_{oiI}(T)$-function, then:*
(1) $V_2 = \emptyset$ and V_1 is an independent set.
(2) T does not have any strong support vertex.
(3) All leaves of T belong to the same partite set of T, namely $X_{L(T)}$.
(4) The distance between any pair of vertices in $X_{L(T)}$ is even.
(5) There exists a $\gamma_{oidI}(T)$-function g for which $V_1^g \cup V_3^g = \emptyset$.

Proof. Assume that $\gamma_{oidI}(T) = 2\gamma_{oiI}(T)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{oiI}(T)$-function.

(1) As it is shown in the proof of Theorem 1.1, re-assigning 2 to every vertex of V_1 and 3 to every vertex of V_2 produces an OIDIDF for T implying that $\gamma_{oidI}(T) \leq 2|V_1| + 3|V_2| \leq 2|V_1| + 4|V_2| = 2\gamma_{oiI}(T) = \gamma_{oidI}(T)$ which implies that $V_2 = \emptyset$. If there are two adjacent vertices u and v in V_1, then the function g defined by $g(u) = 1$ and $g(x) = 2f(x)$ for every $x \in V(T) - \{u\}$ is an OIDIDF, and so $\gamma_{oidI}(T) \leqslant w(g) < 2w(f) = 2\gamma_{oiI}(T)$, a contradiction. Thus V_1 is an independent set.

(2) By (1), $V_2 = \emptyset$ and both V_0 and V_1 are independent sets. Assume that T has a strong support vertex u. Then $f(u) = 0$ and $f(x) = 1$ for every leaf neighbor of u. Then the function g defined by $g(u) = 3$, $g(t) = 0$ for every leaf-neighbor of u, and $g(x) = 2f(x)$ for any other vertex is an OIDIDF. Let A be the set of all leaf-neighbors of u. Then $|A| \geqslant 2$, since u is a strong support vertex. Thus

$$\gamma_{oidI}(T) \leqslant w(g) = 3 + 2(|V_1 - A|) \leqslant 3 + 2|V_1| - 4 \leqslant 2|V_1| - 1 < 2|V_1| = 2\gamma_{oiI}(T),$$

a contradiction. Consequently, T does not have any strong support vertex.

(3) Since $V_2 = \emptyset$ and both V_0 and V_1 are independent sets, it follows that V_0 and V_1 are partite sets of T. Since $f(x) = 1$ for every leaf x, we find that all leaves belong to V_1.

(4) Let $X_{L(T)}$ be the partite set of T containing all leaves. Let x, y be two vertices in $X_{L(T)}$, and let $P : x_0x_1....x_ky$ be the shortest path between x and y, where $x_0 = x$. Then clearly $f(x) = f(y) = f(x_{2i}) = 1$ for $i \leq \lfloor \frac{k}{2} \rfloor$, and $f(u) = 0$ for any other vertex of P. Since V_0 is an independent set, we conclude that k is odd. Consequently, $d(x, y)$ is even.
(5) In view of the proof of (1), we have $\gamma_{oidI}(T) = 2\gamma_{oiI}(T) = 2|V_1|$. It is now easy to see that $g = (V_0^g, V_1^g, V_2^g, V_3^g) = (V_0, \emptyset, V_1, \emptyset)$ is a $\gamma_{oidI}(T)$-function for which $V_1^g \cup V_3^g = \emptyset$. □

According to Lemma 2.1, for every tree T with $\gamma_{oidI}(T) = 2\gamma_{oiI}(T)$, we denote by $X_{L(T)}$ the partite set containing all leaves of T. Now we define the family $\mathcal{T}$ of trees as follows. Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence $T_1, \ldots, T_i$ $(i \geq 1)$ of trees such that T_1 is a path P_5 and if $i \geq 2$, then T_{i+1} can be obtained, recursively, from T_i by the following operation.

Operation $\mathcal{O}$: Let $u \in X_{L(T_i)}$. Then T_{i+1} is obtained from T_i by joining u to a leaf of a path P_2.

