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A note on isolate domination

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Abstract

A set S of vertices of a graph G such that $\langle S\rangle$ has an isolated vertex is called an *isolate set* of G. The minimum and maximum cardinality of a maximal isolate set are called the *isolate number* $i_0(G)$ and the *upper isolate number* $I_0(G)$ respectively. An isolate set that is also a dominating set (an irredundant set) is an *isolate dominating set* (*an isolate irredundant set*). The *isolate domination number* $\gamma_0(G)$ and the *upper isolate domination number* $\Gamma_0(G)$ are respectively the minimum and maximum cardinality of a minimal isolate dominating set while the *isolate irredundance number* $ir_0(G)$ and the *upper isolate irredundance number* $IR_0(G)$ are the minimum and maximum cardinality of a maximal isolate irredundant set of G. The notion of isolate domination was introduced in [5] and the remaining were introduced in [4]. This paper further extends a study of these parameters.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, non-trivial, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to the book by Chartrand and Lesniak [2].

The open neighbourhood $N(v)$ of a vertex is the set of all vertices adjacent to v while the closed neighbourhood $N[v]$ is $N(v) \cup \{v\}$. The subgraph induced by a set S of vertices of a graph

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G is denoted by $\langle S\rangle$ with $V(\langle S\rangle) = S$ and $E(\langle S\rangle) = \{uv \in E(G) : u, v \in S\}$. A vertex u is said to be a private neighbour of a vertex $v \in S$ with respect to the set S if $N[u] \cap S = \{v\}$ (In particular, an isolated vertex in $\langle S\rangle$ is a private neighbour of itself with respect to the set S). The private neighbour set of a vertex v with respect to the set S is denoted by $pn[v, S]$.

A set D of vertices of a graph G is said to be a *dominating set* if every vertex in $V - D$ is adjacent to a vertex in D. A dominating set D is said to be a *minimal dominating set* if no proper subset of D is a dominating set. The minimum cardinality of a minimal dominating set of a graph G is called *the domination number* of G and is denoted by $\gamma(G)$. The *upper domination number* $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of G. The minimum cardinality of an independent dominating set is called the *independent domination number*, denoted by $i(G)$ and *the independence number* $\beta_0(G)$ is the maximum cardinality of an independent set of G. A set S is a *total dominating set*, if $N(S) = V$. The *total domination number* $\gamma_t(G)$ equals the minimum cardinality of a total dominating set of G. A set $D \subseteq V(G)$ which is a dominating set of both G and $\overline{G}$ is called a *global dominating set*. The minimum cardinality of a global dominating is called the *global domination number* and is denoted by $\gamma_g(G)$. A set S of vertices is irredundant if every vertex $v \in S$ has at least one private neighbour. The minimum and maximum cardinality of a maximal irredundant set are respectively called *the irredundance number* $ir(G)$ and *the upper irredundance number* $IR(G)$.

A set S of vertices of a graph G such that $\langle S\rangle$ has an isolated vertex is called an *isolate set* of G. The minimum and maximum cardinality of a maximal isolate set are called the *isolate number* $i_0(G)$ and the *upper isolate number* $I_0(G)$. An isolate set that is also a dominating set (an irredundant set) is an *isolate dominating set* (*an isolate irredundant set*). The *isolate domination number* $\gamma_0(G)$ and the *upper isolate domination number* $\Gamma_0(G)$ are respectively the minimum and maximum cardinality of a minimal isolate dominating set while the *isolate irredundance number* $ir_0(G)$ and the *upper isolate irredundance number* $IR_0(G)$ are the minimum and maximum cardinality of a maximal isolate irredundant set of G. An isolate set S of G with $|S| = i_0(G)$ is called an i_0-*set* of G. Similarly, γ_0-set, Γ_0-set, ir_0-set are defined. The notion of isolate domination was introduced in [5] and the remaining were introduced in [4]. An extended chain of inequalities connecting all these parameters has been established in [4] as below:

$$ir(G) \leq ir_0(G) \leq \gamma_0(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_0(G) = \Gamma(G) \leq IR_0(G) = IR(G) \leq I_0(G) \quad (1)$$

This paper further studies these concepts by establishing some relationship among those parameters. We need the following results.

Theorem 1.1 ([4]). *Let S be an isolate set of a graph G. Then, S is a maximal isolate set of G if and only if every vertex in $V - S$ is adjacent to all the isolates of S.*

Theorem 1.2 ([3]). *If G is a graph of order n with no isolates, then $\gamma(G) \leq \frac{n}{2}$.*

Theorem 1.3 ([1]). *For any graph G, $\frac{\gamma(G)}{2} \leq ir(G) \leq \gamma(G) \leq 2ir(G) - 1$.*

Theorem 1.4 ([4]). *Every minimal isolate dominating set of G is a maximal isolate irredudant set of G.*

2. Main Results

In this section we establish some relationships among the isolate domination number and the isolate parameters ir_0 and i_0. We first obtain a bound for i_0 in terms of order and characterizes the extremal graphs.

Theorem 2.1. *For any graph G of order n, we have $1 \leq i_0(G) \leq n$. Further,*

(i) $i_0(G) = 1$ *if and only if* $\Delta(G) = n - 1$.

(ii) $i_0(G) = 2$ *if and only if* $G = H + \overline{K_2}$*, where H is any graph with* $\Delta(H) \leq |V(H)| - 2$.

(iii) $i_0(G) = n$ *if and only if G has an isolated vertex.*

Proof. (i) If $\Delta(G) = n - 1$, then a vertex of degree $n - 1$ forms a maximal isolate set so that $i_0(G) = 1$. On the other hand if $\{u\}$ is a maximal isolate set of G, then every vertex of G other than u must be adjacent to u so that $deg\ u = n - 1$.

(ii) Suppose $i_0(G) = 2$ and S is an i_0-set of G. Then, S is an independent set of G and therefore by Theorem 1.1, we have every vertex of $V - S$ is adjacent to both the vertices of S. Therefore $G = \overline{K_2} + H$, where $H = \langle V - S \rangle$. Further, $\Delta(G) < |V(G)| - 1$ as $i_0(G) > 1$, and so $\Delta(H) < |V(H)| - 2$. Conversely, if $G = \overline{K_2} + H$, where H is any graph with $\Delta(H) \leq |V(H)| - 2$, then $i_0(G) \geq 2$. Further, since the vertices of $\overline{K_2}$ form a maximal isolate of G, the result follows.

(iii) If G itself has an isolated vertex, then $V(G)$ is the only maximal isolate set of G so that $i_0(G) = n$. Further, if $i_0(G) = n$ means $V(G)$ is an isolate set so that there must be an isolated vertex.

□

The following theorems establish some relationships among the isolate parameters i_0, ir_0 and γ_0 with global and total domination numbers.

Theorem 2.2. *For any graph G,* $\gamma_t(G) \leq i_0(G) + 1$ *and the bound is sharp.*

Proof. Let S be a maximal isolate set of G. Then, by Theorem 1.1, every vertex lying in $V - S$ is adjacent to all the isolates of $\langle S \rangle$ and consequently for any vertex $u \in V - S$, the set $S \cup \{u\}$ is a total dominating set of G so that $\gamma_t(G) \leq i_0(G) + 1$. For stars, the value of γ_t is 2 whereas i_0 equals 1. □

Theorem 2.3. *If $diam\ G \geq 5$, then* $\gamma_g(G) \leq \gamma_0(G)$.

Proof. Let G be a graph of diameter at least 5 and let S be a γ_0-set of G. Let us prove that S is a global dominating set of G. That is, we need to verify that S is a dominating set of $\overline{G}$ as well. It is clear that $|S| \geq 2$ for otherwise diameter of G becomes two. Certainly, an isolated vertex of $\langle S \rangle$ will dominate all the vertices of S in $\overline{G}$. Let us now see how the vertices of $V - S$ are dominated in $\overline{G}$ by S. If a vertex $v \in V - S$ is a private neighbour of a vertex u in S with respect to S, then it will be dominated in $\overline{G}$ by a vertex of S other than u (this is possible as $|S| \geq 2$). Therefore, only the vertices of $V - S$ that are not private neighbours of any vertex of S have to be dominated in $\overline{G}$ by S. Now, if there is a vertex in $V - S$ that is adjacent to all the vertices of S in G, then that vertex will not be dominated in $\overline{G}$ by any vertex of S. But we prove that this situation does not occur. Suppose in contrary that there is a vertex $v \in V - S$ that is adjacent in G to all the vertices of S. Then for any two vertices u_1 and u_2 of G, we have the following cases.

(i) If $u_1, u_2 \in S$, then (u_1, v, u_2) is a path connecting u_1 and u_2 and therefore $d(u_1, u_2) \leq 2$.

(ii) Let $u_1, u_2 \in V - S$ and u_1' and u_2' be the vertices in S adjacent to u_1 and u_2 respectively. If $u_1 = u_2$, then $(u_1, u_1' = u_2', u_2)$ is a $u_1 - u_2$ path; otherwise $(u_1, u_1', v, u_2', u_2)$ is a path connecting u_1 and u_2 provided $v \neq u_1, u_2$. Even if $v = u_1$ then $(u_1 = v, u_2', u_2)$ is a required $u_1 - u_2$ path. Therefore $d(u_1, u_2) \leq 4$.