Lemma 2.2. *If $\gamma_{oidI}(T_i) = 2\gamma_{oiI}(T_i)$ and T_{i+1} is obtained from T_i by Operation $\mathcal{O}$, then $\gamma_{oidI}(T_{i+1}) = 2\gamma_{oiI}(T_{i+1})$.*

Proof. Let $v \in X_{L(T_i)}$ and suppose T_{i+1} is obtained from T_i by joining v to the leaf y of a P_2-path $P_2 : yz$ according to Operation $\mathcal{O}$. We first show that $\gamma_{oiI}(T_{i+1}) = \gamma_{oiI}(T_i)+1$. Let $f = (V_0, V_1, V_2)$

be a $\gamma_{oiI}(T_i)$-function. According to Lemma 2.1 (1), $V_2 = \emptyset$ and V_1 is an independent set. If u is a leaf in $X_{L(T_i)}$, then $f(u) = 1$. Since $d(u, v)$ is even, we find that $f(v) = 1$. Then assigning 0 to y and 1 to z produces an OIIDF for T_{i+1} implying that $\gamma_{oiI}(T_{i+1}) \leq w(f) + 1 = \gamma_{oiI}(T_i) + 1$. On the other hand, let g be a $\gamma_{oiI}(T_{i+1})$-function. Clearly, $g(v) + g(z) + g(y) \geqslant 2$. We may assume that $g(z) = 1$, $g(y) = 0$ and $g(v) \geqslant 1$. Therefore $\gamma_{oiI}(T_i) \leq w(g\mid_{T_i}) \leqslant w(g) - 1 = \gamma_{oiI}(T_{i+1}) - 1$. Consequently, $\gamma_{oiI}(T_{i+1}) = \gamma_{oiI}(T_i) + 1$.

We next prove that $\gamma_{oidI}(T_{i+1}) = 2\gamma_{oiI}(T_{i+1})$. First we note that according to Proposition 1.1, $\gamma_{oidI}(T_{i+1}) \leqslant 2\gamma_{oiI}(T_{i+1})$. Conversely let f^* be a $\gamma_{oidI}(T_{i+1})$-function. Clearly $f^*(y) + f^*(z) \in \{2, 3\}$. If $f^*(y) + f^*(z) = 2$, then $f^*(y) = 0$ and $f^*(z) = 2$, and so $f^*|_{T_i}$ is an OIDIDF on T_i and so $\gamma_{oidI}(T_i) \leqslant w(f^*) - 2 = \gamma_{oidI}(T_{i+1}) - 2$. This implies that

$$\gamma_{oidI}(T_{i+1}) \geqslant \gamma_{oidI}(T_i) + 2 = 2\gamma_{oiI}(T_i) + 2 = 2(\gamma_{oiI}(T_i) + 1) = 2\gamma_{oiI}(T_{i+1})$$

as desired. Thus we assume that $f^*(y)+f^*(z) = 3$. Without loss of generality, we may assume that $f^*(y) = 3$ and $f^*(z) = 0$. Then $f^*(v) < 2$, since otherwise we can replace $f^*(y)$ by 0 and $f^*(z)$ by 2 to obtain an OIDIDF on T_{i+1} with weight less than $w(f^*)$ which is a contradiction. If $f^*(v) = 1$, then we define a function g^* by $g^*(v) = g^*(z) = 2, g^*(y) = 0$ and $g^*(t) = f^*(t)$ otherwise. Then $w(g^*) = w(f^*)$ and $g^*\mid_{T_i}$ is an OIDIDF on T_i. Then $\gamma_{oidI}(T_i) \leqslant w(g^*\mid_{T_i}) = w(g^*) - 2$ and so $\gamma_{oidI}(T_{i+1}) \geqslant \gamma_{oidI}(T_i) + 2 = 2\gamma_{oiI}(T_i) + 2 = 2(\gamma_{oiI}(T_i) + 1) = 2\gamma_{oiI}(T_{i+1})$. Thus assume that $f^*(v) = 0$. Note that $f^*(t) \geqslant 1$ for $t \in N(v)$. If $|N(v) - \{y\}| \geqslant 3$, then $f^*\mid_{T_i}$ is an OIDIDF on T_i and so $\gamma_{oidI}(T_i) \leqslant w(f^*\mid_{T_i}) = w(f^*) - 3 < w(f^*) - 2 = \gamma_{oidI}(T_{i+1})$ and we obtain the result as before. If $|N(v) - \{y\}| = 2$, then we may assume that $f^*(y) = 1$ and $f^*(z) = 2$ and define the function f' by $f'(v) = 1, f'(z) = 2$ and $f'(y) = 0$. Then $f'\mid_{T_i}$ is an OIDIDF on T_i and, as before, we find that $\gamma_{oidI}(T_{i+1}) \geqslant 2\gamma_{oiI}(T_{i+1})$. Therefore assume that $|N(v)-\{y\}| = 1$. Let $N(v)-\{y\} = \{w\}$. If $f^*(w) \geqslant 2$, then $f'\mid_{T_i}$, where f' is described above, is an OIDIDF on T_i, and as before the result follows. Thus assume that $f^*(w) = 1$. Then $h\mid_{T_i}$, where h is defined by $h(v) = 2$ and $h(t) = f^*(t)$ otherwise, is an OIDIDF on T_i. Let $N(w) = \{v, w_1, w_2 \ldots w_k\}$, where $k \geq 1$. Let g_1 be a $\gamma_{oidI}(T)$-function on T_i satsfying Lemma 2.1 (5). Then $w(g_1) = 2\gamma_{oiI}(T_i)$, $g_1(w) = 0$ and $g_1(v) = g_1(w_i) = 2$ for $i = 1, 2 \ldots, k$. Assume that there is an integer $j \in \{1, 2 \ldots, k\}$ such that $w(h\mid_{T_{w_j}}) \geq w(g_1\mid_{T_{w_j}})$. Then we change $h(x)$ by $g_1(x)$ for each $x \in V(T_{w_j})$ and define the function h^* by $h^*(w) = 2, h^*(w_j) = h^*(v) = 1, h^*(x) = g_1(x)$ for $x \in V(T_{w_j}) - \{w_j\}$ and $h^*(t) = h(t)$ otherwise. Since $d(v, w_j) = 2$ and $v \in X_{L(T_i)}$, h^* is an OIDIDF on T_i and so $\gamma_{oidI}(T_i) \leqslant w(h^*) \leqslant w(h) - 1 = w(f^*) - 2$. Therefore $\gamma_{oidI}(T_{i+1}) \geqslant \gamma_{oidI}(T_i) + 2$, as desired. Then we assume that $w(h\mid_{T_{w_i}}) \leqslant w(g_1\mid_{T_{w_i}})-1$, for every $i = 1, ..., k$. We now have the following.

$$\gamma_{oidI}(T_i) \leqslant h(v) + h(w) + \sum_{i=1}^{k} h(T_{w_i}) \leqslant 2 + 1 + \sum_{i=1}^{k} g_1(T_{w_i}) - k = 2\gamma_{oiI}(T_i) + 1 - k.$$

If $k \geqslant 2$, then $\gamma_{oidI}(T_i) < 2\gamma_{oiI}(T_i)$, a contradiction. Thus $k = 1$. Then $\deg(w) = 2$. Observe that $f^*(w_1) = 2$, since f^* is an OIDIDF on T_{i+1}. Let f_1 be a function defined by $f_1(w) = f_1(y) = 0$, $f_1(v) = f_1(z) = 2$ and $f_1(t) = f^*(t)$ otherwise. Then $w(f_1) = w(f^*)$ and $f_1\mid_{T_i}$ is an OIDIDF on T_i. Then $\gamma_{oidI}(T_i) \leqslant w(f_1\mid_{T_i}) = \gamma_{oidI}(T_{i+1}) - 2$ and so

$$\gamma_{oidI}(T_{i+1}) \geqslant \gamma_{oidI}(T_i) + 2 = 2\gamma_{oiI}(T_i) + 2 = 2(\gamma_{oiI}(T_i) + 1) = 2\gamma_{oiI}(T_{i+1})$$

as desired. □

We are now ready to characterize trees with outer independent double Italian domination number twice the outer independent Italian domination number.

Theorem 2.1. *For a tree T of order $n \geq 5$, $\gamma_{oidI}(T) = 2\gamma_{oiI}(T)$ if and only if $T \in \mathcal{T}$.*

Proof. First we show that for every tree T of the family $\mathcal{T}$, $\gamma_{oidI}(T) = 2\gamma_{oiI}(T)$. We proceed by induction on the order $n \geqslant 5$ of a tree $T \in \mathcal{T}$. Clearly $6 = \gamma_{oidI}(P_5) = 2\gamma_{oiI}(P_5) = 2(3)$. This establishes the base step. For the inductive hypothesis, assume that for every tree T' of order n', $(5 \leqslant n' < n)$ that belongs to the family $\mathcal{T}$, we have $\gamma_{oidI}(T') = 2\gamma_{oiI}(T')$. Let T be a tree of order n that belongs to the family $\mathcal{T}$. Then T is obtained from a sequence $T_1, \ldots, T_i$ $(i \geq 1)$ of trees such that T_1 is a path P_5 and if $i \geq 2$, then T_{i+1} can be obtained, recursively, from T_i by Operation $\mathcal{O}$. Then by the inductive hypothesis and Lemma 2.2 we obtain that $\gamma_{oidI}(T) = 2\gamma_{oiI}(T)$.