(iii) Let $u_1 \in S$, $u_2 \in V - S$ and u_2' be a vertex in S dominating u_2. Then (u_1, v, u_2', u_2) will be a path connecting u_1 and u_2 and therefore $d(u_1, u_2) \leq 3$.

Therefore the conclusion that we draw is any two vertices of G are at a distance of at most four so that $diam\ G \leq 4$ which is a contradiction to the assumption that $diam\ G \geq 5$. Hence all the non-private neighbours of S in G are dominated in G by the vertices of S and so S is a dominating set of $\overline{G}$ also. Therefore $\gamma_g(G) \leq |S| = \gamma_0(G)$. □

Remark 2.1. The above theorem need not be true for graphs of diameter less than five. For example, for the graphs of diameter 1 (complete graphs) the value of γ_g is its order whereas γ_0 is just 1. The complete bipartite graph $K_{r,s}$, where $3 \leq r \leq s$, is of diameter two such that $\gamma_0(K_{r,s}) = r$ and $\gamma_g(K_{r,s}) = 2$. Further, graphs of diameter 3 and diameter 4 for which the value of γ_0 exceeds the value of γ_g are given in Figure 1.

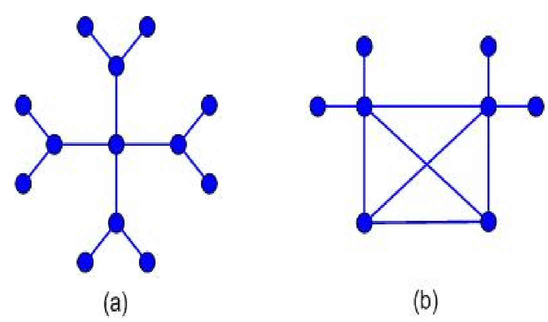

Figure 1. (a) A graph G of diameter 4 for which $\gamma_0(G) = 4 < 5 = \gamma_g(G)$, (b) A graph H of diameter 3 for which $\gamma_0(H) = 3 < 4 = \gamma_g(H)$

Lemma 2.1. *Let S be an i_0-set of a graph G. If there is a vertex in $V - S$ that is adjacent to all the vertices of S, then $diam\ G \leq 3$.*

Proof. If $i_0(G) = 1$, then $\Delta(G) = |V(G)| - 1$ so that $diam\ G \leq 2$. Assume $i_0(G) \geq 2$. Let S be an i_0-set and v be a vertex in $V - S$ that is adjacent to all the vertices of S. Therefore, two vertices of G that belong to S are at a distance of at most two. Now, if x is an isolate of $\langle S \rangle$, it follows from Theorem 1.1 that every vertex in $V - S$ is adjacent to all the isolates of $\langle S \rangle$ and in particular to the vertex x and so any two vertices of G lying in $V - S$ are at a distance of at most two. Suppose u_1 and u_2 are two vertices of G such that $u_1 \in S$ and $u_2 \in V - S$. If $u_1 = x$ or $u_2 = v$ then $d(u_1, u_2) = 1$, otherwise (u_1, v, x, u_2) is an $u_1 - u_2$ path in G so that $d(u_1, u_2) \leq 3$. Thus $diam$ $G \leq 3$. □

Theorem 2.4. *If $diam\ G \geq 4$, then $\gamma_g(G) \leq i_0(G)$.*

Proof. Let G be a graph of diameter at least 4 and S be an i_0-set of G. Then an isolate of $\langle S \rangle$ itself dominates all the vertices of $V - S$ in G so that S is a dominating set of G by Theorem 1.1. Further, it follows from Lemma 2.1 that there is no vertex in $V - S$ that is adjacent to all the vertices of $V - S$. That is, every vertex in $V - S$ has a non-neighbour in S so that the vertices of $V - S$ will be dominated in $\overline{G}$ by S. Certainly, an isolate of $\langle S \rangle$ dominates all the remaining vertices of S in $\overline{G}$. Thus S is a global dominating set of G. Hence the desired result follows. □

The following theorem establishes an upper bound for γ_0 in terms of i_0 for C_4-free graphs with minimum degree at least 2.

Theorem 2.5. *Let G be a C_4-free graph and $\delta(G) \geq 2$. Then $\gamma_0(G) \leq \left\lceil \frac{i_0(G)}{2} \right\rceil$ and the bound is sharp.*

Proof. Let S be an i_0-set of G. We first claim that $\langle S \rangle$ has exactly one isolated vertex. Suppose $\langle S \rangle$ has more than one isolated vertices. Obviously, the set $V - S$ must have at least two vertices; for otherwise the degree of the isolates of $\langle S \rangle$ will be less than 2 which is not true as $\delta(G) \geq 2$. Therefore $|V - S| \geq 2$.