Conversely, assume that T is a tree of order $n \geqslant 5$ with $\gamma_{oidI}(T) = 2\gamma_{oiI}(T)$. We show that $T \in \mathcal{T}$. We proceed by induction on the order $n \geqslant 5$. We first note that if $diam(T) = 2$, then T is a star and clearly $\gamma_{oidI}(T) = 3$, $\gamma_{oiI}(T) = 2$ and $\gamma_{oidI}(T) \neq 2\gamma_{oiI}(T)$. If $diam(T) = 3$, then T is a double star in which $\gamma_{oidI}(T) \in \{5, 6\}$, $\gamma_{oiI}(T) \in \{3, 4\}$ and $\gamma_{oidI}(T) \neq 2\gamma_{oiI}(T)$. Therefore we assume that $diam(T) \geqslant 4$ and so $n \geqslant 5$. If $n = 5$, then $T \cong P_5 \in \mathcal{T}$. This establishes the base step of the induction. For the induction hypothesis assume that every tree T' of order n' $(5 \leqslant n' < n)$ with $\gamma_{oidI}(T') = 2\gamma_{oiI}(T')$ belongs to the family $\mathcal{T}$. Let T be a tree of order n with $\gamma_{oidI}(T) = 2\gamma_{oiI}(T)$. We show $T \in \mathcal{T}$. We consider a diametrical path $x_1x_2\ldots x_dx_{d+1}$ in T and root T at x_1. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{oiI}(T)$-function. By Lemma 2.1 (parts (1) and (2)), $\deg(x_d) = 2$, $V_2 = \emptyset$, V_1 and V_0 are independent sets. Clearly, $f(x_{d+1}) = f(x_{d-1}) = 1$ and $f(x_d) = 0$. Thus $X_{L(T)} = \{x : f(x) = 1\}$. Note that x_{d-1} does not have any leaf neighbor, since V_1 is an independent set.

Let $T' = T - \{x_d, x_{d+1}\}$. We first show that $\gamma_{oiI}(T) = \gamma_{oiI}(T') + 1$. Observe that $f\mid_{T'}$ is an OIIDF on T'. Thus $\gamma_{oiI}(T') \leqslant \gamma_{oiI}(T) - 1$. Assume that $f' = (V_0', V_1', V_2')$ is a $\gamma_{oiI}(T')$-function. If $f'(x_{d-1}) \geqslant 1$, then the function g defined by $g(x_d) = 0, g(x_{d+1}) = 1$ and $g(t) = f'(t)$ otherwise, is an OIIDF on T and so $\gamma_{oiI}(T) \leqslant \gamma_{oiI}(T') + 1$. Thus we assume that $f'(x_{d-1}) = 0$. We consider the following cases.

Case 1. $\deg(x_{d-1}) \geqslant 3$. Since x_{d-1} does not have any leaf neighbor, we may assume that x_{d-1} has a child $u \neq x_d$ which is a support vertex. By Lemma 2.1 $\deg(u) = 2$. Let w be the child of u. Then $f'(u) + f'(w) = 2$. Then we replace $f'(x_{d-1})$ and $f'(w)$ by 1 and $f'(u)$ by 0 to obtain a $\gamma_{oiI}(T')$-function f'' with $f''(x_{d-1}) \geq 1$. Then, as before, the function g defined by $g(x_d) = 0, g(x_{d+1}) = 1$ and $g(t) = f''(t)$ otherwise, is an OIIDF on T and so $\gamma_{oiI}(T) \leqslant \gamma_{oiI}(T') + 1$, as desired.

Case 2. $\deg(x_{d-1}) = 2$. Observe that $f'(x_{d-2}) = 2$. Suppose that x_{d-2} has a leaf neighbor v. Then $f(x_{d-2}) = 0$ and $f(v) = 1$. Also x_{d-3} doesn't have a leaf neighbor, since $d(x_{d-3}, x_{d+1})$ is even. Now we define the function f_1 by $f_1(x_{d-2}) = 3$, $f_1(v) = 0$, $f_1(x_{d-1}) = f_1(x_{d-3}) = 1$ and $f_1(t) = 2f(t)$ otherwise. Then f_1 is an OIDIDF on T and so $\gamma_{oidI}(T) \leqslant w(f_1) = 2w(f) - 1 < 2\gamma_{oiI}(T)$, a contradiction. Thus x_{d-2} does not have a leaf neighbor, and so every child of x_{d-2} with value 0 (if any), is adjacent to at least one other vertex with value at least 1. Then we change $f'(x_{d-2})$ and $f'(x_{d-1})$ to 1 to obtain a $\gamma_{oiI}(T')$-function f'' with $f''(x_{d-1}) \geq 1$. Then the function

g defined by $g(x_d) = 0, g(x_{d+1}) = 1$ and $g(t) = f''(t)$ otherwise, is an OIIDF on T and so $\gamma_{oiI}(T) \leqslant \gamma_{oiI}(T') + 1$, as desired.

We conclude that $\gamma_{oiI}(T) = \gamma_{oiI}(T') + 1$. Thus $\gamma_{oidI}(T) = 2\gamma_{oiI}(T) = 2(\gamma_{oiI}(T') + 1)$. Then $\gamma_{oidI}(T) - 2 = 2\gamma_{oiI}(T')$.

We next show that $\gamma_{oidI}(T') = \gamma_{oidI}(T) - 2$. Assume that g is a $\gamma_{oidI}(T')$-function. If $g(x_{d-1}) \geqslant 1$, then we set $g(x_d) = 0$ and $g(x_{d+1}) = 2$ to extend g to an OIDIDF on T. Therefore $\gamma_{oidI}(T) \leqslant \gamma_{oidI}(T') + 2$. Assume that $g(x_{d-1}) = 0$. If $\deg_T(x_{d-1}) = 2$, then $g(x_{d-2}) = 3$ and $g(x_{d-3}) + g(x_{d-4}) \geqslant 1$. Note that $diam(T') \geqslant 4$, since $\gamma_{oidI}(T') = 2\gamma_{oiI}(T')$. Since $\deg(x_{d-2}) = 2$, we may change $g(x_{d-2})$ to 2, and set $g(x_{d-1}) = 1$, $g(x_d) = 0$ and $g(x_{d+1}) = 2$ to extend g to an OIDIDF on T and as before find that $\gamma_{oidI}(T) \leqslant \gamma_{oidI}(T') + 2$. Thus assume that $\deg_T(x_{d-1}) \geqslant 3$. Since x_{d-1} does not have a leaf neighbor, it has a child y of degree 2. Let z be the child of y. Then $g(y) + g(z) = 3$. Now we change $g(x_{d-1})$ to 1, $g(y)$ to 0 and $g(z)$ to 2, and as before obtain that $\gamma_{oidI}(T) \leqslant \gamma_{oidI}(T') + 2$. We conclude that $\gamma_{oidI}(T) \leqslant \gamma_{oidI}(T') + 2$. Next we show that $\gamma_{oidI}(T) \geqslant \gamma_{oidI}(T') + 2$. Assume that h is a $\gamma_{oidI}(T)$-function. If $h(x_{d-1}) \geqslant 2$, then $h(x_{d+1}) + h(x_d) = 2$ and $h\ |_{V(T')}$ is an OIDIDF on T', implying that $\gamma_{oidI}(T') \leqslant \gamma_{oidI}(T) - 2$. Assume that $h(x_{d-1}) = 1$. If $\sum_{(v \in N[x_{d-1}] - x_d)} h(v) \geqslant 3$, then we may assume that $h(x_d) = 0$ and $h(x_{d+1}) = 2$, in which the result follows as before. Thus assume that $\sum_{(v \in N[x_{d-1}] - x_d)} h(v) < 3$. Then $h(x_d) + h(x_{d+1}) = 3$ and we may assume that $h(x_{d+1}) = h(x_{d-1}) = 2$ and $g(x_d) = 0$ in which the result follows as before. We now assume that $h(x_{d-1}) = 0$. If $\deg(x_{d-1}) \geqslant 3$, then x_{d-1} has a child y of degree two. Assume that z is the child of y. Since $h(x_{d-1}) = 0$, we have $h(t) \geqslant 1$ for every vertex $t \in N(x_{d-1})$. Furthermore, $h(x_{d+1}) + h(x_d) = h(y) + h(z) = 3$. Let h' be a function defined by $h'(x_{d-1}) = h'(y) = 1$, $h'(x_d) = 0$, $h'(z) = h'(x_d) = 2$ and $h'(t) = h(t)$ otherwise. Then $h'\ |_{T'}$ is an OIDIDF on T' and so $\gamma_{oidI}(T') \leqslant w(h') - 2 = w(h) - 2$ as required. Thus assume that $\deg(x_{d-1}) = 2$. If $h(x_{d-2}) \geqslant 2$, then we change $h(x_{d-1})$ to 1, $h(x_d)$ to 0 and $h(x_{d+1})$ to 2, and as before we find that $\gamma_{oidI}(T') \leqslant \gamma_{oidI}(T) - 2$. It remains to assume that $h(x_{d-2}) = 1$. Then $h(x_{d-3}) \geqslant 2$, since $\deg(x_{d-2}) = 2$. Now we change $h(x_{d+1})$ and $h(x_{d-1})$ to 2 and $h(x_d)$ and $h(x_{d-2})$ to 0 to obtain h'. Then $h'\ |_{T'}$ is an OIDIDF on T' and the result is obtained as before. Hence, $\gamma_{oidI}(T') = \gamma_{oidI}(T) - 2 = 2\gamma_{oiI}(T')$. We deduce $T' \in \mathcal{T}$ by the induction hypothesis. Since $f(x_{d-1}) = 1$, we have $x_{d-1} \in X_{L(T)}$ and so T is formed by Operation $\mathcal{O}$ on T'. □