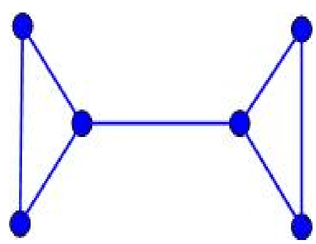

Figure 2. A C_4-free graph G with $\delta(G) = 2$ and $\gamma_0(G) = \left\lceil \frac{i_o(G)}{2} \right\rceil$

Now, by Theorem 1.1 that every isolate of $\langle S \rangle$ is adjacent to all the vertices of $V - S$ and so any two isolates of $\langle S \rangle$ together with any two vertices of $V - S$ will form a cycle of length 4. This is a contradiction and hence the claim follows. Therefore the set $\langle S - \{v\} \rangle$ will have no isolated vertices, where v is the isolated vertex of S. By Theorem 1.2 that the cardinality of a γ-set D of $\langle S - \{v\} \rangle$ is less than or equal to $\frac{|S|-1}{2}$. Now, the isolated vertices of S together with the set D will form an isolate dominating set of G and hence $\gamma_0(G) \leq |D| + 1 \leq \frac{|S|-1}{2} + 1 = \frac{|S|+1}{2} \leq \left\lceil \frac{i_0(G)+1}{2} \right\rceil$. For the graph of Figure 2 the bound is attained. □

Corollary 2.1. *If G is a C_4-free graph with $\delta(G) \geq 2$, then $\gamma_0(G) \leq \left\lceil \frac{n-\delta+1}{2} \right\rceil$.*

Proof. The result follows from the fact that $i_0(G) \leq n - \delta$. □

Theorem 1.3 gives a bound for $\gamma(G)$ in terms of $ir(G)$. Similar to this, in the following theorem, we find an upper bound for $\gamma_0(G)$ in terms of $ir_0(G)$. It follows from Theorem 1.3 and Chain 1 that $\gamma(G) \leq 2ir(G) - 1 \leq 2ir_0(G) - 1$. Thus we obtain a bound for $\gamma(G)$ in terms of the isolate irredundance number ir_0. The following theorem provides a similar result for γ_0.

Theorem 2.6. *For any graph G, $\gamma_0(G) \leq 2(ir_0(G) - 1)$.*

Proof. Let $ir_0(G) = k$ and let $S = \{v_1, v_2, v_3, \ldots, v_t, v_{t+1}, \ldots, v_k\}$ be an ir_0-set of G, where $v_{t+1}, v_{t+2}, \ldots, v_k$ are isolates of $\langle S \rangle$. Since S is irredundant, $pn[v_i, S] \neq \phi$, for $1 \leq i \leq k$. Let $S' = \{u_1, u_2, \ldots, u_t\}$ where $u_i \in pn[v_i, S]$ for $1 \leq i \leq t$. Now, we claim that the set $S'' = S \cup S'$ is an isolate dominating set of G. Since $v_{t+1}, v_{t+2}, \ldots v_k$ are the isolates of $\langle S'' \rangle$, it is enough to prove that S'' is a dominating set of G. If not, then there must be at least one vertex $w \in V - S''$ which is not dominated by S''. This means that $w \notin N[x]$, for any vertex $x \in S''$ and therefore $pn[w, S \cup \{w\}] \neq \phi$. Hence the set $S \cup \{w\}$ is an isolate irredundant set which contradicts the assumption that S is a maximal irredundant set. Therefore S'' is an isolate dominating set. Even though S'' is an isolate dominating set it cannot be a minimal isolate dominating set; for otherwise by Theorem 1.4, it will be a maximal isolate irredundant set, which would again contradicts the maximality of S. Therefore $\gamma_0(G) \leq |S''| - 1 \leq 2(ir_0(G) - 1)$. □

3. Open Problems

We close the paper with the following interesting problems.

(i) Find a class of graphs for which all the parameters in the chain 1 are distinct.

(ii) It is proved in Theorem 2.2 that $\gamma_t(G) \leq i_0(G) + 1$. Find a characterization of graphs for which $\gamma_t(G) = i_0(G) + 1$.

(iii) The problem of characterizing C_4-free graphs G with $\delta(G) \geq 2$ for which $\gamma_0(G) = \left\lceil \frac{i_0(G)}{2} \right\rceil$ seems to be challenging.

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