3. New lower bounds

In this section we prove two new lower bounds on the outer independent double Italian domination number.

Theorem 3.1. *For every graph G of order n with minimum degree $\delta = \delta(G) > 0$ and maximum degree $\Delta = \Delta(G)$, $\gamma_{oidI}(G) \geqslant \lfloor \frac{n\delta}{\delta+\Delta} \rfloor + 1$ and this bound is sharp.*

Proof. Let G be a graph of order n with minimum degree $\delta > 0$, and $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{oidI}(G)-$function. Let m be the number of edges having one end point in V_0 and the other end point in $V_1 \cup V_2 \cup V_3$. We then have $m \geq \delta|V_0|$, since V_0 is independent and $f(N(u)) \geq 3$ for each vertex $u \in V_0$. On the other hand each vertex in $V_1 \cup V_2 \cup V_3$ has at most Δ neighbors in V_0. Thus,

$m \leq \Delta(|V_1|+|V_2|+|V_3|)$, and so $\delta(n-|V_1|-|V_2|-|V_3|) \leqslant \Delta(|V_1|+|V_2|+|V_3|)$ which implies that $\delta n \leqslant (\Delta+\delta)(|V_1|+|V_2|+|V_3|)$. Therefore,

$$\frac{\delta n}{\Delta+\delta} \leqslant |V_1|+|V_2|+|V_3| = \gamma_{oidI}(G) - |V_2| - 2|V_3|.$$

If $V_2 \cup V_3 \neq \emptyset$, then $\gamma_{oidI}(G) \geqslant \frac{n\delta}{\Delta+\delta} + |V_2| + 2|V_3| \geqslant \frac{n\delta}{\Delta+\delta} + 1$, as desired. Thus assume that $V_2 \cup V_3 = \emptyset$. Then each vertex in V_1 is adjacent to at least two vertices in V_1 and each vertex in V_0 is adjacent to at least three vertices in V_1. Then if $|V_0| > 0$, then $\Delta \geq 3$ while if $|V_0| = 0$ then $\Delta \geq 2$. Furthermore, counting the number of edges having one end point in V_0 and the other end point in V_1 implies that $\delta|V_0| \leqslant (\Delta-2)|V_1|$. Since $n = |V_0|+|V_1|$, we find that $\delta(n-|V_1|) \leqslant (\Delta-2)|V_1|$. This implies that $\gamma_{oidI}(G) = |V_1| \geq \frac{\delta n}{\Delta+\delta-2} > \frac{\delta n}{\Delta+\delta}$. Since $\gamma_{oidI}(G)$ is an integer we deduce that $\gamma_{oidI}(G) \geqslant \lfloor \frac{n\delta}{\delta+\Delta} \rfloor + 1$. To see the sharpness, for each integer $n \geq 3$, let G_n be a graph obtained from a cycle C_n by adding $2n$ new vertices and joining each new vertex to all vertices of C_n. Note that $|V(G_n)| = 3n$, $\delta(G_n) = n$, $\Delta(G_n) = 2n+2$ and $\gamma_{oidI}(G_n) = n = \lfloor \frac{3n^2}{3n+2} \rfloor + 1$. □

We now introduce a family of graphs as follows. Let $\mathcal{G}$ be the class of all graphs G such that $G \in \mathcal{G}$ if and only if G is obtained from a graph H of order at least three and minimum degree at least two by adding at least $|V(H)| - \delta(H)$ new vertices and joining each new vertex to all vertices of H.

Proposition 3.1. *If G is a graph of order $n \geq 1$ with minimum degree δ, $\gamma_{oidI}(G) \geq \delta$, with equality holds if and only if $G \in \mathcal{G}$.*

Proof. The result is obvious if $\delta \leq 2$. Thus assume that $\delta \geq 3$. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{oidI}(G)$-function. If $V_0 = \emptyset$, then $\gamma_{oidI}(G) \geq n \geq \delta + 1$, since $\delta \leq n-1$. Thus assume that $V_0 \neq \emptyset$. Let $x \in V_0$. Therefore $N(x) \subseteq V_1 \cup V_2 \cup V_3$, since V_0 is an independent set. Also $|N(x)| \geq \delta$. Note that

$$\gamma_{oidI}(G) = |V_1| + 2|V_2| + 3|V_3| = (\sum_{i=1}^{3} |V_i|) + |V_2| + 2|V_3|.$$

If $(V_2 \cup V_3) \neq \emptyset$, then $\gamma_{oidI}(G) \geq \left(\sum_{i=1}^{3} |V_i|\right) + 1 \geq \deg(x) + 1 \geq \delta + 1$. Thus assume that $V_2 \cup V_3 = \emptyset$. Then $\gamma_{oidI}(G) = |V_1|$. Since V_0 is an independent set, we find that $\gamma_{oidI}(G) = |V_1| \geq \delta$.

We next prove the equality part. Assume that $\gamma_{oidI}(G) = \delta$. Clearly $\gamma_{oidI}(G) \geq 2$. It is easy to see that $\gamma_{oidI}(G) = 2$ if and only if $G = K_1$. Since K_1 has minimum degree 0, from $\gamma_{oidI}(G) = \delta$ we obtain that $\delta \geq 3$. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{oidI}(G)$-function. Following the above argument for the first part of the proof, we obtain that $V_2 \cup V_3 = \emptyset$ and $\gamma_{oidI}(G) = |V_1| = \delta$. Let $H = G[V_1]$. Since V_0 is independent and $|V_1| = \delta$, every vertex of V_0 is adjacent to all vertices of H. Since any vertex of V_1 is adjacent to at least two other vertices of V_1, we obtain that H has minimum degree at least 2. Suppose that $|V_0| < |V(H)| - \delta(H)$. Let $v \in V_1$ be a vertex with $\deg_H(v) = \delta(H)$. Then

$$\deg_G(v) = \deg_H(v) + |V_0| < |V(H)| = \delta(G).$$

This is a contradiction. Thus $|V_0| \geq |V(H)| - \delta(H)$. We conclude that $G \in \mathcal{G}$.

Conversely assume that $G \in \mathcal{G}$. Thus G is obtained from a graph H of order $m \geq 3$ with $V(H) = \{y_1, ..., y_m\}$ and $\delta(H) \geq 2$ by adding $t \geq m - \delta(H)$ new vertices $x_1, ..., x_t$, and adding edges $x_i y_j$, for $i = 1, ..., t$ and $j = 1, ..., m$. Note that $\deg_G(x_i) = m$ for $i = 1, ..., t$, and $\deg_G(y_j) = \deg_H(y_j) + t \geq \delta(H) + t \geq m$ for $j = 1, ..., m$. Thus $\delta(G) = m$. Then $\gamma_{oidI}(G) \geq m$ by the first part of the proposition. Now let g be a function on G defined by $g(y_j) = 1$ for each $j = 1, ..., m$ and $g(x_i) = 0$ for each x_i, $i = 1, ..., t$. Then g is an OIDIDF for G, implying that $\gamma_{oidI}(G) \leq m$. Consequently, $\gamma_{oidI}(G) = m = \delta(G)$. □

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